Week 8 notes

Aaron Fenyes
University of Toronto

Chaos, fractals, and dynamics
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Term test 2

Rough range:

- Up to week 5 notes.
- Up to homework 3.

1.2 A warm-up for the lower range

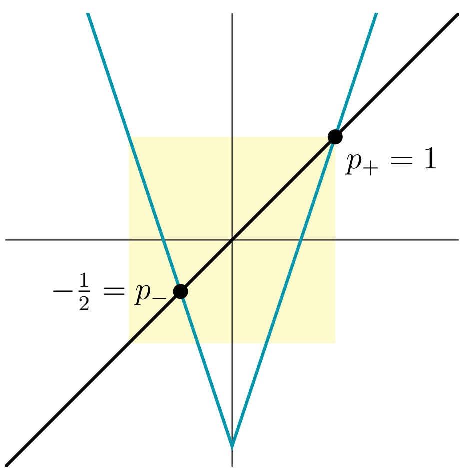
Textbook references:

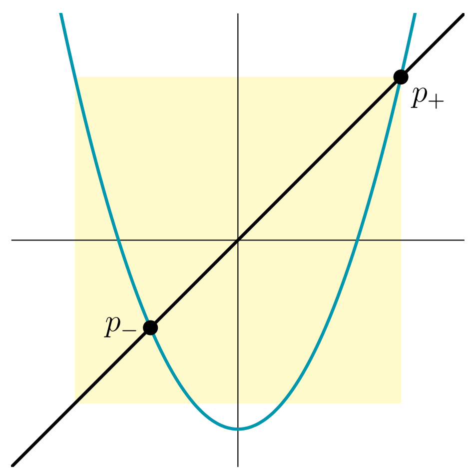
- Chapter 7 of *A First Course in Chaotic Dynamical Systems*
- Section 1.5 of *An Introduction to Chaotic Dynamical Systems*

1.2.1 The V map

Consider the dynamical map $V(x) = 3|x| - 2$ on the state space $\mathbb{R}$. This map is very similar to Q_c with $c \in (-\infty, -2]$, but can be understood much more concretely.

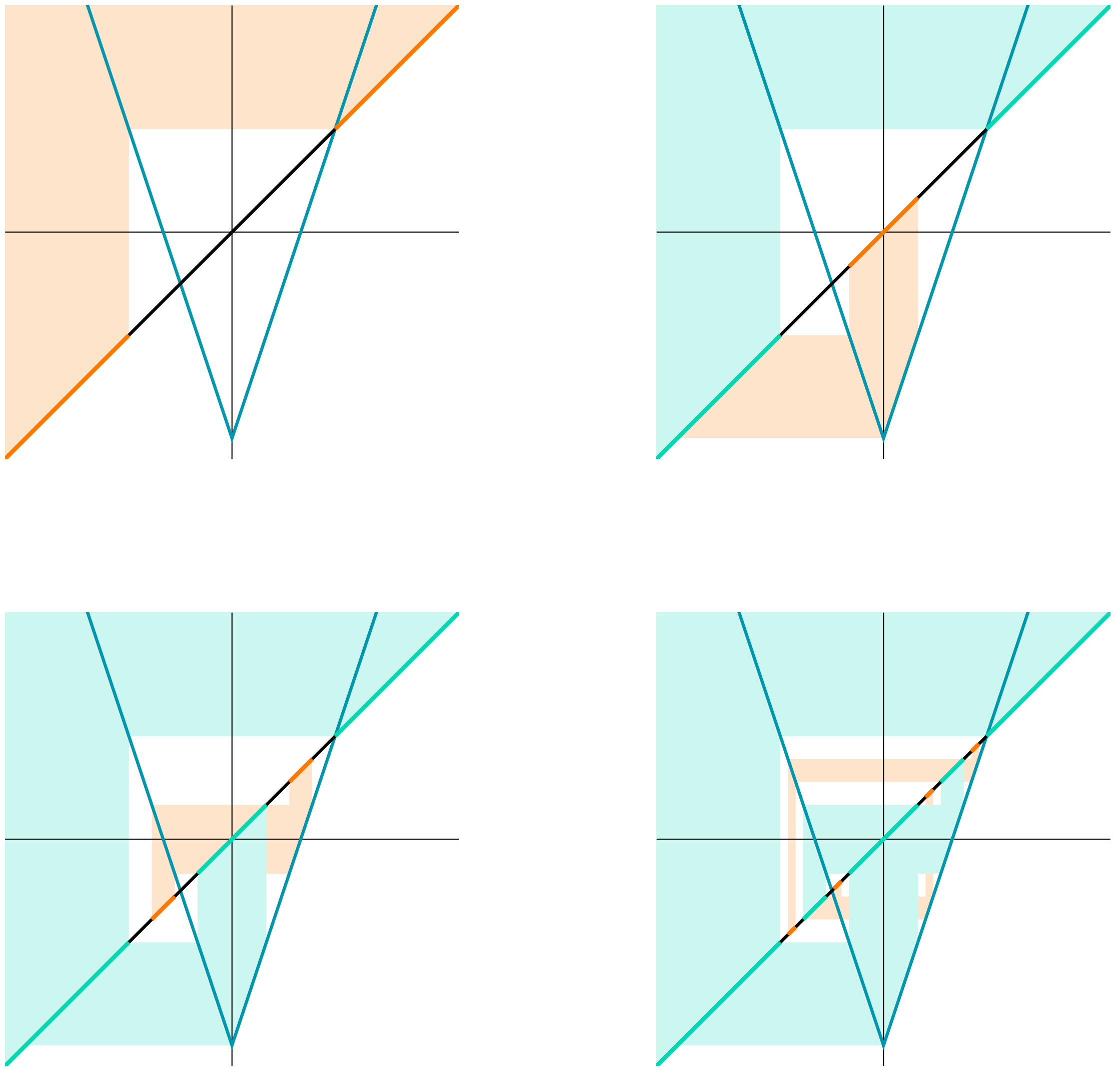
The graph of V looks a lot like the graph of Q_c. It touches the diagonal twice, at the points $p_- = -\frac{1}{2}$ and $p_+ = 1$. It pokes out below the bottom of the $[-p_+, p_+]$ box.





