Week 2 notes

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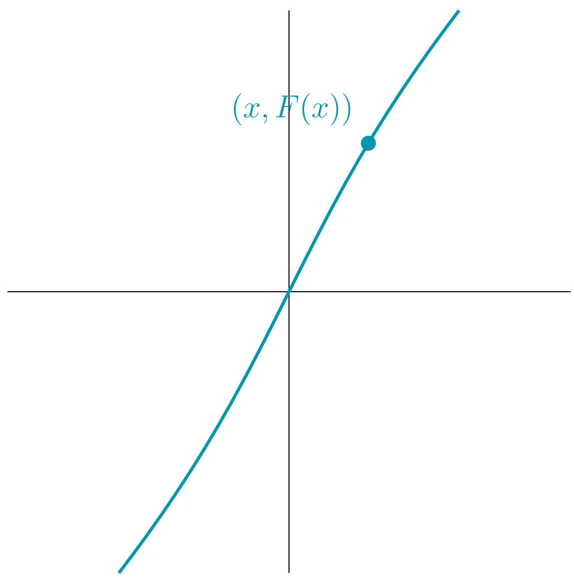
Chaos, fractals, and dynamics
MAT 335, Winter 2019

1 Graphical analysis of dynamics on $\mathbb{R}$

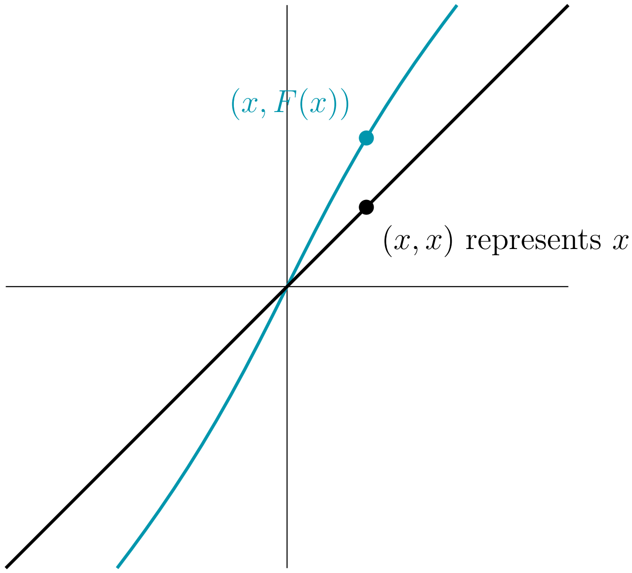
Over the next few weeks, we'll learn a bunch of techniques for understanding the orbits of dynamical systems. The first one is a clever way of looking at the graph of the dynamical map. We'll start by learning how to use it for dynamical systems with state space $\mathbb{R}$. Later, we'll apply it to systems with state space $\mathbb{T}$ as well.

1.1 Computing orbits graphically

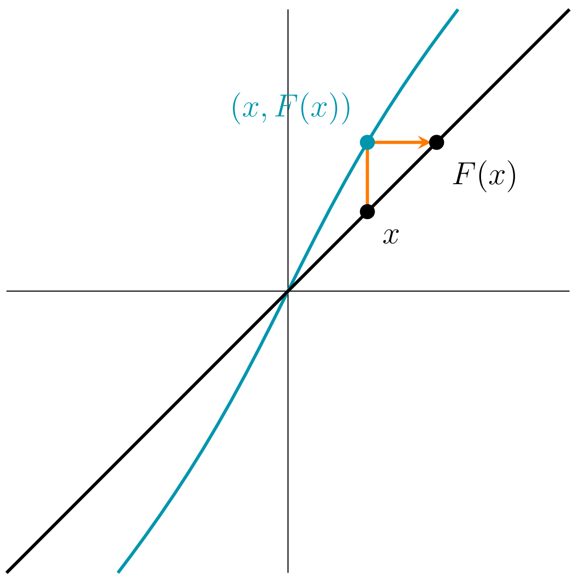
Pick any dynamical map $F\colon \mathbb{R} \to \mathbb{R}$. For illustration, let's use $F(x) = x + \arctan(x)$. Draw the graph of F in the plane $\mathbb{R}^2$. The points on the graph have coordinates $(x, F(x))$.



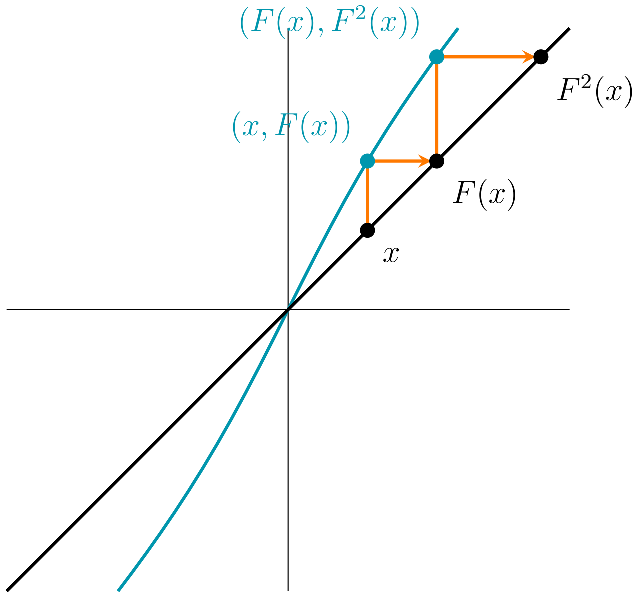
You're already used to thinking of the horizontal and vertical axes of $\mathbb{R}^2$ as copies of the real line, with the point $(x, 0)$ on the horizontal axis or the point $(0, x)$ on the vertical axis representing the point $x \in \mathbb{R}$. In dynamics, it's useful to think of the diagonal axis as a copy of the real line, with the point (x, x) on the diagonal axis representing $x \in \mathbb{R}$.



