Homework 4

Due on Crowdmark
March 25, 11 a.m.

Chaos, fractals, and dynamics
MAT 335, Winter 2019

Show your calculations, and explain your reasoning. Your goal is for the graders to understand how you got your answers, and to be convinced that your reasoning makes sense.

1 Countdown

For this problem, you can take for granted the solution of problem 2 from homework 1. The technique from problem 5 of homework 1 will make your work much easier. I recommend looking at the homework 1 solutions posted on the web site.

a. How many 6-periodic points does the shift map have?

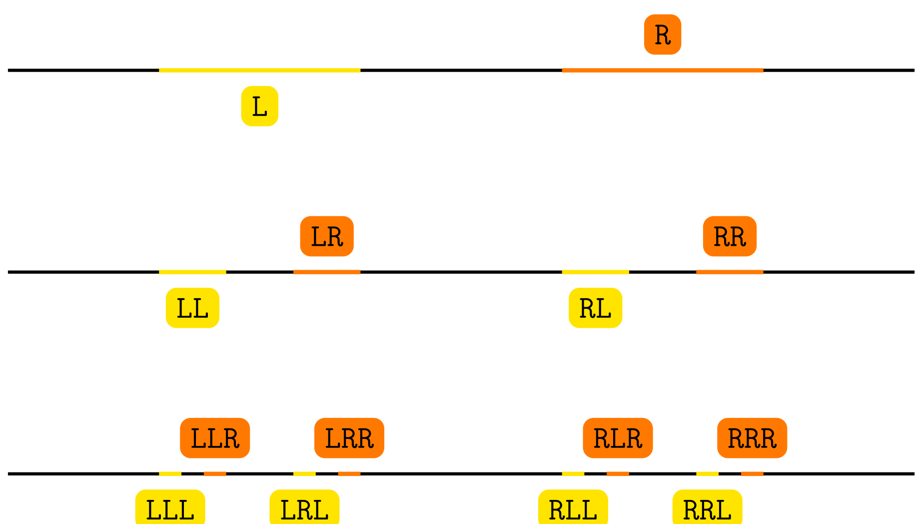
b. Among those points, how many have minimum period 6?

I'm not asking you to list any points—I'm just asking you to count them.

2 Both sides, now

In this problem, we'll learn more about the filled Julia set of the V map. You should refer to the week 8 notes and slides for background information and terminology.

The labeling of nth-level intervals that we used in class make it easy to see where V sends each interval, but it makes it hard to see where each interval sits on the real line. The pictures below show a different labeling, which makes it easy to see where each interval sits.



Removing L_0 and L_1 from $\mathbb{R}$ leaves two first-level intervals. The left and right intervals are called $H_{\mathtt{L}}$ and $H_{\mathtt{R}}$, respectively. Removing L_2 splits each first-level interval into a pair of second-level intervals. The left and right halves of $H_{\mathtt{L}}$ are called $H_{\mathtt{LL}}$ and $H_{\mathtt{LR}}$, respectively. The left and right halves of $H_{\mathtt{R}}$ are called $H_{\mathtt{RL}}$ and $H_{\mathtt{RR}}$, respectively. In general, the left and right halves of $H_{\square}$ are called $H_{\square\mathtt{L}}$ and $H_{\square\mathtt{R}}$, respectively.

As we saw in class, V sends each $(n+1)$st-level interval to an nth-level interval. In parts a–b, you'll use the $\mathtt{L}, \mathtt{R}$ labeling to describe where V sends each first and second-level interval. No justification is required, since you can find the answers just by looking at the week 8 slides.

a. Figure out where V sends each second-level interval. I'll give you one answer for free: V sends $H_{\mathtt{LL}}$ to $H_{\mathtt{R}}$.

b. Figure out where V sends each third-level interval. I'll give you one answer for free: V sends $H_{\mathtt{LLL}}$ to $H_{\mathtt{RR}}$.

Let's use the shorthand $\{\mathtt{L}, \mathtt{R}\}^{\mathbb{N}}$ for the set of sequences of Ls and Rs. We can define a function $\rho\colon K \to \{\mathtt{L}, \mathtt{R}\}^{\mathbb{N}}$ in the following way.

$$\text{the } n\text{th digit of } \rho(x) \text{ is } \begin{cases} \mathtt{L} & \text{if } x \text{ is in an } n\text{th-level interval whose label ends with } \mathtt{L} \\ \mathtt{R} & \text{if } x \text{ is in an } n\text{th-level interval whose label ends with } \mathtt{R} \end{cases}$$

For this definition, let's call the starting digit of a sequence the 1st digit.

c. Describe a dynamical map $G\colon \{\mathtt{L}, \mathtt{R}\}^{\mathbb{N}} \to \{\mathtt{L}, \mathtt{R}\}^{\mathbb{N}}$ with the property that

$$G(\rho(x)) = \rho(V(x)) \quad \text{for all } x \in K.$$

d. Suppose $\rho(x) = \overline{\mathtt{RRL}}$. Find the itinerary $\tau(x)$.