1.2.2 The filled Julia set of the V map

Let's say K is the filled Julia set of V. We can carve out K from $\mathbb{R}$ in the same way that we carved out K_c. We start by defining L_0 as $\mathbb{R} \setminus [-p_+, p_+]$. The points that enter L_0 after n steps, but not before, form a subset $L_n \subset \mathbb{R}$. You can see L_0 and a graphical calculation of L_1, L_2, L_3 in the slides, and the pictures below.



The reason we can understand V more concretely than Q_c is that, when we're carving out K, we can find simple, concrete descriptions of the sets $L_1, L_2, L_3, \ldots$. The set L_{n+1} consists of the middle thirds of the intervals that remain when you remove $L_0, \ldots, L_n$ from $\mathbb{R}$. Explicitly,

$$\begin{aligned}
L_0 &= (-\infty, -1) \cup (1, \infty) \\
L_1 &= (-\tfrac{1}{3}, \tfrac{1}{3}) \\
L_2 &= (-\tfrac{7}{9}, -\tfrac{5}{9}) \cup (\tfrac{5}{9}, \tfrac{7}{9}) \\
L_3 &= (-\tfrac{25}{27}, -\tfrac{23}{27}) \cup (-\tfrac{13}{27}, -\tfrac{11}{27}) \cup (\tfrac{11}{27}, \tfrac{13}{27}) \cup (\tfrac{23}{27}, \tfrac{25}{27})
\end{aligned}$$

1.2.3 An itinerary function for the V map

Now that we know what K looks like, our next goal is to find a semiconjugacy from the shift map to $V \colon K \to K$. We'll do this using the idea of an *itinerary function*, which you met in homework 3 (problem 2).

Removing L_0 and L_1 from $\mathbb{R}$ leaves two intervals. Let's call the left one I_0 and the right one I_1, as shown in the slides. The set K divides naturally into two parts: the part inside

I_0 and the part inside I_1. Define a function $\tau \colon K \to \mathbf{2}^{\mathbb{N}}$ in the following way.

$$\text{the } n\text{th digit of } \tau(x) \text{ is } \begin{cases} 0 & \text{if } V^n(x) \in I_0 \\ 1 & \text{if } V^n(x) \in I_1 \end{cases}$$

Like we did in the homework, we'll call the starting digit of a sequence the 0th digit, and we'll use the convention that $V^0(x) = x$. The function τ is an example of an *itinerary function* [see homework 3, problem 2, for another]. Intuitively, the sequence $\tau(x)$ tells you when the orbit of x visits the left and right parts of K.

