# Homework 3

Due on Crowdmark
March 4, 11 a.m.

Chaos, fractals, and dynamics
MAT 335, Winter 2019

*Show your calculations, and explain your reasoning. Your goal is for the graders to understand how you got your answers, and to be convinced that your reasoning makes sense.*

## Marking guide

*For each problem, in the "Solution" heading, I describe how to split the problem into gradable pieces, and how many points each piece is worth. Give the work for each gradable piece* 0/3*,* 1/3*,* 2/3*, or* 3/3 *of the points available, according to the guidelines below. Record your mark for each piece on the page; I recommend using the built in ✘ and ✔ symbols in Crowdmark as shown.*

✘ **0/3** Does not show much understanding of what's going on.

✔ **1/3** Shows a basic understanding of what's going on, but doesn't get very far down any promising path to a solution.

✔✔ **2/3** Gets pretty far down a promising path to a solution, but doesn't get all the way there, due to a conceptual error, major computation errors, or significant omissions.

✔✔✔ **3/3** Gets basically all the way to a solution, with only superficial errors or omissions.

### 1 Standardizing quadratic maps

In week 1, when we first met the dynamical maps $Q_u(x) = x^2 + u$ on the state space $\mathbb{R}$, I introduced them as the "standard quadratic maps." Now that we've learned about semiconjugacy, I can explain why I chose that name.

a. Find a semiconjugacy from the quadratic map $F(x) = x^2 + 6x + 5$ to the standard quadratic map $Q_{-1}$. *(Corrected: the previous version had* 3 *as the coefficient of* $x$*.)*

   HINT: Look for constants $a$, $b$ that make $\psi(x) = ax+b$ a semiconjugacy from $F$ to $Q_{-1}$. You should make sure you found a semiconjugacy by checking the four properties a semiconjugacy needs to have. You can take it as given that $\psi(x) = ax+b$ is continuous for any choice of $a$ and $b$, but you should think about why this is true.

b. Find a semiconjugacy from the quadratic map $G(x) = 2x^2 - 3$ to a standard quadratic map $Q_u$.

   HINT: Look for constants $a$, $b$, $u$ that make $\psi(x) = ax + b$ a semiconjugacy from $G$ to $Q_u$.

c. Show that every quadratic map $P(x) = Ax^2+2Bx+C$, with $A \neq 0$, is semiconjugate to a standard quadratic map $Q_u$. Write formulas for the semiconjugacy and the constant $u$ in terms of $A$, $B$, $C$.

Each of the semiconjugacies you'll find in this problem has a special property: it's invertible, and its inverse is a semiconjugacy too. A semiconjugacy that goes both ways like this is called a *conjugacy*. If two dynamical systems are connected by a conjugacy, they're the same for all practical purposes. So, this problem demonstrates that every quadratic map is the same, for all practical purposes, as one of our standard quadratic maps.

**Solution (3 points for a; 3 points for b; 6 points for c)**

a. The function $\psi(x) = x + 3$ is continuous, onto, and one-to-one, so it's a semiconjugacy from $F$ to $Q_{-1}$ as long as the functions $\psi \circ F$ and $Q_{-1} \circ \psi$ are the same. Using the formulas for $F$, $Q_{-1}$, and $\psi$, we see that

$$\begin{aligned} Q_{-1}(\psi(x)) &= (x+3)^2 - 1 \\ &= (x^2 + 6x + 9) - 1 \\ &= x^2 + 6x + 8, \end{aligned}$$

while

$$\begin{aligned} \psi(F(x)) &= F(x) + 3 \\ &= x^2 + 6x + 8. \end{aligned}$$

Hence, $\psi(x) = x + 3$ is indeed a semiconjugacy from $F$ to $Q_{-1}$.

b.

c. The function $\psi(x) = 2x$ is continuous, onto, and one-to-one, so it's a semiconjugacy from $G$ to $Q_u$ as long as the functions $\psi \circ G$ and $Q_u \circ \psi$ are the same. Using the formulas for $G$, $Q_u$, and $\psi$, we see that

$$\begin{aligned} Q_u(\psi(x)) &= (2x)^2 + u \\ &= 4x^2 + u, \end{aligned}$$

while

$$\begin{aligned} \psi(G(x)) &= 2G(x) \\ &= 4x^2 - 6. \end{aligned}$$

The formulas for $\psi \circ G$ and $Q_u \circ \psi$ match when $u = -6$. Hence, $\psi(x) = 2x$ is a semiconjugacy from $G$ to $Q_{-6}$.

d. Let's look for constants $a$, $b$, $u$ that make $\psi(x) = ax + b$ a semiconjugacy from $P$ to $Q_u$. There are four properties we need $\psi$ to have.