We can calculate $F(x)$ graphically by moving vertically from the point x on the diagonal to the graph of F, and then moving horizontally back to the diagonal.

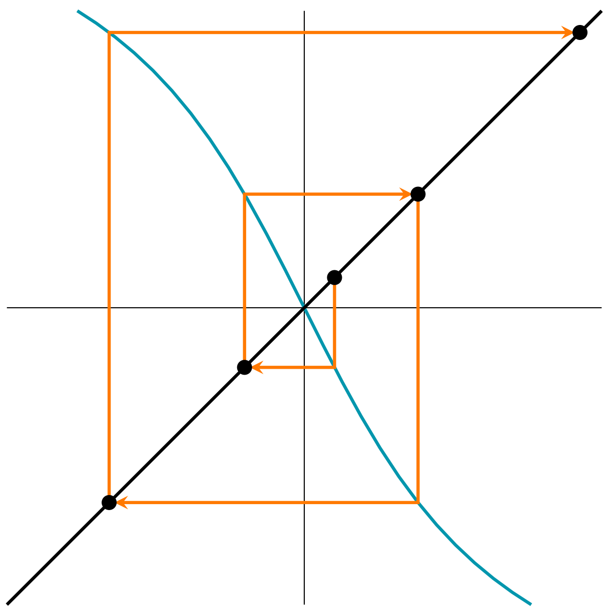


Repeat the process to find $F^2(x)$, $F^3(x)$, and so on, hopping along the orbit of x.



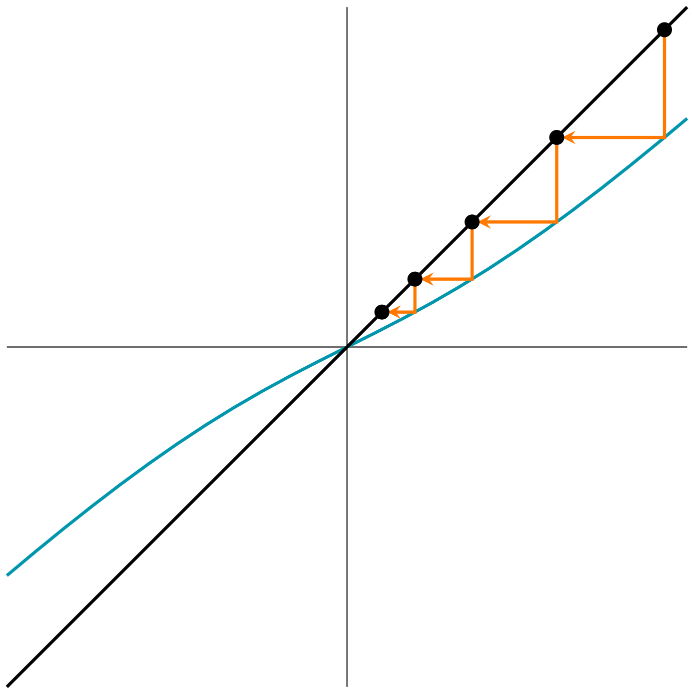
1.1.1 More examples of graphical orbit calculations

- An orbit of $-2\arctan(x)$.



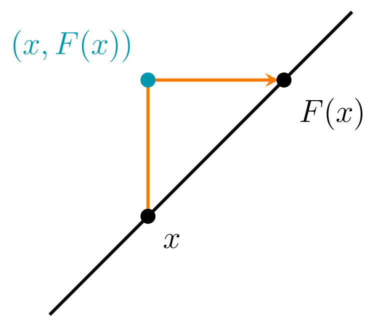
  In this case, the path we trace out as we compute the orbit resembles the spiraling path of a spider weaving a web. For this reason, pictures of graphical orbit calculations are often called *cobweb plots*.

- An orbit of $x - \frac{1}{2}\arctan(x)$.



1.2 Finding fixed points graphically

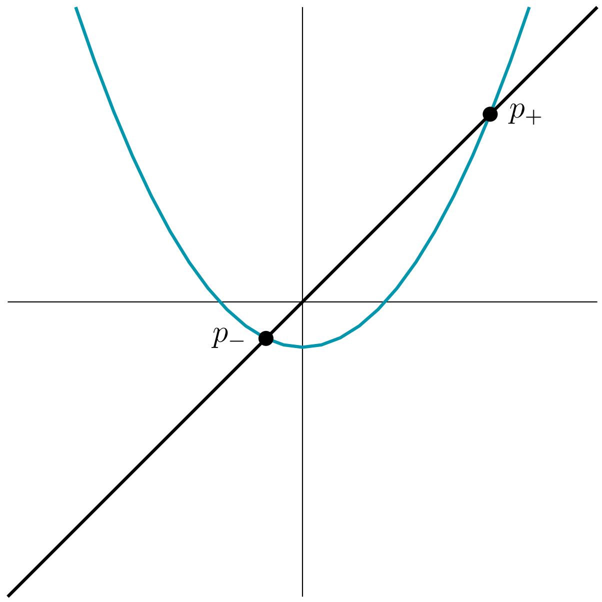
Which points on the diagonal represent fixed points of F? A fixed point is a point x with the property that $F(x) = x$. Let's look back at our graphical calculation of $F(x)$.