I'd like to convince you that τ is a semiconjugacy from $V \colon K \to K$ to the shift map. Furthermore, τ is invertible, and its inverse is a semiconjugacy too. In other words, τ is a *conjugacy*—that's a term we learned from homework 3. As I said in the homework, two dynamical systems connected by a conjugacy are the same for all practical purposes. So, if we can convince ourselves that τ is a conjugacy, we'll understand $V \colon K \to K$ just as well as we understand the shift map—and we understand the shift map *very* well.

1.2.4 Dividing up the filled Julia set

There are lots of steps involved in showing that τ is a conjugacy, but they all rest on one key trick: dividing K into pieces according to the first few digits of the itinerary.

Removing L_0 and L_1 from $\mathbb{R}$ left us with the two intervals I_0 and I_1. Each one maps to $[-p_+, p_+]$ when you apply V. Removing L_2 divides each of the intervals I_0 and I_1 into two "second-level" intervals. We can name them according to what happens to them when you apply V.

The 1st half of	I_0	maps to	I_1,	so we call it	I_{01}.
The 2nd half of	I_0	maps to	I_0,	so we call it	I_{00}.
The 1st half of	I_1	maps to	I_0,	so we call it	I_{10}.
The 2d half of	I_1	maps to	I_0,	so we call it	I_{11}.

Removing L_3 divides each of the second-level intervals into two "third-level" intervals. We can name them in a similar way.

The 1st quarter of	I_0	maps to	I_{11},	so we call it	I_{011}.
The 2nd quarter of	I_0	maps to	I_{10},	so we call it	I_{010}.
The 3rd quarter of	I_0	maps to	I_{00},	so we call it	I_{000}.
The 4th quarter of	I_0	maps to	I_{01},	so we call it	I_{001}.
The 1st quarter of	I_1	maps to	I_{01},	so we call it	I_{101}.
The 2nd quarter of	I_1	maps to	I_{00},	so we call it	I_{100}.
The 3rd quarter of	I_1	maps to	I_{10},	so we call it	I_{110}.
The 4th quarter of	I_1	maps to	I_{11},	so we call it	I_{111}.

If you know which nth-level interval a point $x \in K$ is inside, you know the first n digits of $\tau(x)$. For example,

- $\frac{9}{26}$ is inside I_1, so $\tau(\frac{9}{26})$ looks like 1________.
- $\frac{9}{26}$ is inside I_{10}, so $\tau(\frac{9}{26})$ looks like 10________.

- $\frac{9}{26}$ is inside $I_{\mathtt{101}}$, so $\tau(\frac{9}{26})$ looks like $\mathtt{101}$____.

Each nth-level interval has width $2/3^n$.

1.2.5 Our itinerary function is a conjugacy

I'd like to convince you that τ is a conjugacy from $V\colon K \to K$ to the shift map $S\colon \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$. Instead of giving you a full justification, I'll give you a sketch that illustrates the basic ideas. Then, if you talk about it with your classmates, you should be able to come up with a full justification on your own.

First, I have to show τ is a semiconjugacy from V to S. That means I have to convince you τ has the following properties.

- *Desired property.* We can find out what S does to $\tau(x)$ by looking at what V does to x. In symbols,

  $$S(\tau(x)) = \tau(V(x)) \quad \text{for all } x \in K.$$

  Justification. To show that the binary sequences $S(\tau(x))$ and $\tau(V(x))$ are equal, we just have to show that all their digits match. We can write the digits of $S(\tau(x))$ in terms of the digits of $\tau(x)$ using the definition of the shift map:

  $$\text{the } n\text{th digit of } S(\tau(x)) \text{ is the } (n+1)\text{st digit of} \tau(x).$$