HINT: Look at the orbit of $\rho(x)$ under G.

3 Here it goes again

When you were doing problem 2a of homework 3, you may have noticed that the orbit of 1 under R_2 seems to cut the circle into finer and finer pieces as time goes on. This problem will help you see why that's happening. By the end of the it, you should be convinced that R_2 has a dense orbit, like I claimed in class. You might also have some ideas about the strange almost-repeating patterns that appear in the itinerary of 1.

For each angle $\alpha \in \mathbb{R}$, the rotation map $R_\alpha\colon \mathbb{T} \to \mathbb{T}$ is defined by the formula $R_\alpha(\theta) \equiv \theta + \alpha$. The rotation map family has a funny feature: the iterates of a rotation map are also rotation maps. As a warm-up, convince yourself that if the angle $\alpha \in (-\pi, \pi)$ describes the point $R_2^m(0)$, then $R_\alpha = R_2^m$.

In parts a–d, we'll investigate the rotation map R_2 using concrete calculations. I encourage you to use a calculator for these parts.

a. Find the first number $m_1 \geq 1$ for which $R_2^{m_1}(0)$ falls in $[-1, 1]_{\text{through } 0}$.

The "through 0" is to specify that I mean the interval that runs from -1 to 1 through 0, rather than the one that runs through π.

For convenience, let's express $R_2^{m_1}(0)$ as an angle $\alpha_1 \in (-\pi, \pi)$. Notice that $R_{\alpha_1} = R_2^{m_1}$.

b. Find the first number $m_2 \geq 1$ for which $R_{\alpha_1}^{m_2}(0)$ falls in $[-\frac{\alpha_1}{2}, \frac{\alpha_1}{2}]_{\text{through } 0}$.

Express $R_{\alpha_1}^{m_2}(0)$ as an angle $\alpha_2 \in (-\pi, \pi)$. Notice that $R_{\alpha_2} = R_{\alpha_1}^{m_2} = R_2^{m_1 m_2}$.

c. Find the first number $m_3 \geq 1$ for which $R_{\alpha_2}^{m_3}(0)$ falls in $[-\frac{\alpha_2}{2}, \frac{\alpha_2}{2}]_{\text{through } 0}$.

   Express $R_{\alpha_1}^{m_m}(0)$ as an angle $\alpha_3 \in (-\pi, \pi)$. Notice that $R_{\alpha_3} = R_{\alpha_1}^{m_2} = R_2^{m_1 m_2 m_3}$.

d. Suppose I start cutting the circle at the points $0,\ R_2(0),\ R_2^2(0),\ R_2^3(0),\ \ldots$. If I want to be sure all the pieces have size $|\alpha_2|$ or smaller, how many cuts should I make? It's okay if you ask me to make more cuts than necessary.

   HINT: You can find enough information to solve this part in the numbers m_1, m_2, m_3.

Now that our calculations have revealed a pattern, let's think more abstractly about how it continues.

e. I've chosen an angle $\alpha \in (-\pi, \pi)$, but I won't tell you what it is. Argue that there's a number $m \geq 1$ for which R_α^m falls in $[-\frac{\alpha}{2}, \frac{\alpha}{2}]_{\text{through } 0}$.

Once you've done this, you can define more numbers $m_3, m_4, m_5, \ldots$ and more angles $\alpha_3, \alpha_4, \alpha_5, \ldots$ in the following way:

- Define m_{j+1} to be the first number in $\{1, 2, 3, \ldots\}$ for which $R_{\alpha_j}^{m_{j+1}}$ falls in $[-\frac{\alpha_j}{2}, \frac{\alpha_j}{2}]_{\text{through } 0}$.
- Express $R_{\alpha_j}^{m_{j+1}}(0)$ as an angle $\alpha_{j+1} \in (-\pi, \pi)$.

f. Argue that the sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$ limits to 0.

   HINT 1: To find out what you need to do, look up the definition of a limit in Section 1.5 of the week 3 notes.

   HINT 2: The sequence $1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$ limits to 0. You can use this fact without justifying it.

g. Argue that the orbit of 0 under R_2 is a dense subset of $\mathbb{T}$.

   HINT: If you you're trying to make the orbit of 0 hit an open ball of radius greater than $\frac{\alpha_N}{2}$, how many steps should you go to be sure you've succeeded? You can find enough information to answer this question in the numbers $m_1, \ldots, m_{N+1}$.