Term Test 2

March 12
7:10 – 9:00 p.m.

Chaos, fractals, and dynamics
MAT 335, Winter 2019

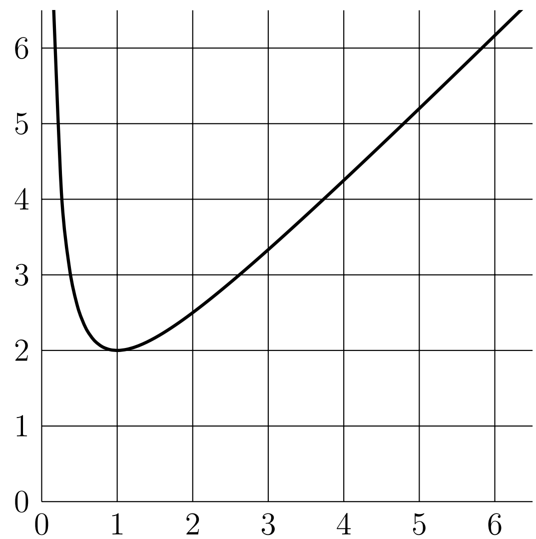
Instructions

- There are 6 problems, worth a total of 20★.
- For pacing, aim to complete at least 17★ carefully.
- Show your calculations, and explain your reasoning. Your goal is for the graders to understand how you got your answers, and to be convinced your reasoning makes sense.
- Do not discuss the test until 6 p.m. Wednesday, since there will be an alternate sitting.
- Do not turn the page until you're told to start the test.
- Before the test starts, write your name and student number above.
- Only the work written in this booklet will be marked. Don't write near the staple.
- If your work on a problem continues into the extra space at the end of the booklet, say so in the space for the problem.
- If you need scrap paper for scratch work, raise your hand.
- If you have a question or need to use the washroom, raise your hand.
- You may use a ruler for graphical analysis. No other computing devices are allowed.

1. Define a function $\alpha\colon (0,\infty) \to [2,\infty)$ with the formula

$$\alpha(x) = x + \frac{1}{x}.$$

Here's the graph of α.



Define $P(x) = x^2$, and recall that $Q_{-2}(x) = x^2 - 2$.

★★★ •. Is α a semiconjugacy from $P\colon (0,\infty) \to (0,\infty)$ to $Q_{-2}\colon [2,\infty) \to [2,\infty)$?

If it is, convince the grader that it has all the defining properties of a semiconjugacy. If it isn't, convince the grader that it lacks at least one of the defining properties.

You can take it for granted that α is continuous.

SUGGESTION: If you can show that α lacks one of the defining properties, you don't have to check the others. However, if you have time left over at the end of the test, you should come back and check the others anyway. That might help you get partial credit if you made a mistake.

Marking

- Since we're taking continuity for granted, there are three properties left to verify. Give 3 points per property, for a total of 9 points.

Sample solution

Yes. To prove it, we need to show that α has the four defining properties of a semiconjugacy.

- We can take it for granted that α is continuous.

- To show that α is onto, consider the equation

$$\begin{aligned} \alpha(x) &= y \\ x + \frac{1}{x} &= y \\ x - y + \frac{1}{x} &= 0 \\ x^2 - yx + 1 &= 0. \end{aligned}$$

  For any $y \in [2, \infty)$, we can find a number $x \in (0, \infty)$ with $\alpha(x) = y$ by choosing[1]

$$x = \frac{1}{2}\left(y + \sqrt{y^2 - 4}\right).$$

  The square root makes sense because $y \geq 2$, so $y^2 - 4 \geq 0$.

- Let's show that α is few-to-one. Specifically, we'll show that α is at most two-to-one. We saw above that the equation $\alpha(x) = y$ can be rewritten as $x^2 - yx + 1 = 0$. For each choice of y, this becomes a quadratic equation for x, which has at most two solutions. Hence, for each $y \in [2, \infty)$, there are at most two values of $x \in (0, \infty)$ with $\alpha(x) = y$.

