Fact sheet

For term test 2
March 12

Chaos, fractals, and dynamics
MAT 335, Winter 2019

Attraction and repulsion

Say p is a fixed point of a dynamical map F.

- A *basin of attraction* for p is an open ball U with the following properties.
  - ◇ U contains p.
  - ◇ Every orbit starting in U stays in U forever.
  - ◇ Every orbit starting in U limits to p.

  (The second property is equivalent to the property that $F(U) \subset U$, using our shorthand from class.)

- A *region of repulsion* for p is an open ball U with the following properties.
  - ◇ U contains p.
  - ◇ Every orbit starting in U eventually leaves U, unless it starts at p.
    (It only has to leave once; it can come back later.)

Continuity

Formal definition. Consider a function $\psi\colon W \to X$. Pick any point $w \in W$. The function ψ is *continuous at* w if we can keep the output of ψ within any "target" open ball around $\psi(w)$ by keeping its input within a small enough open ball around w.

The function ψ is *continuous* if it's continuous at every point in W.

Semiconjugacy

The function $\psi\colon W \to X$ is a *semiconjugacy* from the dynamical map $E\colon W \to W$ to the dynamical map $F\colon X \to X$ if it has the following properties.[1]

- We can find out what F does to a point by looking at what E does to its label. In symbols,

  $$F(\psi(w)) = \psi(E(w)) \quad \text{for all labels } w \in W.$$

  In a picture,

  $$\begin{tikzcd} W \arrow[r, "E"] \arrow[d, "\psi"'] \arrow[dr, phantom, "/\!\!/"] & W \arrow[d, "\psi"] \\ X \arrow[r, "F"'] & X \end{tikzcd}$$

[1] The version handed out during the test had a typo: E and F were switched. I announced a correction.

- Every point in X has a label. In other words, every point $x \in X$ can be expressed as $\psi(w)$ for some $w \in W$. A function ψ with this property is called *onto*.
- Each point in X has a limited number of labels. Specifically, we can fix a maximum m and say that that each point in X has at most m labels in W. A function ψ with this property is called *at most m-to-one.*
- The function ψ is continuous.

Fact. *If ψ is a semiconjugacy from $E\colon W \to W$ to $F\colon X \to X$, it's also a semiconjugacy from E^n to F^n, for any number n of iterations.*

The binary representation

The *binary representation* of angles is a function $\phi\colon \mathbf{2}^{\mathbb{N}} \to \mathbb{T}$. Using $w_1, w_2, w_3, w_4, \ldots$ to denote the digits of a point $w \in \mathbf{2}^{\mathbb{N}}$, we can write

$$\phi(w) \equiv 2\pi\left(\frac{w_1}{2^1} + \frac{w_2}{2^2} + \frac{w_3}{2^3} + \frac{w_4}{2^4} + \ldots\right).$$

Fact. *The binary representation is a semiconjugacy from the shift map $S\colon \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$ to the doubling map $D\colon \mathbb{T} \to \mathbb{T}$.*

Fact. *For any $t \in (-1, 1)$,*

$$1 + t + t^2 + t^3 + t^4 + \ldots = \frac{1}{1-t}.$$