Week 7 notes

Aaron Fenyes
University of Toronto

Chaos, fractals, and dynamics
MAT 335, Winter 2019

1 Revisiting the dynamics of quadratic maps

1.1 Dynamics in different ranges of c

At the beginning of the course, we played with the standard quadratic maps $Q_c = x^2 + c$ on the state space $\mathbb{R}$, and we noticed that different values of c led to very different kinds of behavior. Now that we've learned about graphical analysis and semiconjugacy, we can understand the behavior of Q_c, and the transitions between different kinds of behavior, for a wide range of c values.

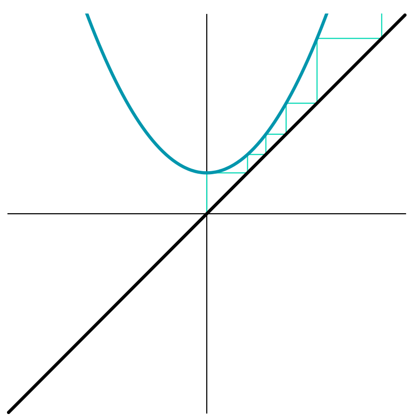
1.1.1 The "upper range," $c \in (-1.4011551\ldots, \infty)$

Graphical analysis is all you need

When c is above the weird-looking value $-1.4011551\ldots$, it's possible to understand all the orbits of Q_c just using graphical analysis. As c decreases, the orbits get more and more complicated. Follow along in the slides as we descend.

- $c \in (0.25, \infty)$

  No fixed points. All orbits fly off to the right.



↓ *At $c = 0.25$, the graph of Q_c touches the diagonal (see slides).*

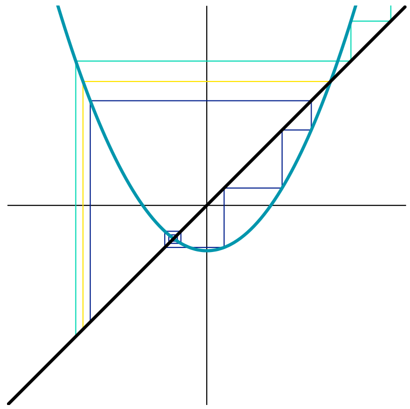
When c goes below 0.25, the graph of Q_c crosses the diagonal twice, once with slope just shallower than 1 and once with slope just steeper than 1. These crossings show that Q_c has developed two new fixed points—one attracting and one repelling.

- $c \in (-0.75, 0.25)$

  One repelling fixed point, $p_+ = \frac{1}{2}(1 + \sqrt{1 - 4c})$.

  One attracting fixed point, $p_- = \frac{1}{2}(1 - \sqrt{1 - 4c})$.

Orbits starting outside $[-p_+, p_+]$ fly off to the right. The interval $(-p_+, p_+)$ is a basin of attraction for p_+. The orbits of $-p_+$ and p_+ are eventually fixed at p_+.



↓ *When c is just above -0.75, the graph of Q_c crosses the diagonal at p_- with slope just shallower than -1, and the graph of Q_c^2 crosses the diagonal at p_- with slope just shallower than 1 (see slides).*

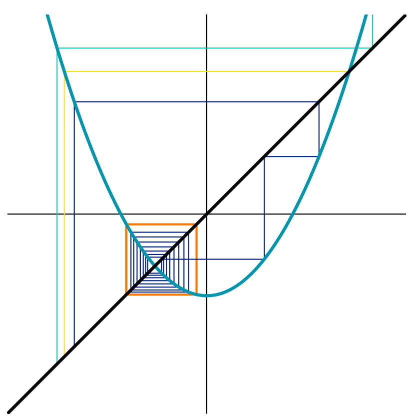
When c goes below -0.75, the slope of Q_c at p_- becomes steeper than -1, so the fixed point p_- goes from attracting to repelling. The slope of Q_c^2 at p_- becomes steeper than 1, and two new crossings with slope just shallower than 1 appear beside the old crossing. These new crossings show that Q_c has developed a new attracting 2-periodic orbit.

- $c \in (-1.25, -0.75)$

  Two repelling fixed points, p_+ and p_-.

  One attracting 2-periodic orbit, which alternates between the points $q_\pm = \frac{1}{2}(-1 \pm \sqrt{-3-4c})$.