- To show that $Q_{-2} \circ \alpha = \alpha \circ P$, we compare the formulas

$$\begin{aligned} \alpha(P(x)) &= \alpha(x^2) \\ &= x^2 + \frac{1}{x^2} \end{aligned}$$

  and

$$\begin{aligned} Q_{-2}(\alpha(x)) &= Q_{-2}\left(x + \frac{1}{x}\right) \\ &= \left(x + \frac{1}{x}\right)^2 - 2 \\ &= \left(x^2 + 2 + \frac{1}{x^2}\right) - 2 \\ &= x^2 + \frac{1}{x^2}. \end{aligned}$$

[1]The choice $x = \frac{1}{2}\left(y - \sqrt{y^2 - 4}\right)$ also works, but it's less obvious in this case that x is in $(0, \infty)$.

Continue problem 1.

2. Consider the following dynamical system.

 State space: $\mathbf{2}^{\mathbb{N}}$.

 Dynamical map: The first `01` turns into `10`. (If there's no `01`, nothing changes.)

 Let's call this map F. As a demonstration, here's what happens the first few times you apply F to two points in $\mathbf{2}^{\mathbb{N}}$.

$$
\begin{aligned}
w &= \texttt{00110001}\ldots & w' &= \texttt{001}\overline{\texttt{0}} \\
F(w) &= \texttt{0}\underline{\texttt{10}}\texttt{10001}\ldots & F(w') &= \texttt{0}\underline{\texttt{10}}\overline{\texttt{0}} \\
F^2(w) &= \underline{\texttt{10}}\texttt{010001}\ldots & F^2(w') &= \underline{\texttt{10}}\texttt{0}\overline{\texttt{0}} \\
F^3(w) &= \texttt{10}\underline{\texttt{10}}\texttt{0001}\ldots & F^3(w') &= \texttt{100}\overline{\texttt{0}} \\
F^4(w) &= \texttt{1}\underline{\texttt{10}}\texttt{00001}\ldots & F^4(w') &= \texttt{100}\overline{\texttt{0}} \\
F^5(w) &= \texttt{110000}\underline{\texttt{10}}\ldots & F^5(w') &= \texttt{100}\overline{\texttt{0}}
\end{aligned}
$$

 a. Is $B_{\overline{0}}(2^0)$ a region of repulsion for the fixed point $\overline{\texttt{0}}$?
 If it is, convince the grader that it has all the defining properties of a region of repulsion. If it isn't, convince the grader that it lacks at least one of the defining properties.

 b. Is $B_{\overline{1}}(2^0)$ a region of repulsion for the fixed point $\overline{\texttt{1}}$?
 If it is, convince the grader that it has all the defining properties of a region of repulsion. If it isn't, convince the grader that it lacks at least one of the defining properties.
 HINT: Think about whether $B_{\overline{1}}(2^0)$ contains any fixed points other than $\overline{\texttt{1}}$.

★ c. Find a point in $\mathbf{2}^{\mathbb{N}}$ which is not eventually fixed. Convince the grader that it's not eventually fixed.

Parts 2a and 2b are worth ★★★ in total.

Marking

- In parts a and b:
  - ◇ 3 marks for understanding the definition of a region of repulsion.
  - ◇ 3 marks for showing that $B_{\overline{0}}$ is a region of repulsion for $\overline{\texttt{0}}$.
  - ◇ 3 marks for showing $B_{\overline{1}}$ is not a region of repulsion for $\overline{\texttt{1}}$.
- 3 marks for part c.

Sample solution

 a. Yes. To see why, first observe that $B_{\overline{0}}(2^0)$ is the set of sequences that start with `0`. Keeping this in mind, we'll prove that $B_{\overline{0}}(2^0)$ has the two defining properties of a region of repulsion for $\overline{\texttt{0}}$.

  - Since $\overline{\texttt{0}}$ starts with `0`, it's in $B_{\overline{0}}(2^0)$.

