Linearizing nonlinear systems of equations

Aaron Fenyes (`afenyes@math.toronto.edu`)

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Example: predatory and prey populations

Model

Put some yeast in a jar full of nutrients. Left to themselves, the yeast reproduce at a rate proportional to their population Y. Their population growth is described by the equation

$$Y' = Y.$$

Take another jar full of Paramecia, which are microscopic predators. Without prey to eat, the Paramecia are dying at a rate proportional to their population. Their population P shrinks according to the equation

$$P' = -P.$$

Now, let the Paramecia into the jar of yeast, so they can eat the yeast. The rate at which they eat is proportional to YP. The more Paramecia there are, the faster they can catch the yeast. The more yeast there are, the easier they are to catch. The populations of yeast and paramecia are now linked by the system of equations

$$\begin{aligned} Y' &= Y - YP \\ P' &= -P + YP. \end{aligned}$$

Linearization around $(0, 0)$

The system above is nonlinear: Y' and P' aren't just weighted sums of Y and P. When we zoomed in on the phase portrait, however, we saw that near $(0, 0)$ it looks like the phase portrait of a linear system with a saddle.

To understand what's going on, suppose the population vector (Y, P) is very close to the critical point $(0, 0)$. To be fancy, we can write

$$\begin{bmatrix} Y \\ P \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} y \\ p \end{bmatrix},$$

with the assumption that

$$\begin{bmatrix} y \\ p \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Rewriting our system of equations in terms of the new variables y and p, we get

$$\begin{aligned}(0+y)' &= (0+y) - (0+y)(0+p)\\ (0+p)' &= -(0+p) + (0+y)(0+p),\end{aligned}$$

which simplifies to

$$\begin{aligned}y' &= y - yp\\ p' &= -p + yp.\end{aligned}$$

Since y and p are very small, their product yp is much smaller than either of them. Hence, the approximation

$$\begin{aligned}y' &\approx y\\ p' &\approx -p\end{aligned}$$

is very accurate. In summary, near the critical point $(0,0)$, our original nonlinear system of equations acts a lot like the linear system

$$\begin{bmatrix} y \\ p \end{bmatrix}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y \\ p \end{bmatrix}.$$

The phase portrait of this *linearized* system is a saddle, explaining why the phase portrait of the original system looks like a saddle near $(0,0)$.

The general solution of the linearized system is

$$\begin{bmatrix} y \\ p \end{bmatrix} = Ae^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + Be^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

for constants A and B. No matter what the initial conditions are, the yeast population grows exponentially, while the Paramecium population shrinks exponentially. The Paramecia can't catch the yeast fast enough to sustain their population, or to slow down the yeast's exponential growth.

Linearization around $(1,1)$

Our nonlinear system has another critical point at $(1,1)$. When we zoom in on the phase portrait near that critical point, we see what looks like the phase portrait of a linear system with a center. [*Show picture in notebook.*] Once again, we can understand what's going on using linearization.

Suppose

$$\begin{bmatrix} Y \\ P \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} y \\ p \end{bmatrix},$$

with

$$\begin{bmatrix} y \\ p \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Rewriting our system of equations in terms of the new variables y and p, we get

$$\begin{aligned}(1+y)' &= (1+y) - (1+y)(1+p)\\ (1+p)' &= -(1+p) + (1+y)(1+p),\end{aligned}$$

which simplifies to

$$\begin{aligned}y' &= -p - yp\\ p' &= y + yp.\end{aligned}$$

Like before, we observe that yp is very small compared to y and p, so the approximation

$$\begin{aligned}y' &\approx -p\\ p' &\approx y\end{aligned}$$

is very accurate. In summary, near the critical point $(1,1)$, our original nonlinear system of equations acts a lot like the linear system

$$\begin{bmatrix} y \\ p \end{bmatrix}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ p \end{bmatrix}.$$

The phase portrait of this *linearized* system is a center, explaining why the phase portrait of the original system looks like a center near $(1,1)$.

Example: epidemic

Model

Our predator-prey population model has just a few separate critical points. Most nonlinear systems look like that. Sometimes, though, you might see weirder-looking phase portraits with more critical points.

A few lectures ago, for example, we looked at a model for the spread of an infection, using the variables S, F, R for the populations of susceptible, infected, and recovered people.

$$\begin{aligned}S' &= -3SF\\ F' &= 3SF - 2F\\ R' &= 2F.\end{aligned}$$

Since R doesn't influence S, and F, we don't have to keep track of it to understand how S and F are changing. [*Show phase portrait in notebook.*]

$$\begin{aligned}S' &= -3SF\\ F' &= 3SF - 2F.\end{aligned}$$

For this equation, every single point on the line $F = 0$ is critical. We can get a linear system by zooming in on any of these points.

Linearization around $(1,0)$

Earlier, for example, we zoomed in on the critical point $(1,0)$. Suppose

$$\begin{bmatrix} S \\ F \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} s \\ f \end{bmatrix},$$

with

$$\begin{bmatrix} s \\ f \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Substituting,

$$\begin{aligned} (1+s)' &= -3(1+s)(0+f) \\ (0+f)' &= 3(1+s)(0+f) - 2(0+f). \end{aligned}$$

Simplifying,

$$\begin{aligned} s' &= -3f - 3sf \\ f' &= f + 3sf, \end{aligned}$$

so we have the approximation

$$\begin{aligned} s' &\approx -3f \\ f' &\approx f. \end{aligned}$$

The linearized system is

$$\begin{bmatrix} s \\ f \end{bmatrix}' = \begin{bmatrix} 0 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ f \end{bmatrix}.$$

[*Show zoomed-in phase portrait.*] The general solution is

$$\begin{bmatrix} s \\ f \end{bmatrix} = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} + Be^t \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

for constants A and B. Note how the number of infected people grows exponentially.

Example: pendulum

Model

I mentioned at the beginning of the course that the angle of a pendulum is described the second-order nonlinear equation

$$X'' = -\sin X.$$

We've been looking at this system for very small values of X—to be fancy, $X = 0 + x$ for $x \approx 0$. We used the approximation $\sin x \approx x$ to get the linear equation

$$x'' = -x.$$

Let's look at this model in more detail.

Introduce the new variable $V = X'$ to get the nonlinear system

$$\begin{aligned} X' &= \quad V \\ V' &= -\sin X. \end{aligned}$$

The critical points are $(n\pi, 0)$ for integers n. [*Show phase portrait, if possible. It's not in the notebook yet, unfortunately.*]

Linearization around $(0, 0)$

Let's zoom in on the critical point $(0, 0)$, supposing

$$\begin{bmatrix} X \\ V \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} x \\ v \end{bmatrix},$$

with

$$\begin{bmatrix} x \\ v \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Substituting, we get

$$\begin{aligned} X' &= \quad 0 + v \\ V' &= -\sin(0 + x). \end{aligned}$$

Using the linear approximation

$$\begin{aligned} \sin(0 + x) &\approx \sin 0 + x \sin' 0 \\ &= 0 - x \cos 0 \\ &= -x. \end{aligned}$$

for the sine function, we get the approximate linear system

$$\begin{aligned} x' &= \quad v \\ v' &= -x. \end{aligned}$$

Linearization around $(\pi, 0)$

[*Repeat.*]