  Orbits starting outside $[-p_+, p_+]$ fly off to the right. Orbits starting in $[-p_+, p_+]$ approach the 2-periodic orbit, unless they're eventually fixed.



↓ *When c is just above -1.25, the graph of Q_c^2 crosses the diagonal at q_- and q_+ with slope just shallower than -1, and the graph of Q_c^4 crosses the diagonal at q_- and q_+ with slope just shallower than 1 (see slides).*

When c goes below -1.25, the slope of Q_c^2 at $q_\pm$ becomes steeper than -1, so the orbit of q_- and q_+ goes from attracting to repelling. The slope of Q_c^4 at q_- and q_+ becomes steeper than 1, and two new crossings with slope just shallower than 1 appear beside each old crossing. These new crossings show that Q_c has developed a new attracting 4-periodic orbit.

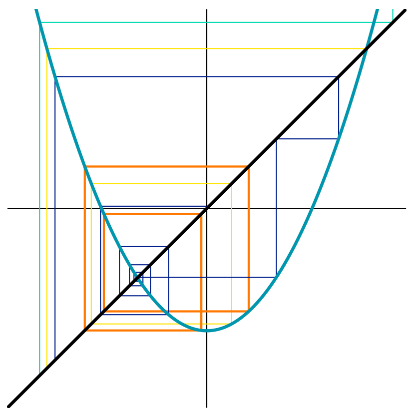
- $c \in (-1.3680989\ldots, -1.25)$

  Two repelling fixed pionts, p_+ and p_-.

  One repelling 2-periodic orbit, which alternates between $q_\pm$.

  One attracting 4-periodic orbit.

  Orbits starting outside $[-p_+, p_+]$ fly off to the right. Orbits starting in $[-p_+, p_+]$ approach the 4-periodic orbit, unless they're eventually 2-periodic or eventually fixed.



This pattern continues all the way to the bottom of the upper range. For each $c \in (-1.4011551\ldots, \infty)$, the map Q_c has repelling periodic orbits with minimum periods $1, 2, 4, \ldots, 2^{n-1}$, and an attracting periodic orbit with minimum period 2^n. When c gets small enough, the orbit with minimum period 2^n becomes repelling, and a new attracting orbit with minimum period 2^{n+1} appears.

Wikipedia has a nice table showing the first few thresholds where new orbits appear.[1]

An orbit of minimum period	appears when c goes below
1	0.25
2	-0.75
4	-1.25
8	$-1.3680989\ldots$
16	$-1.3940462\ldots$
32	$-1.3996312\ldots$
64	$-1.4008286\ldots$
128	$-1.4010853\ldots$
256	$-1.4011402\ldots$
512	$-1.4011519\ldots$
1024	$-1.4011545\ldots$

[1] https://en.wikipedia.org/wiki/Feigenbaum_constants

[Look at Adam Majewski's *bifurcation diagram*,[2] which shows the points with minimum periods 1, 2, 4, 8 at each value of c.]

1.1.2 Focusing on the bounded orbits

If you want to understand $Q_c\colon \mathbb{R} \to \mathbb{R}$, it helps to focus on the points whose orbits don't fly off toward infinity. These points form a subset of $K_c \subset \mathbb{R}$, called the *filled Julia set* of Q_c.

When c is in the "upper range" $(-1.4011551\ldots, \infty)$, we have a very simple description of K_c: it's just the interval $[-p_+, p_+]$. Furthermore, we can understand all the orbits inside K_c using graphical analysis.

1.1.3 The "middle range," $(-2, -1.4011551\ldots)$

In this range, K_c is still just the interval $[-p_+, p_+]$. Unfortunately…

The orbits inside K_c are totally bananas

You can see how complicated the orbits get in the middle range by looking at the slides. The last slide in the upper range shows $Q_{-1.4}$, which has an attracting orbit of minimum period 32. The orbit of 0 isn't exactly simple, but it's much tamer than the orbit of 0 on the next slide. Under $Q_{-1.45}$, at the top of the middle range, the orbit of 0 spreads thickly over a pair of intervals.

1.1.4 The "lower range," $c \in (-\infty, -2]$

Graphical analysis and a semiconjugacy are all you need

In most of this range, K_c isn't an interval anymore, but we can still use graphical analysis to get a pretty simple description of it. Furthermore, we'll find a semiconjugacy from the shift map to $Q_c\colon K_c \to K_c$. You can use that semiconjugacy to understand all the orbits inside K_c.

