Lecture 7: The Riemann sphere, take one: algebra.

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In this lecture, we'll apply the results of the previous lecture (structure of holomorphic maps) to the study of the Riemann sphere. Particularly, we will determine the field of meromorphic functions $\mathcal{M}(\mathbb{P}^1)$ and the group of automorphisms $Aut(\mathbb{P}^1)$.

But first, a correction from last time. Not a correction of mathematics per se, but a correction of notation. Remember, for a map $f : X \to Y$ of Riemann surfaces and a point $x \in X$ such that f is not constant in a neighborhood of x (equivalently: f is not constant on the connected component of X containing x), there is a uniquely determined natural number m such that f is locally equivalent (via biholomorphisms) to

$$z \mapsto z^m.$$

Last time we denoted this integer m by $v_x(f)$ and called it the "valuation" of f at x. But that was a mistake. We should have denoted it by $m_x(f)$ and called it the "multiplicity" of f at x. The reason is, today we'll introduce the concept which actually is called valuation, and it will be different from (though closely related to) this multiplicity.

Without further ado, let's discuss valuation. The general context is that of a non-zero meromorphic function

$$f : X \to \mathbb{P}^1$$

on a connected Riemann surface X. Suppose given a point $x \in X$. Then we will define an integer

$$v_x(f) \in \mathbb{Z}$$

which is supposed to measure "the order of vanishing of f at x", where it being negative means that f has a pole at x.

The definition is as follows. Represent f locally in a chart around x by a meromorphic function $F : U \to \mathbb{P}^1$, where U is an open subset of $\mathbb{C}$ containing 0, with 0 corresponding to x. Then there is exists an integer v such that we can write

$$F(z) = z^v \cdot G(z)$$

with $G(z)$ holomorphic and non-zero in a neighborhood of 0. (Proof: write F as a quotient of holomorphic functions, and pull out as many z's as possible from the power series expansions of the numerator and denominator.) Then we set $v_x(f) = v$.

We need to make sure that this is well-defined, i.e. any integer v as above only depends on f and x. That will follow from the following result, which gives an alternate description of $v_x(f)$:

Lemma 0.1. *Let $f : X \to \mathbb{P}^1$ be a nonzero meromorphic function on a connected Riemann surface, and let $x \in X$. Then:*

1. *If $f(x) \notin \{0, \infty\}$, we have $v_x(f) = 0$.*

2. *If $f(x) = 0$, we have $v_x(f) = m_x(f)$.*

3. *If $f(x) = \infty$, we have $v_x(f) = -m_x(f)$.*

Proof. The first point is obvious: if $f(x) \neq 0, \infty$, then we can't locally represent f by $z^v \cdot G(z)$ with $G(0) \neq 0$ unless $v = 0$. For the second point, we need to recall that in the proof of the local structure theorem, given such a decomposition $F(z) = z^v \cdot G(z)$ with $G(0) \neq 0$, we were able to make a change of charts so that F became $z \mapsto z^v$. Thus $v = m_x(f)$, as claimed. The third claim is exactly the same, except that we need to work in the chart around $\infty \in \mathbb{P}^1$, which means taking the reciprocal of f. That's what introduces the negative sign. □

Here are the two basic properties of this association $f \mapsto v_x(f)$, both easily verified from the definition:

- $v_x(f \cdot g) = v_x(f) + v_x(g)$;
- $v_x(f + g) \geq \min\{v_x(f), v_x(g)\}$.

In the second property one should assume that $f \neq -g$ so that $f + g$ is also a nonzero meromorphic function. But actually, it's slightly more convenient to formally declare that $v_x(0) = \infty$: then both of these identities are valid for arbitrary meromorphic f, g. Besides, it makes sense that the zero function should vanish to order ∞ at all points.

Thus, to a meromorphic function f on a connected Riemann surface X, we've assigned a collection of integers $v_x(f)$, one for each point $x \in X$. These integers record the location and multiplicity of the zeros and poles of f. If we further assume that X is *compact*, then there are some constraints on this system of numbers $v_x(f)$:

Proposition 0.2. *Let $f : X \to \mathbb{P}^1$ be a nonzero meromorphic function on a connected compact Riemann surface X. Then for all but finitely many $x \in X$ we have $v_x(f) = 0$, and moreover*

$$\sum_{x \in X} v_x(f) = 0.$$

(Note: the seemingly infinite sum is actually a finite sum, by the first claim.)

