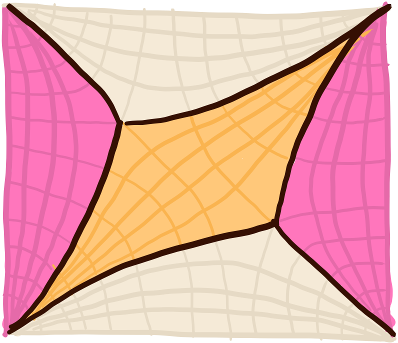
A bumpy ride through the space of holomorphic quadratic differentials

Aaron Fenyes (University of Toronto)

Holomorphic Differentials in Mathematics and Physics
Simons Center, February 2019

Abstract

When you deform a holomorphic quadratic differential on a punctured surface, its large-scale features stay constant, except for occasional jumps.



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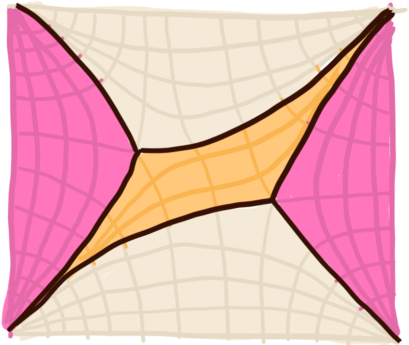

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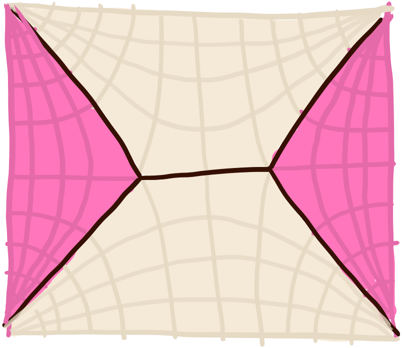

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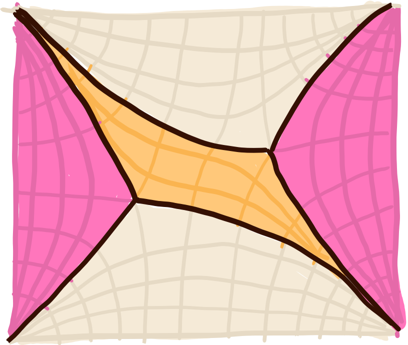

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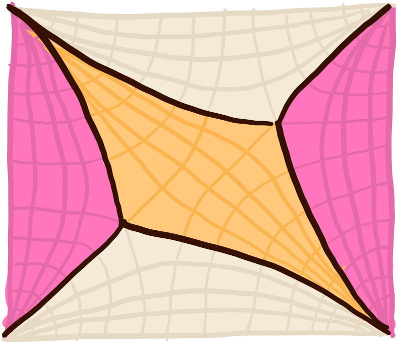

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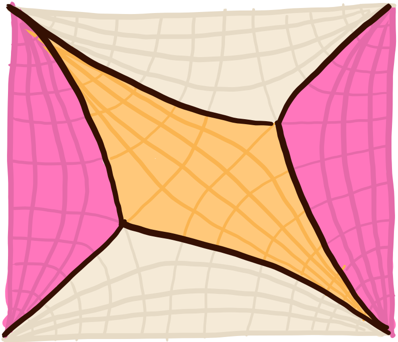

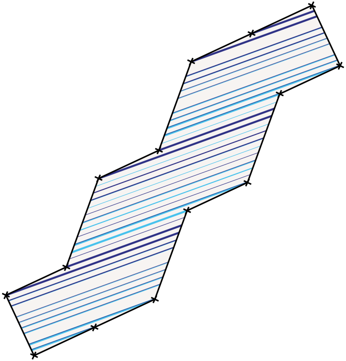

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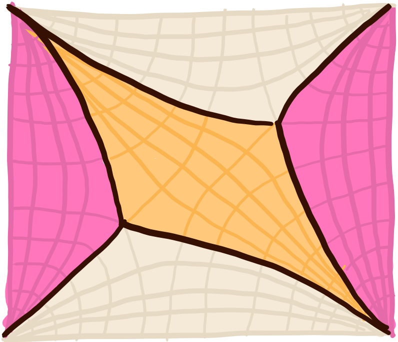