- We also need to show that any orbit starting in $B_{\overline{0}}(2^0)$ eventually leaves $B_{\overline{0}}(2^0)$, unless it starts at $\overline{\texttt{0}}$. To do this, consider any sequence $w \in B_{\overline{0}}(2^0)$ other than $\overline{\texttt{0}}$. There must be a `1` somewhere in w, and applying F over and over will move the first `1` one place closer to the start of the sequence on each step. Eventually, the first `1` will reach the beginning of the sequence. That means the orbit of w will eventually reach a point that's not in $B_{\overline{0}}(2^0)$.

b. No. The sequence $\texttt{1}\overline{\texttt{0}}$ is in $B_{\overline{1}}(2^0)$. Its orbit never leaves $B_{\overline{1}}(2^0)$, because it's a fixed point of F. Hence, $B_{\overline{1}}(2^0)$ contains a point other than $\overline{\texttt{1}}$ whose orbit never leaves $B_{\overline{1}}(2^0)$.

c. The sequence $\overline{\texttt{01}}$ is not eventually fixed. To see why, observe that any sequence in which `01` appears will change when you apply F. Applying F can eliminate at most one occurrence of `01`. The sequence $\overline{\texttt{01}}$ contains infinitely many occurrences of `01`; no matter how many times you apply F, you'll never get rid of them all. As a result, every sequence in the orbit of $\overline{\texttt{01}}$ contains a `01`, so no sequence in the orbit of $\overline{\texttt{01}}$ can be a fixed point.

Continue problem 2.

3. Recall that the shift map $S\colon \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$ erases the starting digit of each sequence, and the doubling map $D\colon \mathbb{T} \to \mathbb{T}$ is given by the formula $D(\theta) \equiv 2\theta$.

★ a. Find a periodic point of the shift map which has minimum period 7.

★★ b. Find a periodic point of the doubling map which has minimum period 7.

HINT: The fact sheet has some facts about the binary representation of angles. You can take these facts for granted in your work.

Marking

- 3 points for part a.
- 6 points for part b.

Sample solution

a. The point $w = \overline{\texttt{0000001}}$ is periodic with period 7. It repeats every seven digits, so removing the first 7 digits gives the same sequence back. Removing fewer initial digits doesn't give the same sequence back.

b. Let's use the binary representation $\phi\colon \mathbf{2}^{\mathbb{N}} \to \mathbb{T}$ to turn w into an angle. We can express $\phi(w)$ as a fraction in the following way.

$$\begin{aligned}
\phi(w) &= 2\pi \left[\frac{0}{2^1} + \ldots + \frac{0}{2^6} + \frac{1}{2^7} + \frac{0}{2^8} + \ldots + \frac{0}{2^{13}} + \frac{1}{2^{14}} + \ldots\right] \\
&= 2\pi \left[\frac{1}{2^7} + \left(\frac{1}{2^7}\right)^2 + \left(\frac{1}{2^7}\right)^3 + \ldots\right] \\
&= \frac{2\pi}{2^7}\left[1 + \frac{1}{2^7} + \left(\frac{1}{2^7}\right)^2 + \left(\frac{1}{2^7}\right)^3 + \ldots\right].
\end{aligned}$$

We can sum this series using the formula from the fact sheet.

$$\begin{aligned}
\phi(w) &= \frac{2\pi}{2^7} \cdot \frac{1}{1 - 1/2^7} \\
&= \frac{2\pi}{128} \cdot \frac{1}{1 - 1/128} \\
&= \frac{2\pi}{128} \cdot \frac{128}{127} \\
&= \frac{2\pi}{127}.
\end{aligned}$$

According to the fact sheet, ϕ is a semiconjugacy from S to D. Therefore, by another fact from the sheet, it's a semiconjugacy from S^7 to D^7. That tells us

$$\begin{aligned}
D^7(\phi(w)) &= \phi(S^7(w)) \\
&= \phi(w).
\end{aligned}$$

In other words, $\phi(w)$ is 7-periodic.