You can see from this picture that $F(x)$ and x are the same if and only if $(x, F(x))$ is on the diagonal. So, the fixed points of F are represented by the points where the graph of F touches the diagonal!

Here's the graph of the map $x^2 - 0.3$ we studied last week. The points where the graph touches the diagonal are just where we expect the fixed points p_+ and p_- to be, based on our earlier calculation that

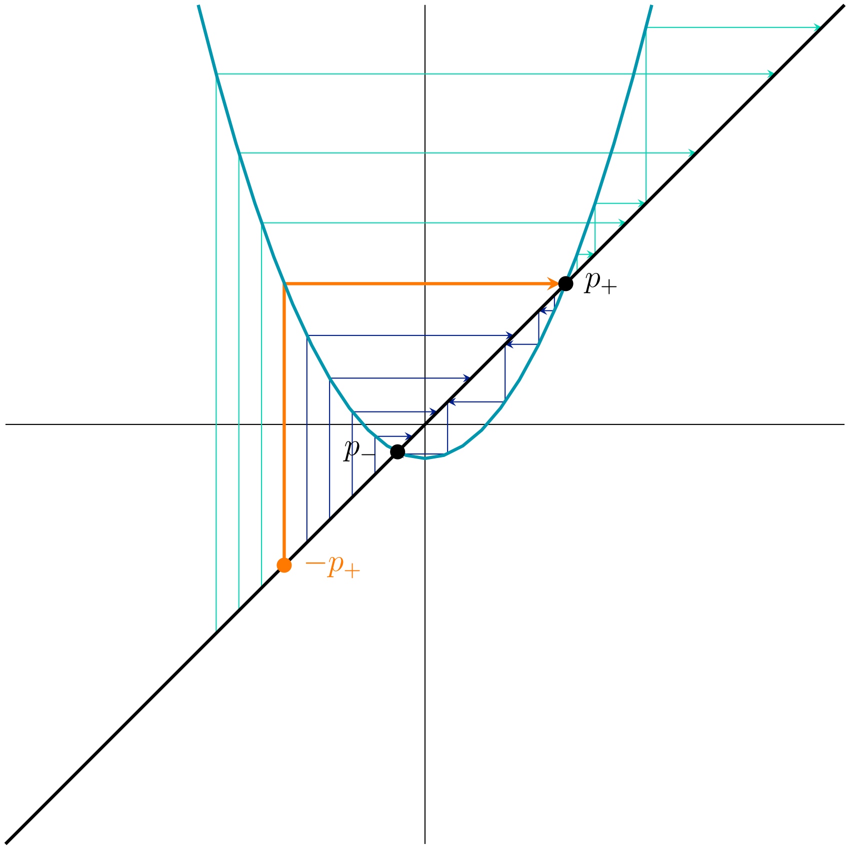
$$\begin{aligned} p_+ &= 1.24161\ldots \\ p_- &= -0.24161\ldots. \end{aligned}$$



1.3 When graphical analysis is all you need

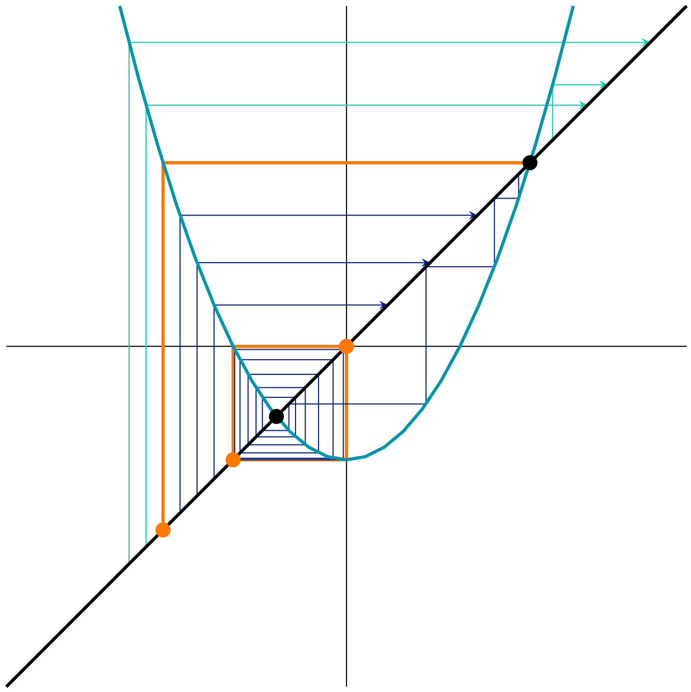
For some dynamical maps $\mathbb{R}$, it's possible to understand all the orbits just using graphical analysis. The maps $x^2 - 0.3$ and $x^2 - 1$ are both examples.

1.3.1 A full graphical understanding of $x^2 - 0.3$



The first thing you might notice about the graph is that all the points to the right of p_+ have orbits that fly off to the right forever. The points to the left of $-p_+$ leap over to the right of p_+, and then fly off. The points in the interval $(-p_+, p_+)$, on the other hand, get trapped in a spiral that brings them closer and closer to p_-.

1.3.2 A full graphical understanding of $x^2 - 1$



In the graph, you can see two fixed points. They turn out to be the large and small golden ratios,

$$p_{\pm} = \frac{1}{2}\left(1 \pm \sqrt{5}\right).$$

In decimal,

$$\begin{aligned} p_{+} &= 1.61803\ldots \\ p_{-} &= -0.61803\ldots. \end{aligned}$$

The familiar periodic orbit $0, -1, 0, -1\ldots$ surrounds p_{-}.

