Homework 1

Due on Crowdmark
January 17, 11 a.m.

Chaos, fractals, and dynamics
MAT 335, Winter 2019

Show your calculations, and explain your reasoning. Your goal is for the graders to understand how you got your answers, and to be convinced that your reasoning makes sense.

Marking guide

For each problem, in the "Solution" heading, I describe how to split the problem into gradable pieces, and how many points each piece is worth. Give the work for each gradable piece 0/3, 1/3, 2/3, or 3/3 of the points available, according to the guidelines below. Record your mark for each piece on the page; I recommend using the built in ✘ and ✔ symbols in Crowdmark as shown.

✘ **0/3** Does not show much understanding of what's going on.

✔ **1/3** Shows a basic understanding of what's going on, but doesn't get very far down any promising path to a solution.

✔✔ **2/3** Gets pretty far down a promising path to a solution, but doesn't get all the way there, due to a conceptual error, major computation errors, or significant omissions.

✔✔✔ **3/3** Gets basically all the way to a solution, with only superficial errors or omissions.

1 Friendly formulas for iterates

Consider the map $F(x) = 2x + 1$ on the state space $\mathbb{R}$.

a. Write formulas for $F^2(x)$, $F^3(x)$, and $F^4(x)$ in terms of x.

b. Write a formula for $F^n(x)$ in terms of n and x.

Solution (3 points for a; 3 points for b)

a.

$$\begin{aligned} F^2(x) &= F(F(x)) \\ &= F(2x+1) \\ &= 2(2x+1)+1 \\ &= 4x+2+1 \\ &= 4x+3. \end{aligned}$$

$$
\begin{aligned}
F^3(x) &= F(F^2(x)) \\
&= F(4x + 2 + 1) \\
&= 2(4x + 2 + 1) + 1 \\
&= 8x + 4 + 2 + 1 \\
&= 4x + 7.
\end{aligned}
$$

$$
\begin{aligned}
F^4(x) &= F(F^3(x)) \\
&= F(8x + 4 + 2 + 1) \\
&= 2(8x + 4 + 2 + 1) + 1 \\
&= 16x + 8 + 4 + 2 + 1 \\
&= 16x + 15.
\end{aligned}
$$

b. Our calculations from part a reveal the pattern

$$F^n(x) = 2^n x + 2^{n-1} + \ldots + 2 + 1.$$

You can prove by induction, or by writing in binary, that $2^{n-1} + \ldots + 2 + 1 = 2^n - 1$. That gives us the formula

$$F^n(x) = 2^n x + 2^n - 1.$$

2 Periodic points of the shift map

For the shift map $S\colon \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$, list the periodic points...

a. ... with minimum period 1.

b. ... with minimum period 2.

c. ... with minimum period 3.

Solution (6 points for a, b, c)

We'll use a bar to denote a repeating sequence of digits. So, for example, $\overline{\texttt{010}}$ denotes

$$\texttt{010010010010010}\ldots.$$

a. The 1-periodic points of the shift map are the sequences in which each digit is equal to the next digit. Hence, a 1-periodic sequence is determined by its first digit, and any choice of first digit is possible. The 1-periodic points are therefore

$$\overline{\texttt{0}}, \quad \overline{\texttt{1}}.$$

A 1-periodic point automatically has minimum period 1.

b. The 2-periodic points of the shift map are the sequences in which each digit is equal to the digit two places to the right. Hence, a 2-periodic sequence is determined by its first two digits, and any choice of the first two digits is possible. Writing down the four possible choices of first two digits, and excluding the two that already appeared in our list of 1-periodic sequences, we see that

$$\overline{\texttt{01}}, \quad \overline{\texttt{10}}$$

are the sequences of minimum period 2.

c. By the same reasoning we used in the previous parts, a 3-periodic sequence is determined by its first three digits, and any choice of the first three digits is possible. Writing down the eight possible choices of first three digits, and excluding the two that already appeared in our lists of 1-periodic and 2-periodic sequences, we see that

$$\overline{\texttt{001}}, \quad \overline{\texttt{010}}, \quad \overline{\texttt{011}}, \quad \overline{\texttt{100}}, \quad \overline{\texttt{101}}, \quad \overline{\texttt{110}}$$

are the sequences of minimum period 3.