To confirm that 7 is the minimum period of $\phi(w)$, we just write out the first few steps of the orbit, and observe that the orbit doesn't return to its starting point until the 7th step.

n	0	1	2	3	4	5	6	7
$D^n(\phi(w))$	$\frac{2\pi}{127}$	$\frac{4\pi}{127}$	$\frac{8\pi}{127}$	$\frac{16\pi}{127}$	$\frac{32\pi}{127}$	$\frac{64\pi}{127}$	$\frac{128\pi}{127}$	$\frac{256\pi}{127} \equiv \frac{2\pi}{127}$

Continue problem 3.

4. Define a function $\psi\colon \mathbb{T} \to [-2, 2]$ with the formula $\psi(\theta) = 2\cos(\theta)$. As we learned in class, ψ is a semiconjugacy from the doubling map $D\colon \mathbb{T} \to \mathbb{T}$ to the quadratic map $Q_{-2}\colon [-2, 2] \to [-2, 2]$. You can take this fact for granted in your work.

   Recall that $D(\theta) \equiv 2\theta$ and $Q_{-2}(x) = x^2 - 2$.

   a. The point $2\cos(2\pi/7)$ is a periodic point of Q_{-2}. Find its minimum period. Convince the grader your answer is correct.

   b. The point $2\cos(2\pi/11)$ is a periodic point of Q_{-2}. Find its minimum period. Convince the grader your answer is correct.

   ★ c. Find a 7-periodic point of Q_{-2}. You can express your it in terms of the cosine function, like the points in parts 4a and 4b.

   HINT: Your solution to problem 3 might help you with this part.

   Parts 4a and 4b are worth ★★★ in total.

   Marking

   - [Marking scheme for parts a–b to be determined.]
   - 3 marks for part c.

   Sample solution

   a. Observe that $2\cos(\frac{2\pi}{7}) = \psi(\frac{2\pi}{7})$. We're looking for the first $n > 0$ with

   $$Q^n_{-2}(\psi(\tfrac{2\pi}{7})) = \psi(\tfrac{2\pi}{7}).$$

   Because ψ is a semiconjugacy from D to Q_{-2}, it's also a semiconjugacy from D^n to Q^n_{-2} for each n, as described on the fact sheet. Hence, we're looking for the first $n > 0$ with

   $$\psi(D^n(\tfrac{2\pi}{7})) = \psi(\tfrac{2\pi}{7}).$$

   From the geometric definition of the cosine function, one can see that that $\psi(\alpha) = \psi(\beta)$ if and only if $\alpha \equiv \pm\beta$. Hence, we're looking for the first $n > 0$ with

   $$D^n(\tfrac{2\pi}{7}) \equiv \pm\tfrac{2\pi}{7}.$$

   Now, let's take a look at the orbit of $2\pi/7$ under D.

   $$\begin{array}{r|cccc} n & 0 & 1 & 2 & 3 \\ D^n(\frac{2\pi}{7}) & \frac{2\pi}{7} & \frac{4\pi}{7} & \frac{8\pi}{7} \equiv -\frac{6\pi}{7} & -\frac{12\pi}{7} \equiv \frac{2\pi}{7} \end{array}$$

   We see that 3 is the value of n we're looking for. That means $\psi(\frac{2\pi}{7})$ has minimum period 3.

   b. Repeating the arguments from part 4a, we conclude that we're looking for the first $n > 0$ with

   $$D^n(\tfrac{2\pi}{11}) \equiv \pm\tfrac{2\pi}{11}.$$

   Let's take a look at the orbit of $\frac{2\pi}{11}$ under D.