Proof. If f is constant, then by the lemma $v_x(f) = 0$ for all $x \in X$, and therefore the claims are trivial. Thus we assume f is non-constant. Then since X is compact, f is a non-constant proper map between connected Riemann surfaces. Therefore it has finite fibers, so $f^{-1}(0)$ and $f^{-1}(\infty)$ are finite. This proves the first claim, since by the lemma again only the points in those two sets can have valuation different from 0. For the second claim, we recall the invariance of degree for such a map f, from the previous lecture:

$$deg_0(f) = deg_\infty(f).$$

This means that $\sum_{x \in f^{-1}(0)} m_x(f) = \sum_{x \in f^{-1}(\infty)} m_x(f)$. By rearranging terms and using the lemma, this gives us exactly the required identity. □

Let us give an example. In the previous lecture we saw that a non-zero polynomial gives a meromorphic function f on $\mathbb{P}^1$ with $m_\infty(f) = deg(f)$. Thus we have

$$v_\infty(f) = -deg(f),$$

and for $a \in \mathbb{C}$ we have that $v_a(f)$ is the multiplicity of a as a root of f (i.e., $f(z) = (z-a)^{v_a(f)} \cdot g(z)$ with $g(a) \neq 0$). Thus, in this case, the above proposition is another restatement of the fact that a

non-zero polynomial has exactly as many roots (with multiplicity) as its degree.

Now, let X be a compact connected Riemann surface. We have just seen that a non-zero meromorphic function on X gives rise to the following data:

- For every point $x \in X$, an integer v_x, such that:
- All but finitely many v_x are equal to 0, and the sum of all the v_x is 0.

It is natural to ask about the converse, i.e. the following *Question*: Suppose we are given data as in the two bullet points above. Is there always meromorphic function which gives rise to it?

This is the problem of specifying a meromorphic function by the location and multiplicity of its zeros and poles. It turns out that the answer depends on the nature of X. In fact:

Proposition 0.3. *The answer to the above question is "Yes" if and only if X is isomorphic to $\mathbb{P}^1$.*

For arbitrary (compact connected) X, it turns out that the extent of the "no" answer is controlled by a (higher dimensional) complex torus known as the *Jacobian* of X. (Thus the Jacobian is trivial when $X = \mathbb{P}^1$.) But let's not get into that, at least not right now.

Proof. First let us show that if the answer is "Yes", then X must be isomorphic to $\mathbb{P}^1$. For that, choose two arbitrary distinct points x_0 and x_∞ on X. Then consider the following instance of the above data: we let $v_x = 0$ for $x \neq x_0, x_\infty$, and we set $v_{x_0} = 1$ and $v_{x_\infty} = -1$. If we assume the answer is "yes", then there is a meromorphic function $f : X \to \mathbb{P}^1$ with $v_x(f) = v_x$ for all $x \in \mathbb{P}^1$. We claim that f must in fact be an isomorphism of Riemann surfaces.

Indeed, since the only point in $f^{-1}(\infty)$ is x_∞, which has multiplicity 1, it follows that $deg_\infty(f) = 1$. Then by invariance of degree, we see that $deg_y(f) = 1$ for all $y \in \mathbb{P}^1$. The only way this can happen is if f is bijective and has multiplicity 1 everywhere, i.e. f is an isomorphism, as claimed.

Now let us show that if X is isomorphic to $\mathbb{P}^1$, then the answer to the question is yes. Since we can transport the initial data as well as the desired meromorphic function f along any isomorphism, it suffices to assume that $X = \mathbb{P}^1$.