- $c = -2$

  We've seen this one before! Let's recall our first two examples of semiconjugacies.

  - ◇ The binary representation $\phi\colon \mathbf{2}^{\mathbb{N}} \to \mathbb{T}$, which is a semiconjugacy from the shift map to the doubling map.

$$\begin{array}{ccc} \mathbf{2}^{\mathbb{N}} & \xrightarrow{S} & \mathbf{2}^{\mathbb{N}} \\ {\scriptstyle\phi}\downarrow & /\!\!/ & \downarrow{\scriptstyle\phi} \\ \mathbb{T} & \xrightarrow[D]{} & \mathbb{T} \end{array}$$

  - ◇ The function $h\colon \mathbb{T} \to [-2, 2]$ given by the formula $h(\theta) = 2\cos(\theta)$, which is a semiconjugacy from the doubling map to $Q_{-2}\colon [-2, 2] \to [-2, 2]$.

[2] https://commons.wikimedia.org/wiki/File:Bifurcation_diagram_for_real_quadratic_map._Periodic_points_for_periods_1,2,4,and_8_are_shown.png

$$\begin{array}{ccc}
\mathbb{T} & \xrightarrow{D} & \mathbb{T} \\
{\scriptstyle h}\downarrow & /\!/ & \downarrow{\scriptstyle h} \\
[-2,2] & \xrightarrow[Q_{-2}]{} & [-2,2]
\end{array}$$

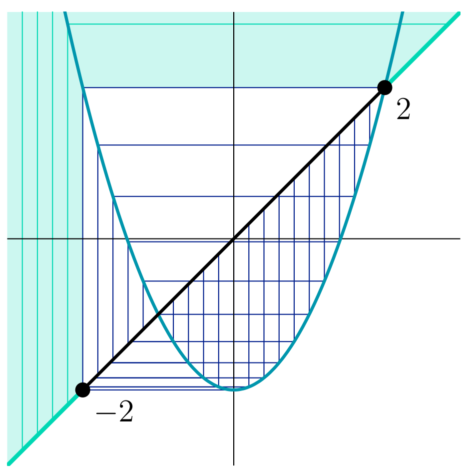
See how the bottom of the first picture matches the top of the second? They're just begging us to stick them together. If we do, everything works out perfectly.

Fact. *The composition of two semiconjugacies is always a semiconjugacy.*

In our case, that means the composition $h \circ \phi$ is a semiconjugacy from the shift map to $Q_{-2} \colon [-2, 2] \to [-2, 2]$.

$$\begin{array}{ccc}
\mathbf{2}^{\mathbb{N}} & \xrightarrow{S} & \mathbf{2}^{\mathbb{N}} \\
{\scriptstyle \phi}\downarrow & /\!/ & \downarrow{\scriptstyle \phi} \\
\mathbb{T} & \xrightarrow{D} & \mathbb{T} \\
{\scriptstyle h}\downarrow & /\!/ & \downarrow{\scriptstyle h} \\
[-2,2] & \xrightarrow[Q_{-2}]{} & [-2,2]
\end{array}$$

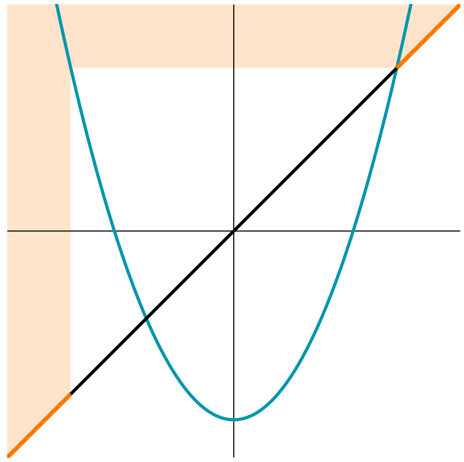
Using graphical analysis and a bit of algebra, you can work out that $K_{-2} = [-2, 2]$.



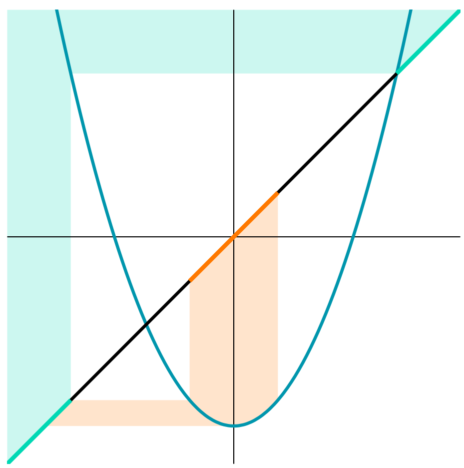
So, we've found a semiconjugacy from the shift map to $Q_{-2} \colon K_{-2} \to K_{-2}$. Later, we'll use this semiconjugacy to understand all the orbits in K_{-2}. For now, though, let's move on to lower values of c.