- We need $\psi$ to be continuous. This is true for any choice of $a$, $b$.
- We need $\psi$ to be onto. This is true as long as $a \neq 0$.
- We need $\psi$ to be few-to-one. When $a \neq 0$, the function $\psi$ is one-to-one.
- We need the functions $\psi \circ P$ and $Q_u \circ \psi$ have to be the same. Using the formulas for $P$, $Q_u$, and $\psi$, we see that

$$\begin{aligned} Q_u(\psi(x)) &= (ax+b)^2 + u \\ &= (a^2)x^2 + (2ab)x + (b^2+u), \end{aligned}$$

while

$$\begin{aligned} \psi(P(x)) &= a(Ax^2 + 2Bx + C) + b \\ &= (aA)x^2 + (2aB)x + (aC+b), \end{aligned}$$

so we need

$$\begin{aligned} a^2 &= aA \\ 2ab &= 2aB \\ b^2 + u &= aC + b. \end{aligned}$$

Recalling that $a$ can't be zero, we can use the first two equations to express $a$ and $b$ in terms of $A$ and $B$.

$$\begin{aligned} a &= A \\ b &= B \end{aligned}$$

Rearranging the third equation then gives us $u$ in terms of $A$, $B$, $C$.

$$\begin{aligned} u &= aC + b(1-b) \\ &= AC + B(1-B). \end{aligned}$$

## 2 An itinerary function

Let's revisit a dynamical system we mentioned briefly in the first week of the course: the rotation map $R_2\colon \mathbb{T} \to \mathbb{T}$, defined by the formula $R_2(\theta) \equiv \theta + 2$. The points in $\mathbb{T}$ whose orbits never hit 0 or $\pi$ form a subset $\Lambda \subset \mathbb{T}$. Let's define a function $\tau\colon \Lambda \to \mathbf{2}^{\mathbb{N}}$ in the following way.

$$\text{the } n\text{th digit of } \tau(\theta) \text{ is } \begin{cases} 0 & \text{if } R_2^n(\theta) \in (0,\pi) \\ 1 & \text{if } R_2^n(\theta) \in (\pi, 2\pi) \end{cases}$$

Let's call the starting digit of a sequence the 0th digit.

You can find the sequence $\tau(\theta)$ by writing down the orbit of $\theta$ and then noting whether each point on the orbit is in the top half or the bottom half of the unit circle. For example, here's a calculation of the first five digits of $\tau(1)$.

| $n$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $R_2^n(1)$ | $1.000\ldots$ | $3.000\ldots$ | $5.000\ldots$ | $0.716\ldots$ | $2.716\ldots$ |
| $n$th digit of $\tau(1)$ | 0 | 0 | 1 | 0 | 0 |

Intuitively, the sequence $\tau(\theta)$ tells you when the orbit of $\theta$ visits the top and bottom halves of the unit circle. A function that gives this kind of information is called an *itinerary function*. Itinerary functions are helpful for understanding the orbits of many dynamical systems, including quadratic maps.

Surprisingly, the function $\tau$ is continuous! This problem will walk you through an argument that $\tau$ is continuous at 1. To understand what you're doing, you should review Section 2 of the week 5 notes.

a. Use a calculator to find the first twenty digits of $\tau(1)$. (The table above gives you the first five digits for free.)

b. Find a radius $r \in (0, \infty)$ small enough that $\tau$ sends every point in $B_1(r)$ into the target ball $B_{\tau(1)}(2^0)$. We'll use the shorthand $\tau(B_1(r)) \subset B_{\tau(1)}(2^0)$ to express this condition.

c. Find a radius $r \in (0, \infty)$ small enough that $\tau(B_1(r)) \subset B_{\tau(1)}(2^{-1})$.

d. Find a radius $r \in (0, \infty)$ small enough that $\tau(B_1(r)) \subset B_{\tau(1)}(2^{-2})$.

e. Convince me that if I gave you a whole number $k \geq 0$, you could find a radius $r \in (0, \infty)$ small enough that $\tau(B_1(r)) \subset B_{\tau(1)}(2^{-k})$. A good way to do this is to describe a step-by-step procedure you would use to find a value of $r$ that works.