Thus suppose given such data on $\mathbb{P}^1$. Define $f : \mathbb{P}^1 \to \mathbb{P}^1$ as follows:

$$f(z) = \prod_{a \in \mathbb{C}} (z - a)^{v_a}.$$

Then f is a rational function in z, and hence is meromorphic. Furthermore, straight from the definitions it follows that $v_x(f) = v_x$ for all $x \in \mathbb{P}^1 \setminus \{\infty\}$. Thus we need only see that $v_\infty(f) = v_\infty$. But we know both that

$$\sum_{x \in \mathbb{P}^1} v_x(f) = 0$$

and that

$$\sum_{x \in \mathbb{P}^1} v_x = 0,$$

so the case $x = \infty$ follows from all the other cases. That finishes the proof. □

The companion question of uniqueness is much less subtle:

Proposition 0.4. *Let X be a compact connected Riemann surface, and let f, g be two nonzero meromorphic functions on X. Then $v_x(f) = v_x(g)$ for all $x \in X$ if and only if there is a nonzero constant λ such that $f = \lambda \cdot g$.*

Proof. It's clear that if $f = \lambda \cdot g$ then $v_x(f) = v_x(g)$ for all $x \in X$. For the converse, consider the meromorphic function $h = f/g$. Then $v_x(h) = 0$ for all $x \in X$. Thus the map $h : X \to \mathbb{P}^1$ misses the point ∞, say. But every nonconstant holomorphic map $X \to \mathbb{P}^1$ has to be surjective: this is a special case of the invariance of degree theorem (a point y is not in the image if and only if $deg_y(h) = 0$). Thus h must be a constant λ (nonzero since f is nonzero), and the claim follows. □

There is a corollary of the (proofs of) the previous two results:

Corollary 0.5. *Every nonzero meromorphic function on $\mathbb{P}^1$ can be uniquely written in the form*

$$f(z) = \lambda \cdot \prod_{a \in \mathbb{C}} (z - a)^{v_a},$$

for some arbitrary function $a \mapsto v_a$ from $\mathbb{C} \to \mathbb{Z}$ which is zero for all but finitely many $a \in \mathbb{C}$, and some arbitrary nonzero constant λ.

In particular, the meromorphic functions on $\mathbb{P}^1$ are exactly the rational functions of z:

$$\mathcal{M}(\mathbb{P}^1) = \mathbb{C}(z).$$

This finishes our discussion of meromorphic functions on $\mathbb{P}^1$. Now we turn to automorphisms of $\mathbb{P}^1$. Here the main claim is the following.

Theorem 0.6. *Let $a, b, c \in \mathbb{P}^1$ be three distinct ordered points. Then there is a unique automorphism $h : \mathbb{P}^1 \to \mathbb{P}^1$ such that*

$$h(0, 1, \infty) = (a, b, c).$$

Thus, the automorphisms of $\mathbb{P}^1$ are in bijection with the triples of distinct ordered points on $\mathbb{P}^1$. In particular, morally speaking, $Aut(\mathbb{P}^1)$ has complex dimension 3 or real dimension 6.

Proof. Suppose given such a, b, c. It will be equivalent to show that there is a unique automorphism H of $\mathbb{P}^1$ with

$$H(a, b, c) = (0, 1, \infty);$$

indeed, the H's and the h's are in bijection by the prescription $h = H^{-1}$.

Now, for $x \in \mathbb{P}^1$, let $v_x = 0$ for $x \neq a, c$, and $v_a = 1$ and $v_c = -1$. By the "yes" answer to the above question for $\mathbb{P}^1$, there is a meromorphic function $H : \mathbb{P}^1 \to \mathbb{P}^1$ with $H(a) = 0$ and $H(c) = \infty$. Moreover as we saw in the first part of the proof of Proposition 0.3, such an H is necessarily an isomorphism, i.e. an automorphism of $\mathbb{P}^1$; and Proposition 0.4 tells us that H is unique up to a nonzero scalar.

Now, we don't know anything about $H(b)$ except that it's different from 0 and ∞ (since H is a bijection). But then by fixing the non-zero scalar we can uniquely ensure that $H(b) = 1$. This finishes the proof. □

This is a nice abstract characterization of the automorphisms of $\mathbb{P}^1$, but for some purposes we want to actually write down a formula. Such a formula falls out from the above proof, since we saw how to produce the required H explicitly in the course of the proof of the "yes" answer for $\mathbb{P}^1$. After some unwinding, the result is the following:

Corollary 0.7. *Every automorphism of $\mathbb{P}^1$ is of the following form:*

$$h(z) = \frac{az + b}{cz + d},$$

where $ad - bc \neq 0$. Moreover, a, b, c and d (which have nothing to do with the previous a, b, c) are uniquely determined up to an overall nonzero scalar factor.