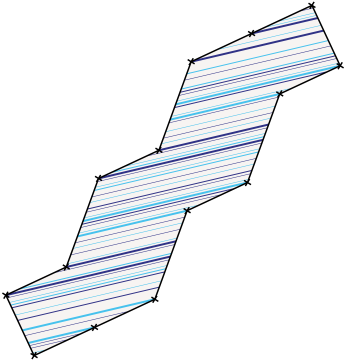

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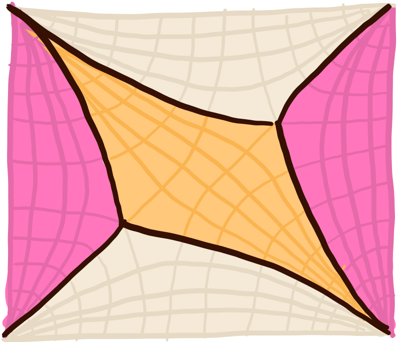

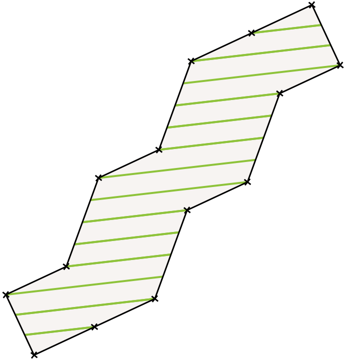

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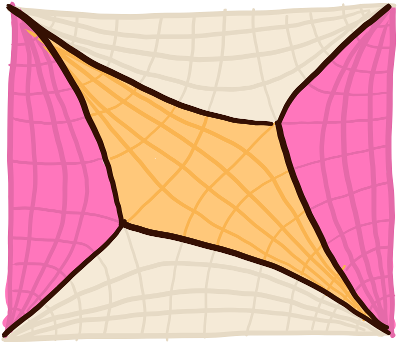

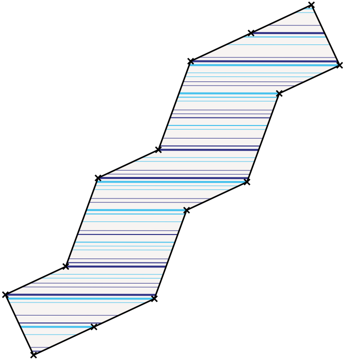

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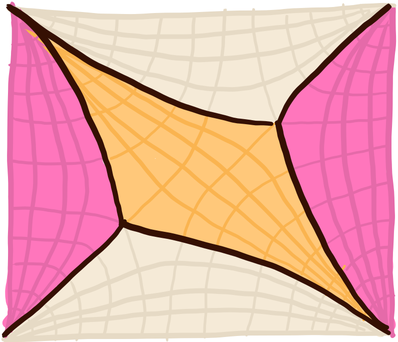

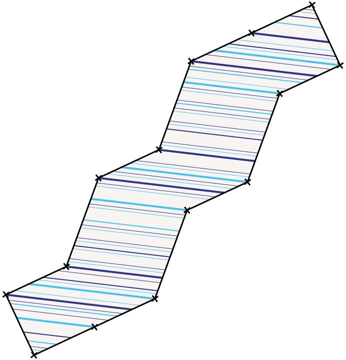

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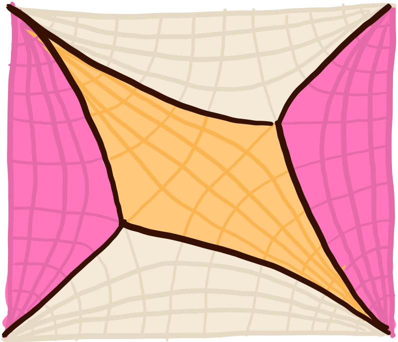

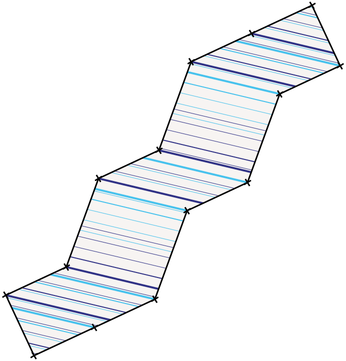

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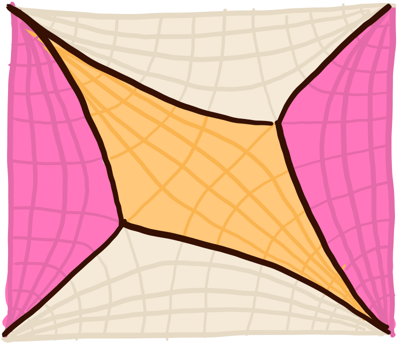