In each of the parts b, c, and d, you need to convince me that your value of $r$ works. This can be done with a short calculation.

### Solution (3 points for each part)

a. The first twenty digits of $\tau(1)$ are `00100100100110110110`.

b. The ball $B_{\tau(1)}(2^0)$ is the set of sequences that match $\tau(1)$ at the 0th digit. That means we want every point in $B_1(r)$ to have an itinerary starting with `0`. The 0th digit of $\tau(\theta)$ is `0` if and only if $\theta$ is in $(0, \pi)$. The ball $B_1(1)$ fits inside the interval $(0, \pi)$, so we can pick $r = 1$.

c. The ball $B_{\tau(1)}(2^{-1})$ is the set of sequences that match $\tau(1)$ at the 0th through 1st digits. That means we want every point in $B_1(r)$ to have an itinerary starting with `00`. We know from the previous part that the 0th digit of $\tau(\theta)$ is `0` as long as $\theta$ is in $B_1(1)$. The 1st digit of $\tau(\theta)$ is `0` if and only if

$$
\begin{aligned}
R_2(\theta) &\in (0, \pi) \\
\theta + 2 &\in (0, \pi) \\
\theta &\in (-2, \pi - 2) \\
&= (-2, 1.141\ldots).
\end{aligned}
$$

The ball $B_1(0.14)$ fits inside both the ball $B_1(1)$ and the interval $(-2, \pi - 2)$, so we can pick $r = 0.14$.

d. The ball $B_{\tau(1)}(2^{-2})$ is the set of sequences that match $\tau(1)$ at the 0th through 2nd digits. That means we want every point in $B_1(r)$ to have an itinerary starting with `001`. We know from the previous parts that the 0th through 1st digits of $\tau(\theta)$ are `00` as long as $\theta$ is in $B_1(0.14)$. The 2nd digit of $\tau(\theta)$ is `1` if and only if

$$\begin{aligned} R_2^2(\theta) &\in (\pi, 2\pi) \\ \theta + 4 &\in (\pi, \pi) \\ \theta &\in (\pi - 4, 2\pi - 4) \\ &= (-0.858\ldots, 2.283\ldots). \end{aligned}$$

The ball $B_1(0.14)$ fits inside the interval $(\pi - 4, 2\pi - 4)$, so we can pick $r = 0.14$ again.

e. Let's say we want to constrain $\theta \in \Lambda$ so that $\tau(\theta)$ will be in $B_{\tau(1)}(2^{-k})$. That means we want $\tau(\theta)$ to match $\tau(1)$ at the 0th through $k$th digits.

Since 1 is in $\Lambda$, none of the points $1, R_2(1), R_2^2(1), \ldots, R_2^k(1)$ are 0 or $\pi$. We can therefore find a radius $r \in (0, \infty)$ small enough that the balls $B_1(r), B_{R_2(1)}, B_{R_2^2(1)}(r), \ldots, B_{R_2^k(1)}(r)$ avoid both 0 and $\pi$.[1] To ensure that $\tau(\theta)$ matches $\tau(1)$ at the 0th through $k$th digits, it's enough to ensure that $R_2^n(\theta)$ is in $B_{R_2^n(1)}(r)$ for each $n \in \{1, \ldots, k\}$. Because of the way $R_2$ moves open balls, this is equivalent to ensuring that $\theta$ is in $B_1(r)$. We can now see, reasoning in the other direction, that $\tau(\theta)$ will match $\tau(1)$ at the 0th through $k$th digits whenever $\theta$ is in $B_1(r)$. We've thus found a radius $r \in (0, \infty)$ small enough that $\tau(B_1(r)) \subset B_{\tau(1)}(2^{-k})$.

[1] Here's a concrete way to do this, suggested by some students during in office hours. (Many thanks for the suggestion! Let me know if you'd like to be credited by name.) For each $n \in \{0, \ldots, k\}$, define $r_n$ as $\min\{d(R_2^n(1), 0), d(R_2^n(1), \pi)\}$. Then choose $r$ to be $\min\{r_1, \ldots, r_k\}$.