3 Eventually fixed points of a quadratic map

Consider the dynamical system with state space $\mathbb{R}$ and dynamical map $G(x) = x^2 - 2$.

a. Find the fixed points of G.

b. Find every point whose orbit reaches a fixed point within four steps.

c. Draw a "family tree" for the points you found in part b, with an arrow from each point y to its "parent" $G(y)$. Don't forget to draw an arrow from each fixed point to itself!

Solution (3 points for a; 3 points for b, c)

a. The fixed points are the solutions of the equation

$$\begin{aligned} G(x) &= x \\ x^2 - 2 &= x \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0. \end{aligned}$$

So, the fixed points are

$$2, \quad -1.$$

b, c The points that reach a fixed point within one step are the solutions of the equations

$$\begin{aligned} G(x) &= 2 \\ x^2 - 2 &= 2 \\ x^2 &= 4 \end{aligned} \qquad\qquad \begin{aligned} G(x) &= -1 \\ x^2 - 2 &= -1 \\ x^2 &= 1. \end{aligned}$$

The solutions are the fixed points and two new points, -2 and 1.

The points that reach -2 or 1 within one step are the solutions of

$$\begin{aligned} G(x) &= -2 \\ x^2 &= 0 \end{aligned} \qquad\qquad \begin{aligned} G(x) &= 1 \\ x^2 &= 3. \end{aligned}$$

The solutions are 0 and $\pm\sqrt{3}$.

The points that reach 0 or $\pm\sqrt{3}$ within one step are the solutions of

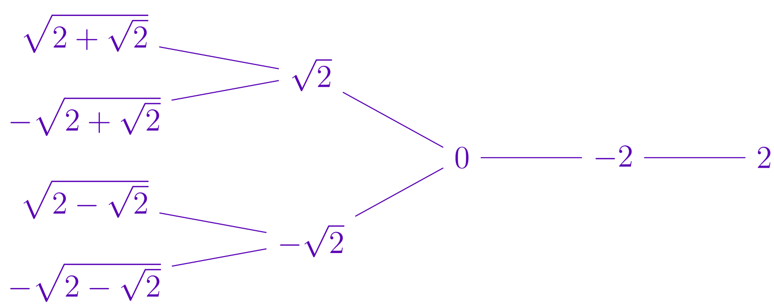
$$\begin{aligned} G(x) &= 0 \\ x^2 &= 2 \end{aligned} \qquad\qquad \begin{aligned} G(x) &= \sqrt{3} \\ x^2 &= 2+\sqrt{3} \end{aligned} \qquad\qquad \begin{aligned} G(x) &= -\sqrt{3} \\ x^2 &= 2-\sqrt{3}. \end{aligned}$$

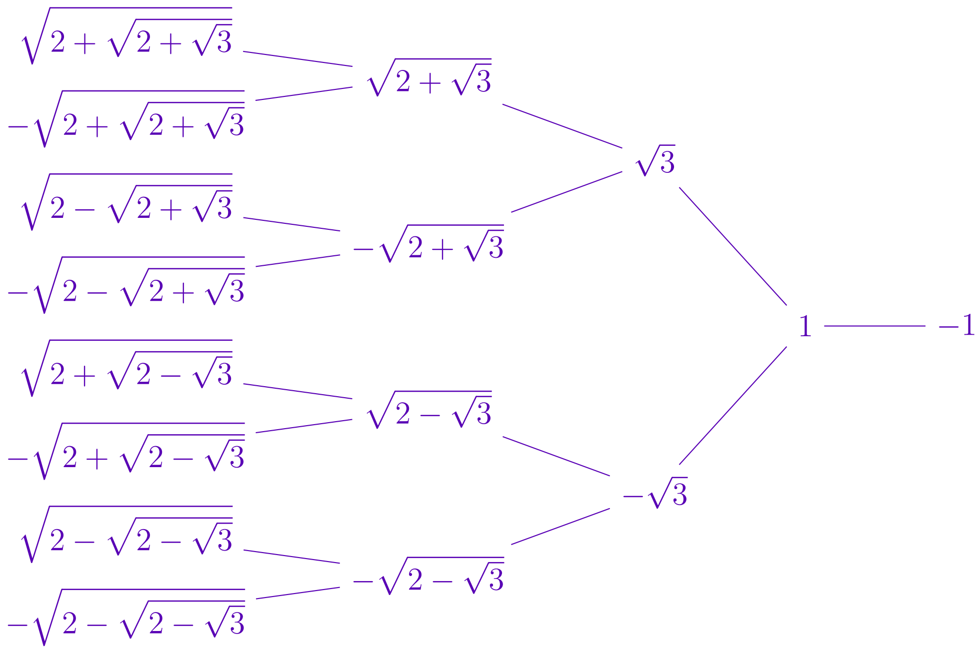
The solutions are $\pm\sqrt{2}$, $\pm\sqrt{2+\sqrt{3}}$, and $\pm\sqrt{2-\sqrt{3}}$.