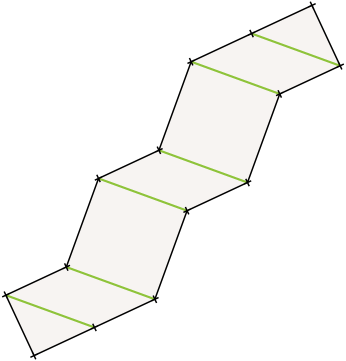

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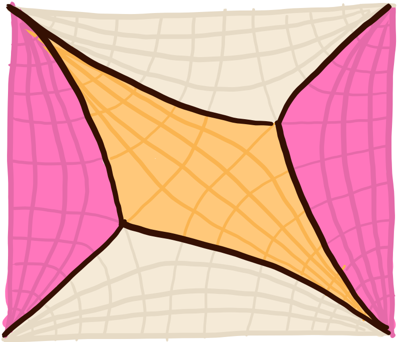

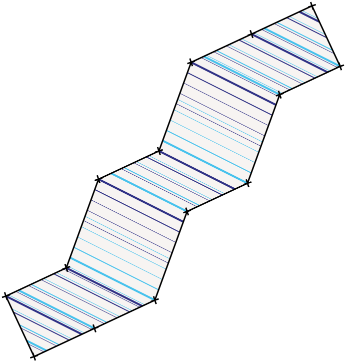

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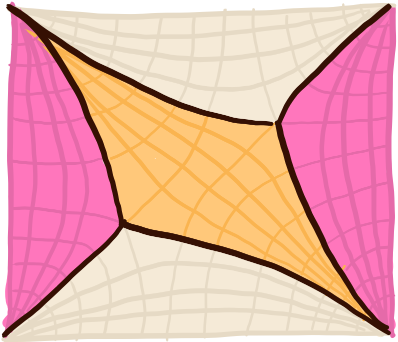

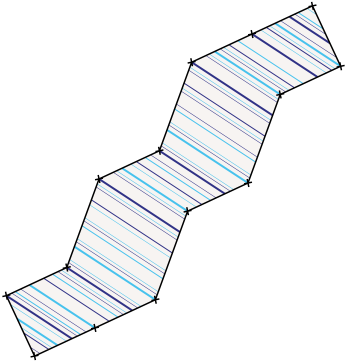

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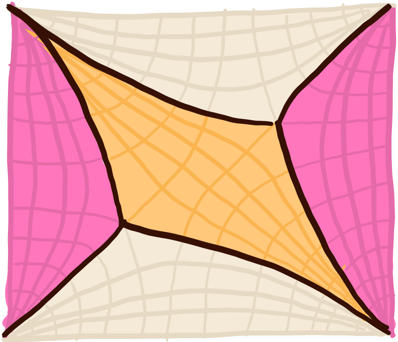

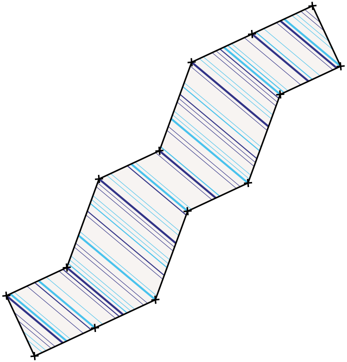

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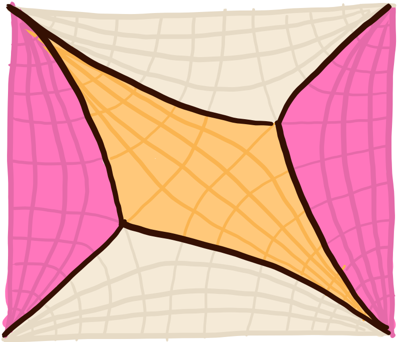