The points in the interval $(-1, 0)$ spiral outward, approaching the periodic orbit. The points in $(-p_{+}, -1)$ and $(0, p_{+})$ make their way into $(-1, 0)$, and then approach the periodic orbit. Just as before, the points to the right of p_{+} fly off to the right. The points to the left of $-p_{+}$ leap over to the right of p_{+} and then fly off.

1.3.3 Seeing similarities

In our first experiment with $Q_{-0.3}(x) = x^2 - 0.3$ and $Q_{-1}(x) = x^2 - 1$, the two maps seemed to act very differently. The former pulled all our test points toward a fixed point, while the latter pulled all our test points into a 2-periodic orbit.

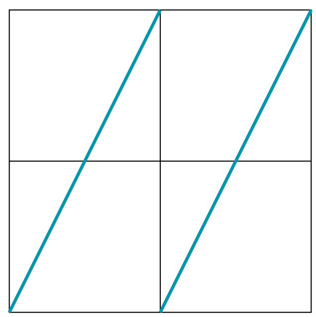
When you look at their cobweb diagrams, though, $Q_{-0.3}$ and Q_{-1} look much more similar than they did before. They both have two fixed points, $p_{\pm}$, and they both send $-p_{+}$ to p_{+}. They both act the same outside the interval $[-p_{+}, p_{+}]$, pushing every orbit off to the right. They also act similarly inside the interval $(-p_{+}, p_{+})$: points on the left side make one big hop to the right, points on the right side tumble down toward p_{-}, and points near p_{-} quickly approach a periodic orbit. The only difference is the minimum period of the periodic orbit: 1 for $Q_{-0.3}$ and 2 for Q_{-1}.

The similarity between the cobweb diagrams of $Q_{-0.3}$ and Q_{-1} is our first glimpse of a relationship that connects many of the standard quadratic maps. We'll return to it later in the course.

2 Graphical analysis of dynamics on $\mathbb{T}$

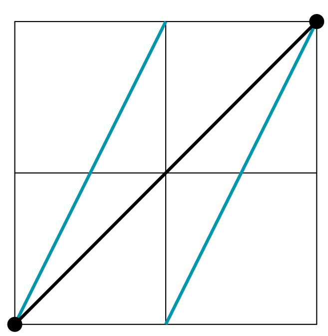
We can analyze the graph of a dynamical map $\mathbb{T}$ just like we did with dynamical maps on $\mathbb{R}$. We just have to keep in mind that angles differing by 2π are equivalent. The easiest way to do this is to represent angles by numbers from 0 to 2π. Then we can graph our dynamical map $F\colon \mathbb{T} \to \mathbb{T}$ on a $[0, 2\pi]$ by $[0, 2\pi]$ square. You can see the equivalence between the angles 0 and 2π by imagining the graph leaving through one side of the square and reappearing in the same place on the opposite side.

Here's the graph of the doubling map, $D(\theta) \equiv 2\theta$.

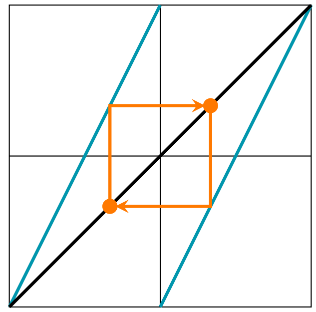


Notice how the graph leaves through the top of the square and reappears in the same place on the bottom. It also leaves through the top right corner of the square and reappears at the bottom left.

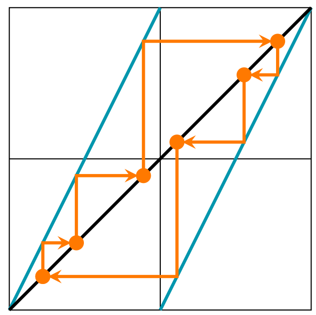
The graph touches the diagonal in one place: the fixed point 0, or equivalently 2π.



Computing the orbits of the doubling map graphically, we can see some periodic orbits that weren't obvious before. Here's one with minimum period 2.



Here's one with minimum period 6.