$$\begin{array}{c|cccccc} n & 0 & 1 & 2 & 3 & 4 & 5 \\ D^n(\frac{2\pi}{11}) & \frac{2\pi}{11} & \frac{4\pi}{11} & \frac{8\pi}{11} & \frac{16\pi}{11} \equiv -\frac{6\pi}{11} & -\frac{12\pi}{11} \equiv \frac{10\pi}{11} & \frac{20\pi}{11} \equiv -\frac{2\pi}{11} \end{array}$$

We see that 5 is the value of n we're looking for. That means $\psi(\frac{2\pi}{11})$ has minimum period 5.

c. In problem 3, we saw that

$$D^7(\tfrac{2\pi}{127}) = \tfrac{2\pi}{127}.$$

Because ψ is a semiconjugacy from D to Q_{-2}, it follows that

$$Q_{-2}^7(\psi(\tfrac{2\pi}{127})) = \psi(\tfrac{2\pi}{127}).$$

In other words, $2\cos(\frac{2\pi}{127})$ is a 7-periodic point for Q_{-2}.

Continue problem 4.

5. Consider the following dynamical system.

State space: $\mathbf{2}^{\mathbb{N}}$.

Dynamical map:

◇ If the sequence starts with `0`, the first `1` turns into a `0`.

◇ If the sequence starts with `1`, the first `0` turns into a `1`.

If the digits are all `0`s, or all `1`s, nothing changes.

Let's call this map C. As a demonstration, here's what happens the first few times you apply C to two points in $\mathbf{2}^{\mathbb{N}}$.

$$\begin{aligned} w &= \texttt{00110001}\ldots \\ C(w) &= \texttt{00}\underline{\texttt{0}}\texttt{10001}\ldots \\ C^2(w) &= \texttt{000}\underline{\texttt{0}}\texttt{0001} \end{aligned} \qquad \begin{aligned} w' &= \texttt{10110001}\ldots \\ C(w') &= \texttt{1}\underline{\texttt{1}}\texttt{110001}\ldots \\ C^2(w') &= \texttt{1111}\underline{\texttt{1}}\texttt{001}\ldots \end{aligned}$$

★ a. Does C have any periodic points other than its fixed points?
If it does, find one. If it doesn't, explain how you know it doesn't.

★★ b. Find a basin of attraction for the fixed point $\overline{\texttt{1}}$. Convince the grader that it has all the defining properties of a basin of attraction.

Marking

- 3 points for a.
- 6 points for b.

Sample solution

a. No. To see why, suppose $w \in \mathbf{2}^{\mathbb{N}}$ starts with `0` and isn't fixed. Since C never changes the first digit, every sequence in the orbit of w will start with `0`.

Since w isn't fixed, it must have a `1` it it somewhere. When you apply C, the first `1` in w turns into a `0`. Applying C more times can never change that `0` back to a `1`, because C can never change a digit that matches the starting digit. Hence, w can't be periodic.

Based on the argument above, we conclude that if w starts with `0` and isn't fixed, then w isn't periodic. We can use a similar argument, with the roles of `0` and `1` reversed, for a sequence starting with `1`.

b. The ball $U = B_{\overline{\texttt{1}}}(2^{-1})$ is a basin of attraction for $\overline{\texttt{1}}$. To prove it, we need to show that U has the three defining properties of a basin of attraction. Before we start, observe that U is the set of all sequences that start with `1`.

- It's clear that U contains $\overline{\texttt{1}}$, because U is an open ball centered on $\overline{\texttt{1}}$.[2]

[2] An open ball always contains its center, because the center has distance zero from itself.

- To show that every orbit starting in U stays in U forever, observe that C never changes the starting digit of a sequence.
- To show that every orbit starting in U limits to $\overline{\texttt{1}}$, consider any $w \in U$. We want to show that for any radius 2^{-n}, the orbit of w has a tail that always stays inside $B_{\overline{\texttt{1}}}(2^{-n})$.
  Since w is in U, it starts with `1`. Applying C to a sequence n times ensures that the first $n+1$ digits all match. Hence, the orbit tail

  $$C^n(w), C^{n+1}(w), C^{n+2}(w), \ldots$$

  always stays inside $B_{\overline{\texttt{1}}}(2^{-n})$.