Here is a corollary of the corollary:

Corollary 0.8. *Every automorphism of $\mathbb{P}^1$ has a fixed point. In particular, any group $\Gamma \subset \mathbb{P}^1$ acting freely has to be the identity $\{id\}$.*

To prove this, one just investigates the equation $h(z) = z$.

Let us return to the original corollary, and make some remarks. Note that the condition $ad - bc \neq 0$ is equivalent to saying that the determinant of the 2×2 matrix formed by a, b, c and d is nonzero. In fact, this matrix language is a good one to use, since, as you can verify, composition of such automorphisms h corresponds exactly to matrix multiplication on the coefficients a, b, c, d.

We can explain this unexpected appearance of linear algebra by giving a new, more symmetric description of the points of $\mathbb{P}^1$. You see, we defined $\mathbb{P}^1$ to be $\mathbb{C} \cup \{\infty\}$. This is an asymmetric description of $\mathbb{P}^1$, since it singles out the point ∞ as being special, whereas, as we have just seen, any three distinct points of $\mathbb{P}^1$ can be carried to any other three distinct points by an automorphism of $\mathbb{P}^1$, so there are no special points of $\mathbb{P}^1$, considered as an abstract Riemann surface. Note that this fact also makes our explicit description of the automorphisms,

$$h(z) = \frac{az + b}{cz + d}$$

somewhat of a nuisance to work with, since ∞ has to be treated separately, e.g. we need to specify that $h(\infty) = \frac{a}{c}$, and $h(z) = \infty$ if $cz + d = 0$.

But now let us fix this by giving a symmetric description of $\mathbb{P}^1$, together with a resulting symmetric description of the automorphisms.

Proposition 0.9. *There is a canonical bijection between $\mathbb{P}^1$ and the set of one-dimensional complex linear subspaces of the two-dimensional complex vector space $V = \mathbb{C} \oplus \mathbb{C}$.*

Proof. The bijection ϕ is defined as follows: we set

$$\phi(\infty) = [1, 0],$$

meaning the one-dimensional vector space spanned by $(1, 0) \in V$; and for $z \in \mathbb{C}$ we set

$$\phi(z) = [z, 1].$$

The inverse to ϕ is defined by

$$[a, b] \mapsto \frac{a}{b},$$

where we mean ∞ if $b = 0$. Note that this makes sense: $\frac{a}{b}$ only depends on the vector space spanned by $(a, b) \neq (0, 0)$. (A loose description of this inverse is that it sends a (complex) line through the origin to its (complex) *slope*.) It's easy to see that these maps are inverse to one another, and hence define the required bijection. □

The automorphisms of $\mathbb{P}^1$ are easy to see in terms of this new description. Indeed, an invertible 2×2 complex matrix A acts by linear automorphisms of V, by matrix multiplication. Being a linear automorphism, it sends one-dimensional subspaces to one-dimensional subspaces, and so does its inverse; thus A induces a bijection from $\mathbb{P}^1$ to itself. If the entries of A are a, b, c, d, then it's simple to verify that this new symmetric description matches the previous

$$h(z) = \frac{az + b}{cz + d} :$$

indeed, that follows from the fact that A acting on the column vector $(z, 1)$ gives $(az + b, cz + d)$.

This new description also turns some of our earlier claims into simple linear algebra facts: for example it's fairly clear that a 2×2 invertible matrix sends every one-dimensional subspace into itself, and hence induces the identity map on $\mathbb{P}^1$, if and only if A is given by multiplication by a nonzero constant; that's why this was the discrepancy in representing an automorphism by a, b, c, d. Furthermore, the fact that every automorphism of $\mathbb{P}^1$ has a fixed point becomes the fact that every invertible complex matrix has a non-zero eigenvalue (with corresponding nonzero eigenvector).

To sum up, we have seen that the group of automorphisms of $\mathbb{P}^1$ is the same as the group

$$PGL_2(\mathbb{C})$$

of invertible 2×2 complex matrices, modulo nonzero complex scalars. This also matches our previous dimension count: there are four complex dimensions coming from the entries of the matrix; then one gets subtracted because of the non-zero scalars, leaving 3 complex dimensions, or 6 real dimensions.