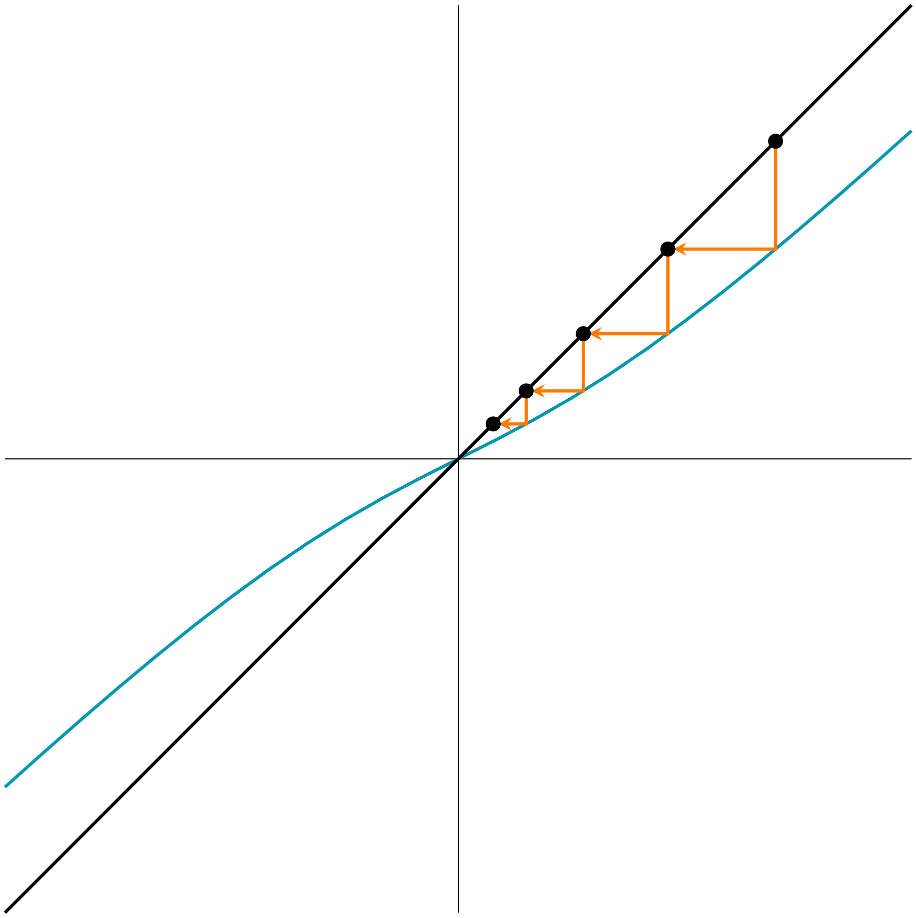


The doubling map turns out to have tons of periodic orbits. Later, using more sophisticated tools, we'll be able to find them all.

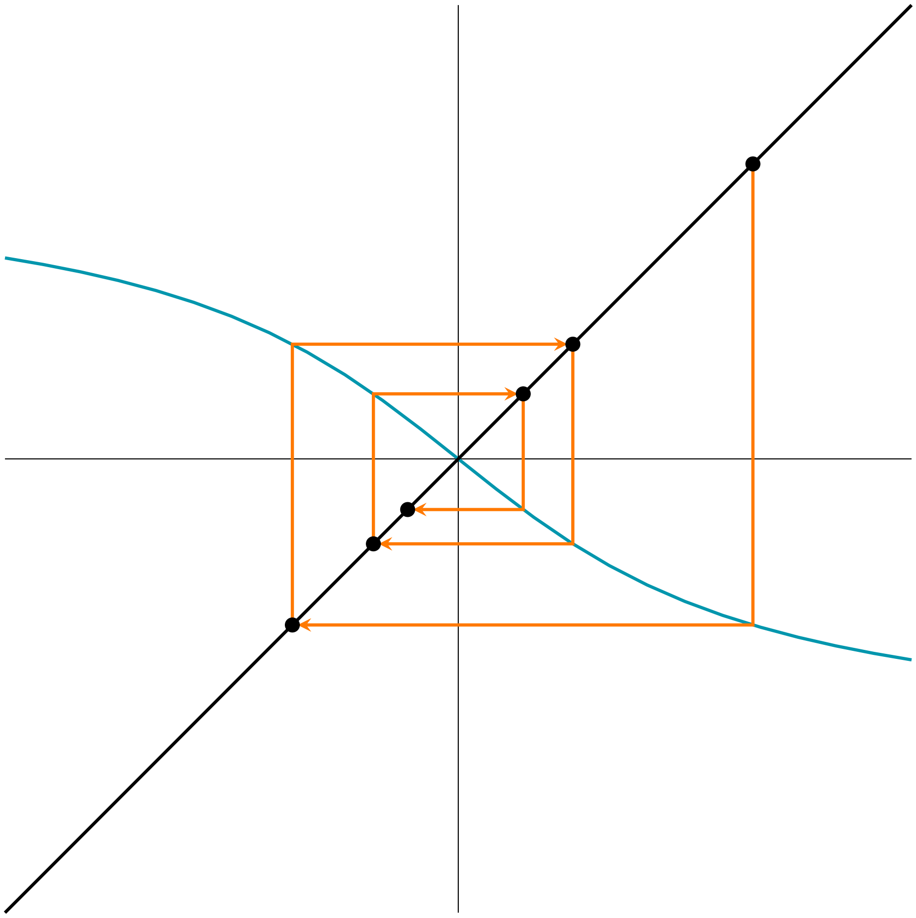
3 Orbits near a fixed point

Let's take another look at the orbits we calculated graphically last class, plus one more.

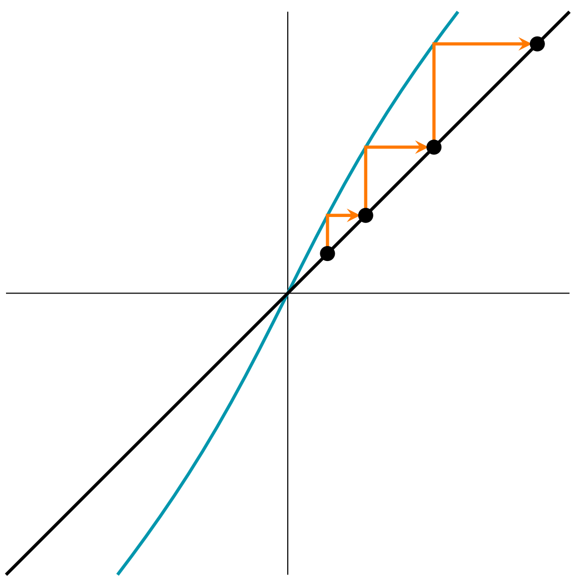
- An orbit of $x - \frac{1}{2}\arctan(x)$.