  Let's see if we can get the same expression for the digits of $\tau(V(x))$. We'll start from the definition of τ:

  $$\begin{aligned} \text{the } n\text{th digit of } \tau(V(x)) \text{ is } & \begin{cases} \mathtt{0} & \text{if } V^n(V(x)) \in I_{\mathtt{0}} \\ \mathtt{1} & \text{if } V^n(V(x)) \in I_{\mathtt{1}} \end{cases} \\ \text{which is } & \begin{cases} \mathtt{0} & \text{if } V^{n+1}(x) \in I_{\mathtt{0}} \\ \mathtt{1} & \text{if } V^{n+1}(x) \in I_{\mathtt{1}} \end{cases} \\ & \text{which is the } (n+1)\text{st digit of } \tau(x). \end{aligned}$$

  It's now apparent that all the digits of $S(\tau(x))$ and $\tau(V(x))$ match, so $S(\tau(x)) = \tau(V(x))$. This conclusion holds for every $x \in K$, because our argument didn't make any assumptions about the value of x.

- *Desired property.* The function τ is continuous.

  Idea for justification. Let me convince you that τ is continuous at $\frac{9}{26}$. I need to show that I can keep $\tau(x)$ within any "target" open ball around $\overline{\mathtt{10}}$ by keeping x close enough to $\frac{9}{26}$.

  The open ball $B_{\overline{\mathtt{10}}}(2^{-n})$ is the set of sequences which match $\overline{\mathtt{10}}$ for the first $n+1$ digits.

I can keep $\tau(x)$ within	____	by keeping x in	____.
	$B_{\overline{\mathtt{10}}}(2^0)$		$I_{\mathtt{1}}$
	$B_{\overline{\mathtt{10}}}(2^1)$		$I_{\mathtt{10}}$
	$B_{\overline{\mathtt{10}}}(2^2)$		$I_{\mathtt{101}}$
	$B_{\overline{\mathtt{10}}}(2^3)$		$I_{\mathtt{1010}}$

For each row of the table, I can find a ball around $\frac{9}{26}$ that stays within the listed interval by looking at the endpoints of the interval.

- *Desired property.* Every point in $\mathbf{2}^{\mathbb{N}}$ has a label in K. In other words, τ is onto.

  Idea for justification. We need to show that every infinite binary sequence is the itinerary of some point in K. As an example, let's find a point whose itinerary is $\overline{\mathtt{10}}$. Any point which is inside all of the intervals $I_{\mathtt{1}}, I_{\mathtt{10}}, I_{\mathtt{101}}, I_{\mathtt{1010}}, \ldots$ will have the itinerary we want. Using graphical analysis to work out where these intervals are, we can express their endpoints concretely.

$$\begin{aligned} I_{\mathtt{1}} &= [\tfrac{1}{3}, \tfrac{1}{3}] \\ I_{\mathtt{10}} &= [\tfrac{1}{3}, \tfrac{4}{9}] \\ I_{\mathtt{101}} &= [\tfrac{1}{3}, \tfrac{10}{27}] \\ &\vdots \end{aligned}$$

  Using some basic facts about limits, it's possible to find a point that's inside all these intervals.[1]

- *Desired property.* Each point in $\mathbf{2}^{\mathbb{N}}$ has a limited number of labels in K. In other words, τ is at most m-to-one, for some m.

  Idea for justification. It turns out that τ is one-to-one: there's only one point with each itinerary.

  As an example, let me convince you that there's only one point with the itinerary $\overline{\mathtt{10}}$. Suppose I tell you $\tau(x) = \overline{\mathtt{10}}$.

Then you know x is in	, which determines x to within
$I_{\mathtt{1}}$	$2/3$
$I_{\mathtt{10}}$	$2/3^2$
$I_{\mathtt{101}}$	$2/3^3$
$I_{\mathtt{1010}}$	$2/3^4$
$\vdots$	$\vdots$

  As we go down the list, the "wiggle room" in the right column shrinks toward zero, telling us that $\tau(x)$ completely determines x.

[1] The key fact is the "monotone convergence theorem," which you may have learned in a calculus course. The lower endpoints of the intervals form a sequence which is always increasing or standing still, but never goes above a certain level. The monotone convergence theorem says a sequence like this always has a limit, a. The upper endpoints form a sequence which is always decreasing or standing still, but never goes below a certain level. The monotone convergence theorem says a sequence like this always has a limit, b. Any point in $[a, b]$ will be in all the intervals.