Week 4 notes

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Chaos, fractals, and dynamics
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1 Introducing semiconjugacy

1.1 Overview

Today, I'd like to step back and take stock of what we've learned so far. We've identified some interesting features that dynamical systems can have. [Write list from Section 1.1.1.] We've also developed a good technique—graphical analysis—for understanding the orbits of dynamical systems on the real line and the unit circle. [Start list from Section 1.1.2.]

This week, we'll meet our second major technique for understanding orbits. It's based on a special kind of relationship between dynamical systems, called "semiconjugacy." [Finish list from Section 1.1.2.]

1.1.1 Features dynamical systems can have.

- Eventually fixed points.
- Fixed points.
- Periodic orbits.

$\left.\begin{array}{l}\text{Fixed points.}\\ \text{Periodic orbits.}\end{array}\right\}$ Attracting, repelling, or neither.

1.1.2 Techniques for understanding the orbits of dynamical systems

- Graphical analysis.

★ New! Semiconjugacy: a special kind of relationship between dynamical systems.

1.2 First example: the doubling map and the shift map

To understand what semiconjugacy is like, and what it's good for, let's start with an example: a semiconjugacy relationship between the doubling map and the shift map.

1.2.1 Representing angles as binary sequences

We can express any angle in the form $2\pi t$, with $t \in [0, 1]$. A convenient way to describe t is to write down a decimal representation:

$$\begin{aligned}\frac{1}{6} &= 0.1666\ldots \text{(decimal)}\\ &= \frac{1}{10} + \frac{6}{100} + \frac{6}{1000} + \frac{6}{10000} + \ldots\end{aligned}$$

Here we're using the familiar base-10 numeral system, where each digit is weighted by a power of 10. There's a tenths place, a hundredths place, a thousandths place, a ten-thousandths place, and so on.

Another way to describe t is to write down a *binary* representation:

$$
\begin{aligned}
\frac{1}{6} &= 0.001010\ldots\text{(binary)} \\
&= \frac{0}{2} + \frac{0}{4} + \frac{1}{8} + \frac{0}{16} + \frac{1}{32} + \frac{0}{64} + \ldots \\
&= 0\,\text{𝅗𝅥} + 0\,\text{♩} + 1\,\text{♪} + 0\,\text{𝅘𝅥𝅯} + 1\,\text{𝅘𝅥𝅰} + \ldots
\end{aligned}
$$

Here we're using the base-2 numeral system, where each digit is weighted by a power of 2. There's a halves place, a quarters place, an eighths place, a sixteenths place, and so on. If you're a musician, you might be familiar with this system from reading sheet music.

We can think of the binary representation of angles as a function ϕ that turns a sequence of `0`s and `1`s into a point on the unit circle:

$$\phi(w) \equiv 2\pi(\text{the number with binary digit sequence } w).$$

In symbols, we'll write $\phi\colon \mathbf{2}^{\mathbb{N}} \to \mathbb{T}$, as shorthand for "$\phi$ is a map from the space of binary sequences to the unit circle."

Examples [Point out the binary representations of $0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$ on the unit circle, noting when there are multiple representations.]

1.2.2 Representing the doubling map as the shift map

The binary representation of angles makes the doubling map really easy to compute. When you double a number, each digit of its binary representation moves one place to the left.

$$
\begin{aligned}
D(2\pi\, 0.001010\ldots) &\equiv 2 \cdot 2\pi\, 0.001010\ldots \\
&\equiv 2\pi\, 0.01010\ldots
\end{aligned}
$$

If a `1` moves into the ones place, you can turn it back to a `0`, because that just changes the angle by 2π.

$$
\begin{aligned}
D(2\pi\, 0.111010\ldots) &\equiv 2\pi\, 1.11010\ldots \\
&\equiv 2\pi + 2\pi\, 0.11010\ldots \\
&\equiv 2\pi\, 0.11010\ldots
\end{aligned}
$$

