Week 9 notes

Aaron Fenyes
University of Toronto

Chaos, fractals, and dynamics
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1 Chaos in the shift map

Textbook references:

- Chapter 10 of *A First Course in Chaotic Dynamical Systems*
- Section 1.8 of *An Introduction to Chaotic Dynamical Systems*

Over the last two weeks, studying the dynamics of quadratic maps in the "lower region" brought us back to the the shift map. We studied the shift map at the beginning of the course because it's simple to define, and it does a lot of simple things. Its fixed, eventually fixed, and periodic points are easy to see, and you should be able to convince yourself without much trouble that all its fixed and periodic points are repelling.

Today I want to show you that the shift map also does some very complex, subtle things—things that resemble certain kinds of complexity that we often see in nature. A lot of that complexity is summed up in the following properties:

- It has "sensitive dependence on initial conditions."

  You can totally change the long-term behavior of an orbit just by nudging it a tiny bit. Turbulent fluids like the Earth's atmosphere have this property, leading to the saying that a butterfly flapping its wings in Brazil might set off a tornado in Texas.[1]

- It's "topologically transitive."

  It can take you from any open ball to any other open ball. Good postal services have this property: you can send a letter from any neighborhood to any other neighborhood. Of course, you do have to make small starting point adjustments to get the letter where you want it to go: you have to write the proper address.

- Its periodic points are "dense."

  Every open ball, no matter how small, has periodic points inside it.

Together, these three properties define a distinctive kind of dynamical behavior, called *chaos*. The shift map on $\mathbf{2}^{\mathbb{N}}$ is a fundamental example of a chaotic dynamical system.

Before defining these properties in a formal, general way, let me illustrate them for the shift map.

[1] The saying comes from meteorologists Edward Lorenz and Philip Merilees (`https://www.technologyreview.com/s/422809/when-the-butterfly-effect-took-flight/`).

1.0.1 Sensitive dependence on initial conditions

Fact. *If you give me a point $w \in \mathbf{2}^{\mathbb{N}}$ and a radius 2^{-n}, I can find an orbit that starts within distance 2^{-n} of w, but eventually reaches a distance of more than $\frac{1}{2}$ from the orbit of w.*

To put it more formally: in any open ball $B_w(2^{-n})$, no matter how small, I can find a point v with the property that $d(S^k(v), S^k(w)) > \frac{1}{2}$ for some k.

Demonstration. [Ask for a point. Someone picks $\overline{\texttt{01}}$. Ask for a radius. Someone gives 2^{-5}. I pick $v = \texttt{0101011}$.] ☐

1.0.2 Topological transitivity

Fact. *If you give me an "source" open ball U and a "destination" open ball V in $\mathbf{2}^{\mathbb{N}}$, I can find a point $u \in U$ whose orbit eventually enters V.*

Demonstration. [Ask for a source. Someone gives $B_{\overline{\texttt{1}}}(\frac{1}{3})$. Ask for a destination. Someone gives $B_{\overline{\texttt{101}}}(2^{-3})$. I pick $u = \texttt{111011}$.] ☐

1.0.3 Density of periodic points

Fact. *Every open ball includes a periodic point.*

Demonstration. [As an example, take the open ball $B_{\texttt{01001110...}}(2^{-3})\ldots$. It includes the periodic point $\overline{\texttt{0100}}$.]

2 The formal definition of chaos

[Notes to be continued...] ☐