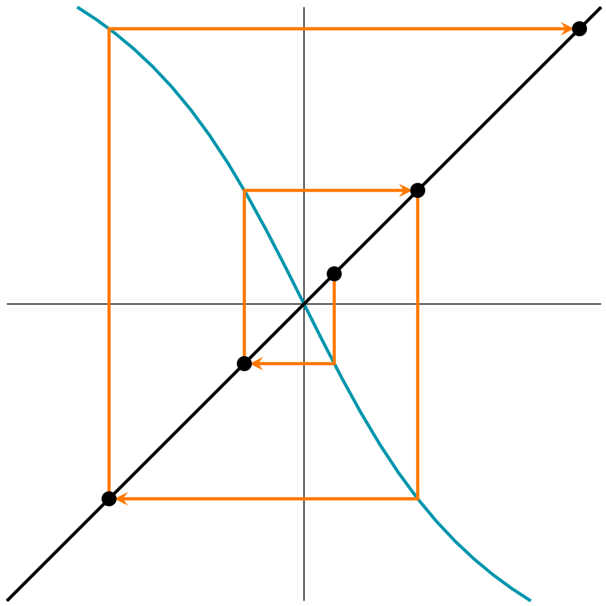
- An orbit of $-\frac{3}{4}\arctan(x)$.



- An orbit of $x + \arctan(x)$.



- An orbit of $-2\arctan(x)$.



These four orbits are all moving near a fixed point, and they seem to fall into a nice pattern. Two move toward the fixed point, and two move away from it. Two stay on one side of the fixed point, and two hop from one side to the other. Our goal for today is to understand these behaviors in a general way, and to predict graphically which behvaior you'll see near a particular fixed point.

3.1 Attracting and repelling fixed points

3.1.1 Definition

Pick any dynamical map $F\colon \mathbb{R} \to \mathbb{R}$. Suppose $p \in \mathbb{R}$ is a fixed point of F.

- A *basin of attraction* for p is an open interval I with the following properties.
  - I contains p.
  - Every orbit starting in I stays in I forever.
  - Every orbit starting in I limits to p.

  If there's a basin of attraction for p, we say p is *attracting.*[1]

  (The second property is equivalent to the property that $F(x) \in I$ for all $x \in I$. Using the shorthand $F(I)$ for the set[2] $\{F(x) : x \in I\}$, you can write this property very efficiently as $F(I) \subset I$.)

- A *region of repulsion* for p is an open interval I with the following properties.
  - I contains p.

[1] The textbook uses the term *weakly attracting.*

[2] Clarify the set-builder notation if any students are unfamiliar with it.

– Every orbit starting in I eventually leaves I, unless it starts at p.
(It only has to leave once; it can come back later.)

If there's a region of repulsion for p, we say p is *repelling*.[3]

3.1.2 Examples

In the four examples we looked at, the fixed point was attracting in two cases and repelling in the other two cases. [Refer back to the cobweb plots, or redraw them quickly.] Here's a basin of attraction or a region of repulsion for each one. [Highlight the middle third or so of the part of the diagonal shown on the graph.]

Last class, we studied the maps $x^2 - 0.3$ and $x^2 - 1$ on $\mathbb{R}$. Each one had two fixed points, $p_\pm$. [Re-draw the cobweb plots from Section 1.3, making sure to label the fixed points.]

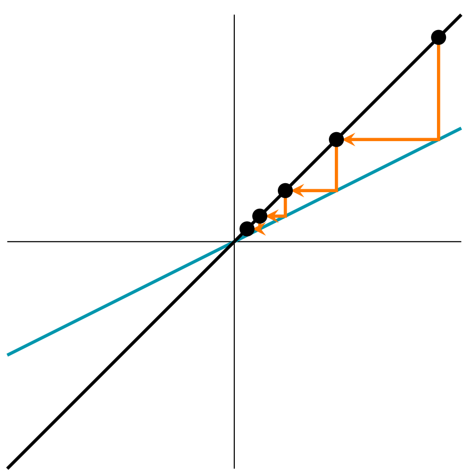
- For $x^2 - 0.3$, the fixed point p_- is attracting. The interval $(-p_+, p_-)$ is a basin of attraction for it, as you can see from the cobweb plot. The fixed point p_+ is repelling.
- For $x^2 - 1$, both fixed points are repelling. To get a region of repulsion for p_-, pick an interval $I \subset (-1, 0)$ with some space between its edges and the edges of $(-1, 0)$. Make sure, of course, that I contains p_-. This works because any orbit starting in I will eventually leave I as it approaches the periodic orbit $0, -1, 0, -1, \ldots$.

3.2 Fixed points of linear and approximately linear functions

3.2.1 Linear functions

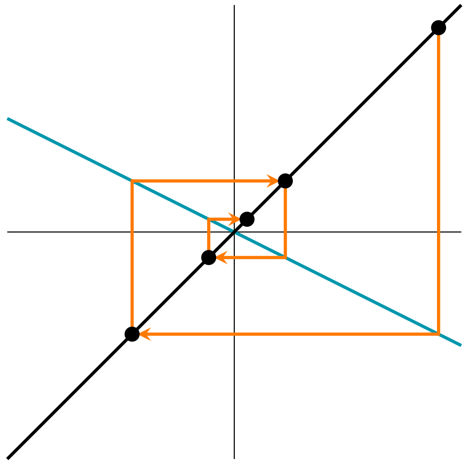
In each of the four basic examples we started with, the graph of F was close to a straight line. The best way to understand these examples is to look at maps whose graphs are actually straight lines.

- $F(x) = \frac{1}{2}x$.