As a result, we can find out what the doubling map does to an angle by looking at what the shift map does to its binary representation. We can express that relationship with a formula:

$$D(\phi(w)) = \phi(S(w)).$$

We can also express it with a picture:

$$
\begin{tikzcd}
\mathbf{2}^{\mathbb{N}} \arrow[r, "S"] \arrow[d, "\phi"'] \arrow[dr, phantom, "/\!\!/"] & \mathbf{2}^{\mathbb{N}} \arrow[d, "\phi"] \\
\mathbb{T} \arrow[r, "D"'] & \mathbb{T}
\end{tikzcd}
$$

Because of this property, and some others we'll learn about later, the function ϕ is an example of a semiconjugacy from the shift map to the doubling map.

1.2.3 What the binary representation is good for

Finding fixed points

Fact. *If w is a fixed point of the shift map, then $\phi(w)$ is a fixed point of the doubling map.*

Explanation. Suppose $S(w) = w$. Then $\phi(S(w)) = \phi(w)$. From the property we saw in Section 1.2.2, $D(\phi(w)) = \phi(w)$. $\square$

As we've seen, the shift map has two fixed points: $\overline{0}$ and $\overline{1}$. Let's see where ϕ takes them.

$$\begin{aligned}
\phi(\overline{0}) &\equiv 2\pi\, 0.\overline{0} \\
&\equiv 2\pi\left(\frac{0}{2} + \frac{0}{2^2} + \frac{0}{2^3} + \frac{0}{2^4} + \dots\right) \\
&\equiv 0.
\end{aligned}$$

$$\begin{aligned}
\phi(\overline{1}) &\equiv 2\pi\, 0.\overline{1} \\
&\equiv 2\pi\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots\right) \\
&\equiv 2\pi \cdot \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots\right) \\
&\equiv 2\pi \cdot \frac{1}{2}\left(\frac{1}{1 - \frac{1}{2}}\right) \\
&\equiv 2\pi \cdot \frac{1}{2} \cdot \frac{2}{1} \\
&\equiv 2\pi \\
&\equiv 0.
\end{aligned}$$

Finding periodic points In week 2, we saw two examples of periodic points for the doubling map. [Draw orbits of $2\pi/3$ and $2\pi/9$.] These periodic points are kind of mysterious.

Fact. *If w is an n-periodic point of the shift map, then $\phi(w)$ is an n-periodic point of the doubling map.*

When you did Homework 1, you probably noticed that there's a simple way to describe the n-periodic points of the shift map: they're the sequences that consist of a repeating n-digit block. Using this description, you can easily find lots of periodic points of the doubling map. (In fact, we'll see later that we can find all the periodic points this way.)

The 2-periodic point of D we saw earlier comes from the 2-periodic point $\overline{\texttt{01}}$ of S.

$$\begin{aligned}
\phi(\overline{01}) &\equiv 2\pi\, 0.\overline{01} \\
&\equiv 2\pi \left(\frac{0}{2} + \frac{1}{2^2} + \frac{0}{2^3} + \frac{1}{2^4} + \frac{0}{2^5} + \frac{1}{2^6} + \dots \right) \\
&\equiv 2\pi \left(\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots \right) \\
&\equiv 2\pi \cdot \frac{1}{4} \left(1 + \frac{1}{4} + \frac{1}{4^2} + \dots \right) \\
&\equiv 2\pi \cdot \frac{1}{4} \left(\frac{1}{1 - \frac{1}{4}} \right) \\
&\equiv 2\pi \cdot \frac{1}{4} \cdot \frac{4}{3} \\
&\equiv \frac{2\pi}{3}
\end{aligned}$$

The 6-periodic point of D we saw earlier comes from the 6-periodic point $\overline{\texttt{000111}}$ of S. To find a formula for this point, first notice that