- $c \in (-\infty, -2)$

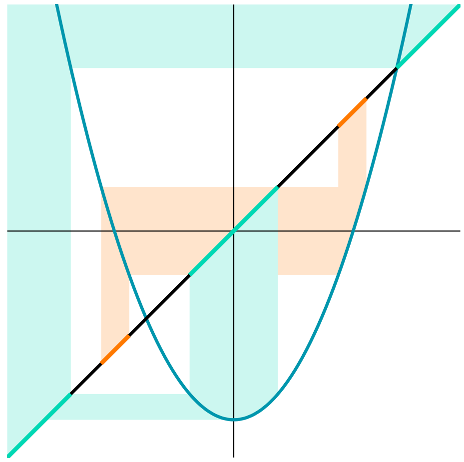
  In this range, the filled Julia set K_c isn't an interval anymore. Let's see what it looks like. Looking at the graph of Q_c, the first thing we notice is that points outside $[-p_+, p_+]$ have orbits that fly off toward infinity. These points aren't in K_c. They form a subset $L_0 \subset \mathbb{R}$.



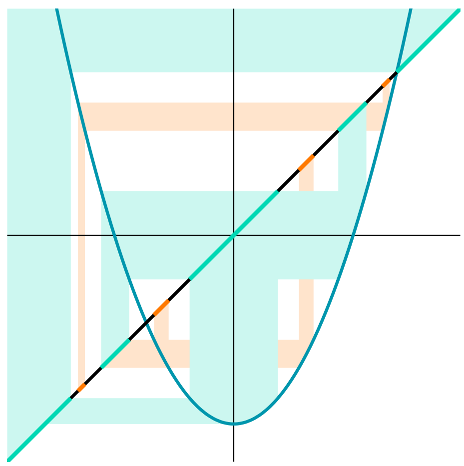
The orbits starting in L_0 aren't the only ones whose orbits fly off toward infinity. If you start close enough to zero, you'll get an orbit that enters L_0 after one step, and then flies off toward infinity from there. The points that enter L_0 after one step, but not before, form a subset $L_1 \subset \mathbb{R}$. Since their orbits fly off toward infinity, these points aren't in K_c either.



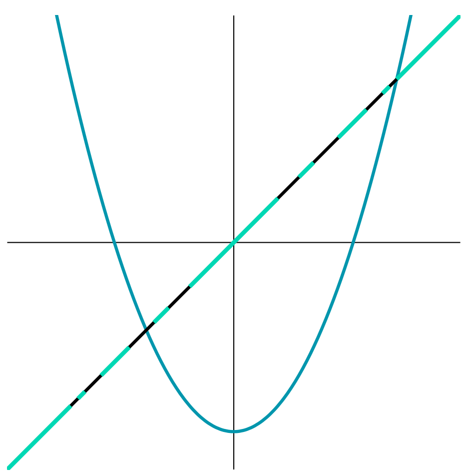
Points that enter L_0 after two steps, but not before, form a subset $L_2 \subset \mathbb{R}$. These points aren't in K_c either.



Points that enter L_0 after three steps, but not before, form a subset $L_3 \subset \mathbb{R}$. These points aren't in K_c either.



Every point whose orbit flies off toward infinity eventually ends up in L_0. That means every point that's not in K_c must be in one of the subsets $L_0, L_1, L_2, L_3 \ldots$. Turning this reasoning around, we learn that K_c is what's left after we remove all the subsets $L_0, L_1, L_2, L_3 \ldots$ from $\mathbb{R}$.



You can get a rough idea of what K_c looks like by removing only the first few L sets—for example, $L_0, \ldots, L_3$ as shown above.

Now that we know what K_c looks like, our next step is to find a semiconjugacy from the shift map to $Q_c \colon K_c \to K_c$. To learn how that semiconjugacy will work, let's take a break to study a dynamical map which is very similar to Q_c with $c \in (-\infty, -2]$, but can be understood much more concretely.