By a similar calculation, the points that reach $\sqrt{2}$, $\pm\sqrt{2+\sqrt{3}}$, or $\pm\sqrt{2-\sqrt{3}}$ within one step are

$$\begin{array}{ccc} \pm\sqrt{2+\sqrt{2}} & \pm\sqrt{2+\sqrt{2+\sqrt{3}}} & \pm\sqrt{2+\sqrt{2-\sqrt{3}}} \\ \pm\sqrt{2-\sqrt{2}} & \pm\sqrt{2-\sqrt{2+\sqrt{3}}} & \pm\sqrt{2-\sqrt{2-\sqrt{3}}} \end{array}$$

Altogether, the points that reach a fixed point within four steps are (student solutions should include arrowheads, and loops for the fixed points)





4 Sweep away the 1s

Here's a new dynamical system.

State space: $\mathbf{2}^{\mathbb{N}}$.

Dynamical map: Each `1` that's followed by a `0` turns into a `0`.

Let's call this map A. As a demonstration, here's what A does to one point in $\mathbf{2}^{\mathbb{N}}$.

$$\begin{aligned} w &= \mathtt{001110011011110100101110}\ldots \\ A(w) &= \mathtt{0011\underline{0}001\underline{0}0111\underline{0}0\underline{0}00\underline{0}011\underline{0}0}\ldots \end{aligned}$$

The changed digits are underlined.

a. Describe all the fixed points of A.

b. Find a point in $\mathbf{2}^{\mathbb{N}}$ which is not eventually fixed. Describe your point carefully, and convince a skeptical grader that it's not eventually fixed.

Solution (3 points for a; 3 points for b)

a. The map A changes every `1` which is followed by a `0`, and doesn't change anything else. Hence, the fixed points of A are the sequences in which `10` never occurs. Using the bar notation from problem 2, these sequences are

$$\mathtt{\overline{1}},\quad \mathtt{0\overline{1}},\quad \mathtt{00\overline{1}},\quad \mathtt{000\overline{1}},\quad \mathtt{0000\overline{1}},\quad \ldots$$

and also

$$\mathtt{\overline{0}}.$$

b. One example is the sequence

$$\underbrace{\texttt{10}}_{\text{block 1}}\ \underbrace{\texttt{110}}_{\text{block 2}}\ \underbrace{\texttt{1110}}_{\text{block 3}}\ \underbrace{\texttt{11110}}_{\text{block 4}}\ \underbrace{\texttt{111110}}_{\text{block 5}}\ \underbrace{\texttt{1111110}}_{\text{block 6}}\ \underbrace{\texttt{11111110}}_{\text{block 7}}\ \ldots.$$

The blocks continue like this forever. Block n consists of n `1`s followed by a `0`.

Here's why this sequence is not eventually fixed. Block n starts out containing n `1`s. It loses a `1` each time you apply A. As a result, block n is changing for the first n steps. This guarantees that, no matter how many times you apply A, there's always a block that's still changing.

5 Dueling periods

Consider a dynamical system with state space Y and dynamical map M.

a. Suppose $a \in Y$ is both 2-periodic and 3-periodic. Find the minimum period of a.

b. Suppose $b \in Y$ is both 3-periodic and 5-periodic. Find the minimum period of b.

Solution (3 points for a; 3 points for b)

a. Since a is 2-periodic, $M(M(a)) = a$. Since a is also 3-periodic,

$$M(\underbrace{M(M(a))}_{a}) = a$$
$$M(a) = a.$$

In other words, a is 1-periodic. A 1-periodic point automatically has minimum period 1.

b. Since b is 3-periodic, $M^3(b) = b$. Since b is also 5-periodic,

$$M^5(b) = b$$
$$M^2(M^3(b)) = b$$
$$M^2(b) = b.$$

In other words, b is 2-periodic. It follows, from part a, that b has minimum period 1.