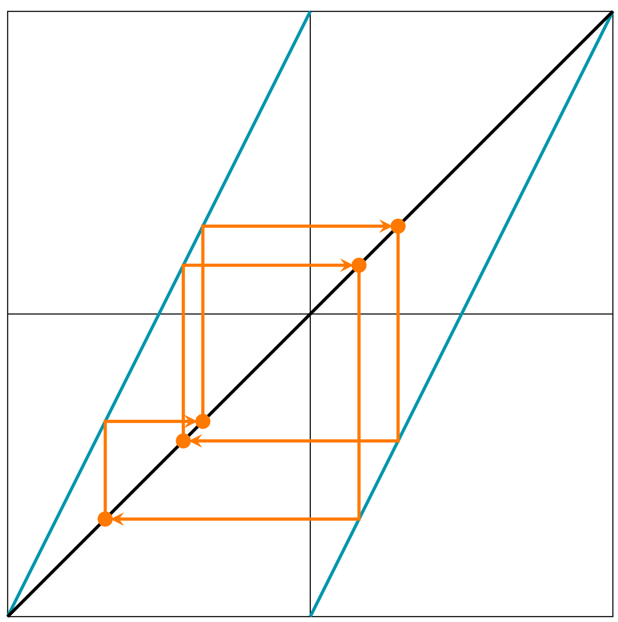
$$\begin{aligned}
0.000111 \text{ (in binary)} &= \frac{0}{2} + \frac{1}{2^2} + \frac{0}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} \\
&= \frac{4}{2^6} + \frac{2}{2^6} + \frac{1}{2^6} \\
&= \frac{7}{64}.
\end{aligned}$$

Then we can see that

$$\begin{aligned}
\phi(\overline{000111}) &\equiv 2\pi\, 0.\overline{000111} \\
&\equiv 2\pi \left(\frac{7}{64} + \frac{7}{64^2} + \frac{7}{64^3} + \frac{7}{64}^4 + \dots \right) \\
&\equiv 2\pi \cdot \frac{7}{64} \left(1 + \frac{1}{64} + \frac{1}{64^2} + \frac{7}{64^3} + \dots \right) \\
&\equiv 2\pi \cdot \frac{7}{64} \left(\frac{1}{1 - \frac{1}{64}} \right) \\
&\equiv 2\pi \cdot \frac{7}{64} \cdot \frac{64}{63} \\
&\equiv 2\pi \cdot \frac{7}{63} \\
&\equiv \frac{2\pi}{9}
\end{aligned}$$

The sequence $\overline{\texttt{00101}}$ gives a periodic point of D we haven't seen before.

$$\begin{aligned}
\phi(\overline{00101}) &\equiv 2\pi\, 0.\overline{00101} \\
&\equiv 2\pi \left(\frac{5}{32} + \frac{5}{32^2} + \frac{5}{32^3} + \dots \right) \\
&\equiv 5 \cdot 2\pi \left(\frac{1}{1 - \frac{1}{32}} \right) \\
&\equiv 5 \cdot 2\pi \, \frac{32}{31} \\
&\equiv 5 \cdot 2\pi \, \left(1 + \frac{1}{31} \right) \\
&\equiv 5 \cdot \frac{2\pi}{31} \\
&\equiv \frac{10\pi}{31}
\end{aligned}$$



This orbit looks really weird! I don't think I ever could've found it without using the semiconjugacy from the shift map to the doubling map.

Finding eventually fixed points

Fact. *If w is an eventually fixed point of the shift map, then $\phi(w)$ is an eventually fixed point of the doubling map.*

The eventually fixed points of the shift map are the sequences that end with $\overline{\texttt{0}}$ or $\overline{\texttt{1}}$. [Point out the simplest few on the unit circle, down to all points at denominator 8 and a few at denominator 16.] These sequences turn out to represent the fractions whose denominators are powers of two. Hence, when t is a fraction whose denominator is a power of two, $2\pi t$ is an eventually fixed point of the doubling map. To be more explicit, we can say the point

$$\frac{2\pi k}{2^n}$$

is eventually fixed for any $k, n \in \{0, 1, 2, 3, \ldots\}$. Note that each eventually fixed point has two binary representations: one ending with $\overline{\texttt{0}}$ and one ending with $\overline{\texttt{1}}$.