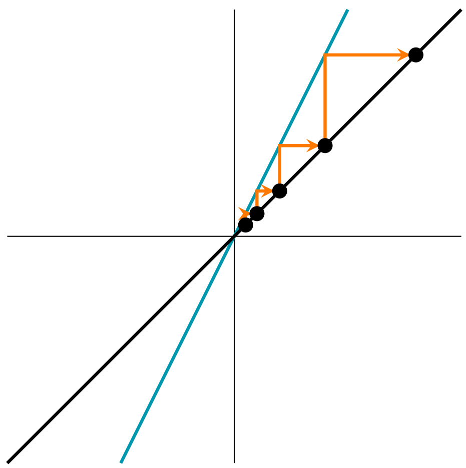


[3]The textbook uses the term *weakly repelling*.

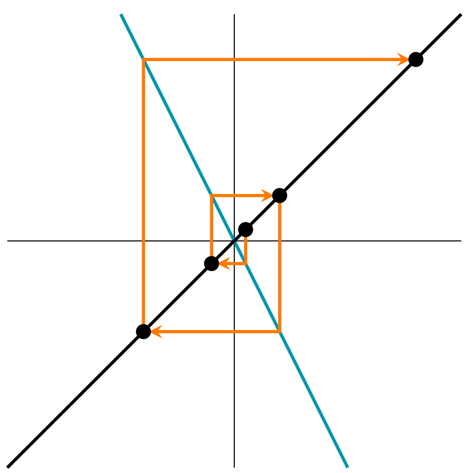
- $F(x) = -\frac{1}{2}x$.



- $F(x) = 2x$.



- $F(x) = -2x$.



Let's take a few minutes to think about these maps. Specifically, let's think about $F(x) = ax$, where a is a constant. This map has a single fixed point: 0.

Get in groups of two or three. Try to convince each other of the following facts.

- If $|a| < 1$, the fixed point 0 of ax is attracting.
- If $|a| > 1$, the fixed point 0 of ax is repelling.

I'll give you two or three minutes to discuss. Raise your hand if you have a question. [Walk around, listen in, talk to people and answer questions. After two or three minutes, or sooner if people look bored, ask if people are more time. Cut off discussion after 5 minutes at most. Ask a few groups to share their arguments with the class.]

3.2.2 Approximately linear functions

In calculus, you learned about differentiable functions. When we say a function F is differentiable at p, we mean its graph near p is close to a straight line:

$$F(p + \Delta x) \approx F(p) + F'(p)\,\Delta x$$
$$\text{when } x \approx 0.$$

Let's say p is a fixed point of F, and F is differentiable on some interval containing p. Under these conditions, the graph of F near p is really close a straight line—close enough that the arguments you came up with earlier still work. That leads us to the following conclusions.

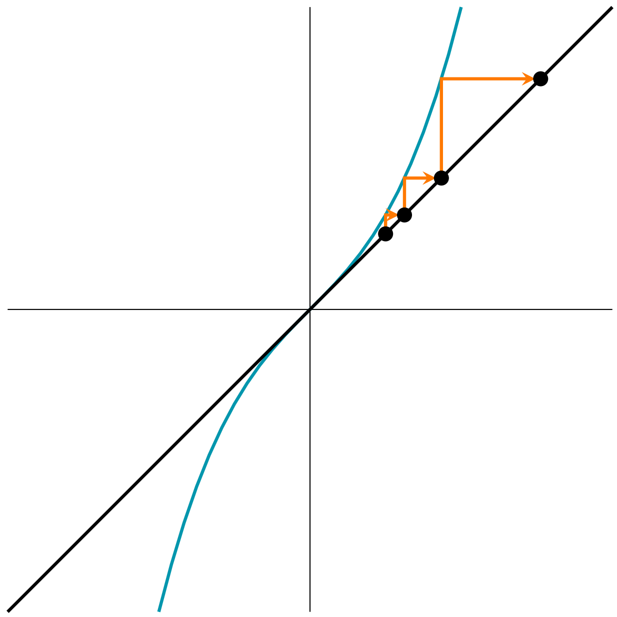
- If $|F'(p)| < 1$, the fixed point p is attracting.
- If $|F'(p)| > 1$, the fixed point p is repelling.

The conditions $|F'(p)| < 1$ and $|F'(p)| > 1$ are traditionally brought together under a common name. When F is differentiable near a fixed point p, and $|F(p)| \neq 1$, we say p is *hyperbolic*. As we just saw, a hyperbolic fixed point is always either attracting or repelling.[4]

3.2.3 Non-hyperbolic fixed points

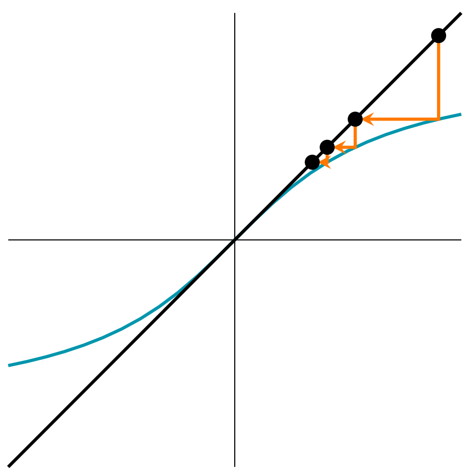
A fixed point doesn't have to be hyperbolic in order to be attracting or repelling. For example, look at the map $F(x) = x + x^3$.

[4] In the textbook, an attracting hyperbolic fixed point is just called *attracting*, and a repelling hyperbolic fixed point is just called *repelling*.



