Variation of parameters for vector differential equations

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General recipe

Say you want to solve the vector differential equation

$$\vec{y}'(t) = P(t)\,\vec{y}(t) + \vec{f}(t),$$

with a chosen initial value $\vec{y}(0)$.

Step 1: Rescale the unknown function

Switch to a new unknown $\vec{Y}$ defined by the equation

$$\vec{y}(t) = \Lambda(t)\,\vec{Y}(t).$$

We'll choose the matrix Λ later.

Step 2: See which choices of Λ are good

Substitute into the original equation to rewrite it in terms of $\vec{Y}$.

$$\Lambda'\vec{Y} + \Lambda\vec{Y}' = P\Lambda\vec{Y} + \vec{f}$$

Rearranging, we get the equation

$$(\Lambda' - P\Lambda)\vec{Y} + \Lambda\vec{Y}' = \vec{f}.$$

If we choose a matrix Λ so that $\Lambda' - P\Lambda = 0$, the equation we're trying to solve simplifies to

$$\Lambda\vec{Y}' = \vec{f}.$$

This equation is easy to solve, as long as Λ is invertible.

Step 3: Find a good Λ

To get an invertible matrix Λ with $\Lambda' - P\Lambda$, find two linearly independent solutions $\vec{a}$ and $\vec{b}$ of the homogeneous equation

$$\vec{u}' = P\vec{u}.$$

Then the matrix

$$\Lambda = \left[\begin{array}{cc} \vec{a} & \vec{b} \end{array} \right]$$

satisfies the equation

$$\Lambda' = P\Lambda.$$

Step 4: Solve for $\vec{Y}$

Now that we have Λ, we can treat the simplified differential equation

$$\Lambda \vec{Y}' = \vec{f}$$

as a system of linear equations for the coordinates $Y_1', Y_2', Y_3' \ldots$. Solve it to express $Y_1', Y_2', Y_3' \ldots$ in terms of Λ and $\vec{f}$. Integrate from the initial values $Y_1(0), Y_2(0), Y_3(0) \ldots$ to get $Y_1, Y_2, Y_3, \ldots$, and then use the definition $\vec{y} = \Lambda \vec{Y}$ to recover the original unknown.

Example

Let's try to find a solution of the second-order inhomogeneous equation

$$x'' + 3x' + 2x = \frac{2}{e^t + e^{-t}},$$

which we used as an example in class. Instead of fixing the initial conditions, we'll just accept any solution.

Rewrite as a vector differential equation

We can rewrite our second-order number-valued equation as the first-order vector-valued differential equation

$$\left[\begin{array}{c} x \\ x' \end{array} \right]' = \underbrace{\left[\begin{array}{cc} 0 & 1 \\ -2 & -3 \end{array} \right]}_{M} \underbrace{\left[\begin{array}{c} x \\ x' \end{array} \right]}_{\vec{y}} + \underbrace{\left[\begin{array}{c} 0 \\ 2/(e^t + e^{-t}) \end{array} \right]}_{\vec{f}}.$$

Step 1: Rescale the unknown function

Switch to a new unknown $\vec{Y}$ defined by the equation

$$\vec{y}(t) = \Lambda(t)\,\vec{Y}(t).$$

We'll choose the matrix Λ later.

Step 2: See which choices of Λ are good

If we choose a matrix Λ so that $\Lambda' - M\Lambda = 0$, the equation we're trying to solve simplifies to

$$\Lambda \vec{Y}' = \begin{bmatrix} 0 \\ 2/(e^t + e^{-t}) \end{bmatrix}.$$

Step 3: Find a good Λ

To get an invertible matrix Λ with $\Lambda' - M\Lambda$, we need to find two linearly independent solutions of the homogeneous equation

$$\vec{u}' = M\vec{u}.$$

Since the coefficient matrix M is constant, we can build solutions from its eigenvectors and eigenvalues in the usual way.

eigenvector	eigenvalue	solution
$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	-1	$e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
$\begin{bmatrix} 1 \\ -2 \end{bmatrix}$	-2	$e^{-2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Then we know the matrix

$$\Lambda = \begin{bmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{bmatrix}$$

satisfies the equation

$$\Lambda' = M\Lambda.$$

Step 4: Solve for $\vec{Y}$

Now we can plug our Λ into the simplified equation

$$\Lambda \vec{Y}' = \begin{bmatrix} 0 \\ 2/(e^t + e^{-t}) \end{bmatrix}.$$

Expressing $\vec{Y}$ in coordinates, we get

$$\begin{bmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{bmatrix} \begin{bmatrix} Y_1' \\ Y_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2/(e^t + e^{-t}) \end{bmatrix}.$$

We can treat this as a system of linear equations for Y_1' and Y_2'. Solve it to see that

$$\begin{aligned} e^{-t}\, Y_1' &= 2/(e^t + e^{-t}) \\ -e^{-2t}\, Y_2' &= 2/(e^t + e^{-t}). \end{aligned}$$

Rearrange to isolate Y_1' and Y_2'.

$$
\begin{aligned}
Y_1' &= \frac{2e^t}{e^t + e^{-t}} \\
Y_2' &= -\frac{2e^{2t}}{e^t + e^{-t}}
\end{aligned}
$$

Let's choose the antiderivatives

$$
\begin{aligned}
Y_1 &= \ln(1 + e^{2t}) \\
Y_2 &= 2[\arctan(e^t) - e^t].
\end{aligned}
$$

Plugging our Λ and our solution for $\vec{Y}$ into the definition

$$\vec{y} = \Lambda \vec{Y},$$

we see that

$$\vec{y} = \begin{bmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{bmatrix} \begin{bmatrix} \ln(1 + e^{2t}) \\ 2[\arctan(e^t) - e^t] \end{bmatrix}.$$

We could get other solutions by adding constants to Y_1 and Y_2.