Continue problem 5.

6. Consider the dynamical system whose state space is $\mathbb{T}$ and whose dynamical map is the doubling map, $D(\theta) \equiv 2\theta$. The points in $\mathbb{T}$ whose orbits never hit 0 or π form a subset $\Lambda \subset \mathbb{T}$. Define a function $\tau\colon \Lambda \to \mathbf{2}^{\mathbb{N}}$ in the following way.

$$\text{the } n\text{th digit of } \tau(\theta) \text{ is } \begin{cases} 0 & \text{if } D^n(\theta) \in (0, \pi) \\ 1 & \text{if } D^n(\theta) \in (\pi, 2\pi) \end{cases}$$

In this problem, we'll call the starting digit of a sequence the 0th digit.

★ a. Find $\tau(2\pi/11)$.

★★ b. Find the 0th digit of $\tau(\phi(\overline{\mathtt{10}}))$, where ϕ is the binary representation of angles. Convince the grader your answer is correct.

See the fact sheet for a reminder of how ϕ is defined.

HINT: Notice that
$$\frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \ldots = \frac{1}{2},$$
and that switching some of the numerators in the series to zero will make the sum smaller.

Marking

- 3 points for a.
- 6 points for b.

Sample solution

a. We can read off the first few digits of $\tau(2\pi/11)$ from first few steps in the orbit of $2\pi/11$.

n	0	1	2	3	4
$D^n(\frac{2\pi}{11})$	$\frac{2\pi}{11}$	$\frac{4\pi}{11}$	$\frac{8\pi}{11}$	$\frac{16\pi}{11}$	$\frac{32\pi}{11} \equiv \frac{10\pi}{11}$
nth digit of $\tau(\frac{2\pi}{11})$	0	0	0	1	0

n	5	6	7	8	9	10
$D^n(\frac{2\pi}{11})$	$\frac{20\pi}{11} \equiv -\frac{2\pi}{11}$	$-\frac{4\pi}{11}$	$-\frac{8\pi}{11}$	$-\frac{16\pi}{11}$	$-\frac{32\pi}{11} \equiv -\frac{10\pi}{11}$	$-\frac{20\pi}{11} \equiv \frac{2\pi}{11}$
nth digit of $\tau(\frac{2\pi}{11})$	1	1	1	0	1	0

The orbit of $2\pi/11$ returns to its starting point after 10 steps. In other words, $2\pi/11$ is a 10-periodic point for the doubling map. It follows that $\tau(2\pi/11)$ is a 10-periodic point for the shift map. Hence, $\tau(2\pi/11) = \overline{\mathtt{0001011101}}$.

b. From the definition of the binary representation, we have

$$\phi(\overline{\mathtt{10}}) = 2\pi\left(\frac{1}{2^1} + \frac{0}{2^2} + \frac{1}{2^3} + \frac{0}{2^4} + \frac{1}{2^5} + \frac{0}{2^6} + \ldots\right).$$

We could just sum the series using the formula on the fact sheet, but here's a more general approach. As described in the hint, notice that

$$\begin{aligned}\frac{0}{2^2}+\frac{1}{2^3}+\frac{0}{2^4}+\frac{1}{2^5}+\frac{0}{2^6}+\ldots &< \frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}+\ldots\\ &= \frac{1}{2}.\end{aligned}$$

Hence,

$$\frac{1}{2^1}+\frac{0}{2^2}+\frac{1}{2^3}+\frac{0}{2^4}+\frac{1}{2^5}+\frac{0}{2^6}+\ldots \in (\tfrac{1}{2}, 1),$$

so $\phi(\overline{\mathtt{10}}) \in (\pi, 2\pi)$. We now see, from the definition of τ, that the 0th digit of $\tau(\phi(\overline{\mathtt{10}}))$ is $\mathtt{1}$.

Continue problem 6.

Extra space

Extra space

Extra space