There's one fixed point, 0. It's not hyperbolic, because $F'(0) = 1$. However, it is repelling. You can figure this out by studying the graph of F.

Similarly, for the map $F(x) = \arctan(x)$, the fixed point 0 is attracting but not hyperbolic.



4 Orbits near a periodic orbit

4.1 Attracting and repelling orbits

Last class, when we did a graphical analysis of the map $x^2 - 1$, we saw that all the orbits starting in $(-p_+, p_+)$ eventually approach the periodic orbit $0, -1, 0, -1, \ldots$. Let's generalize our idea of "attractiveness" to cover this case. We'll take advantage of the fact that fixed points and periodic points are closely related.

Last week, we defined an n-periodic point of F as a point p with the property that $F^n(p) = p$. In other words, an n-periodic point of F is a fixed point of F^n.

- We say a periodic orbit is *attracting* if every point on the orbit is an attracting fixed point of F^n, where n is the minimum period.

- We say a periodic orbit is *repelling* if every point on the orbit is a repelling fixed point of F^n, where n is the minimum period.

As you've probably guessed, $0, -1, 0, -1, \ldots$ is an attracting orbit of the map $x^2 - 1$.

4.2 Hyperbolic orbits

We can generalize the idea of hyperbolicity to orbits in the same way.

- We say a periodic orbit is *hyperbolic* if every point on the orbit is a hyperbolic fixed point of F^n, where n is the minimum period.

We saw earlier that every hyperbolic fixed point is either attracting or repelling. We could tell which is which by looking at the derivative of the map. Using the chain rule, you can prove that F^n has the same derivative at each point of an n-periodic orbit. That lets us generalize our conclusions about hyperbolic fixed points to hyperbolic orbits.

Suppose p is a hyperbolic periodic point with minimum period n.

- If $|(F^n)'(p)| < 1$, the orbit of p is attracting.

- If $|(F^n)'(p)| > 1$, the orbit of p is repelling.

4.2.1 Application

We can use what we just learned about hyperbolic orbits to confirm that $0, -1, 0, -1, \ldots$ is an attracting orbit for $G(x) = x^2 - 1$. The orbit has minimum period 2, so we'd like to show that 0 or -1 is an attracting fixed point of G^2. As a first try, let's check whether 0 is a hyperbolic fixed point of G^2.

Note that $G'(x) = 2x$. By the chain rule,

$$\begin{aligned}(G \circ G)'(0) &= G'(G(0))\, G'(0) \\ &= 2G(0)\, 0 \\ &= 0.\end{aligned}$$

Now we know that $|(G^2)'(0)| < 1$, so 0 is an attracting hyperbolic fixed point of G^2. Hence, the orbit of 0 is an attracting periodic orbit.

5 An application: approximating square roots

Here's a very old method for approximating square roots. Let's say you want to know the square root of $a \in (0, \infty)$. Start with any number $x \in (0, \infty)$ as a first approximation. Apply the function

$$H_a(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$$

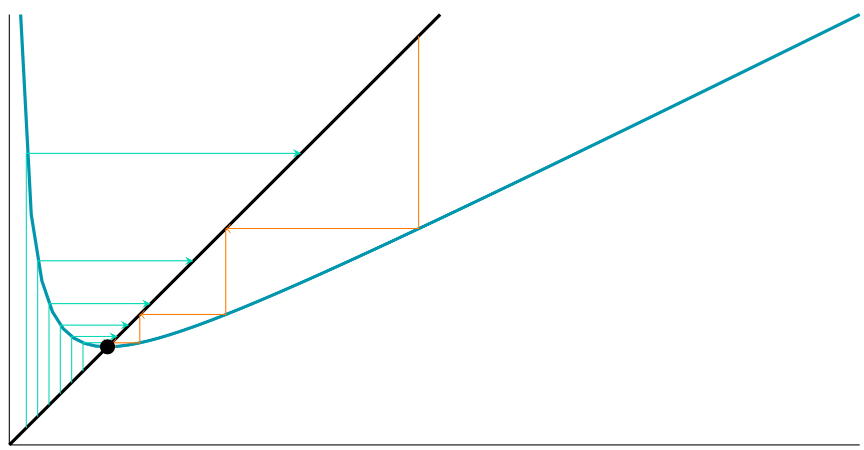
to get the next approximation. If you keep applying H_a, your approximations will eventually get better and better.

Why does this work? Think of the algorithm as a dynamical system, with state space $(0, \infty)$ and dynamical map H_a. The sequence of approximations is the orbit of the first approximation. If $\sqrt{a}$ were an attracting fixed point, with the whole state space as its basin of attraction, that would explain why the approximations always approach $\sqrt{a}$.

The fixed points of H_a are the solutions of the equation

$$\begin{aligned} H_a(x) &= x \\ \frac{1}{2}\left(x + \frac{a}{x}\right) &= x \\ x + \frac{a}{x} &= 2x \\ \frac{a}{x} &= x \\ a &= x^2. \end{aligned}$$

Since the state space is $(0, \infty)$, the only fixed point is $\sqrt{a}$, as we suspected. It turns out to be an attracting hyperbolic fixed point: $H_a'(\sqrt{a}) = 0$, so $|H_a'(\sqrt{a})| < 1$. Looking at a cobweb plot confirms that $\sqrt{a}$ is attracting, and shows that the entire state space is a basin of attraction for it.



Once an orbit gets close to $\sqrt{a}$, it approaches $\sqrt{a}$ very quickly, because the graph of F is almost horizontal near $\sqrt{a}$.