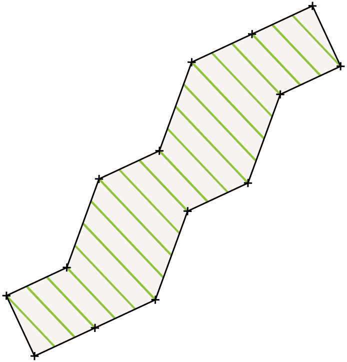

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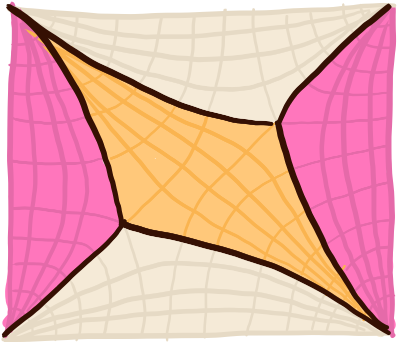

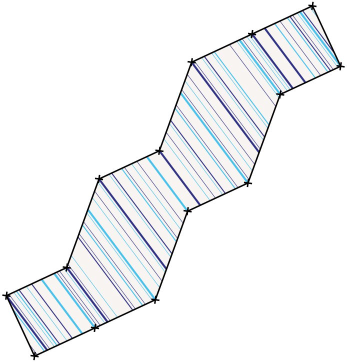

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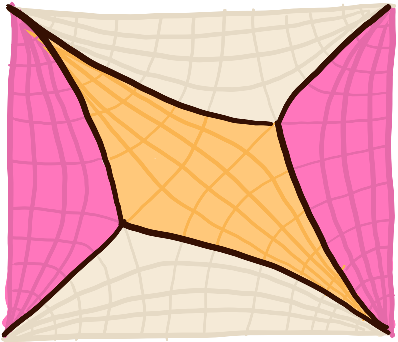
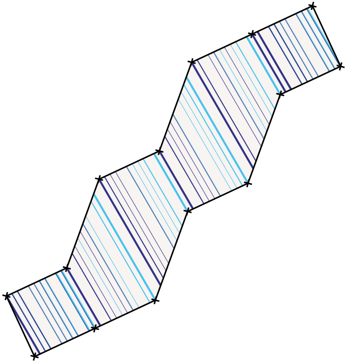


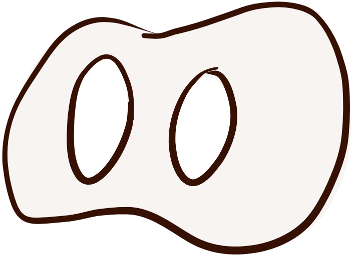


Quadratic differentials
as geometric structures

Take a topological surface, with a number ≥ 2 at each puncture.

A *half-translation structure* on the surface consists of:

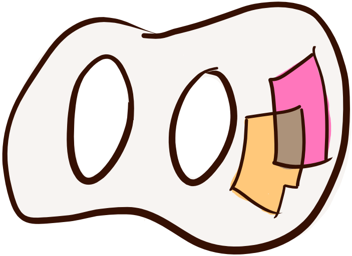
- A complex structure.
- A holomorphic quadratic differential ϕ with a pole of the given order at each puncture.

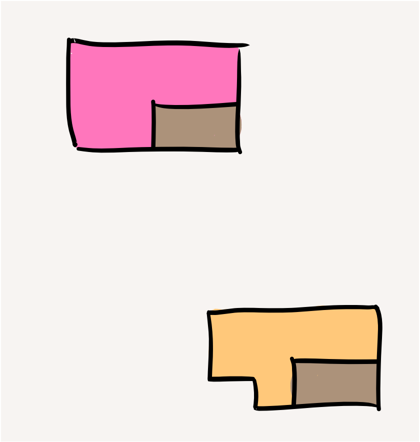


Quadratic differentials
as geometric structures

Holomorphic coordinates s with $ds^2 = \phi$ form an atlas encoding the half-translation structure.

The transition maps are translations and 180° flips.





complex plane

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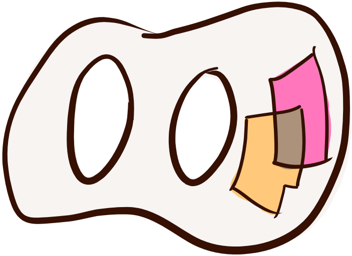
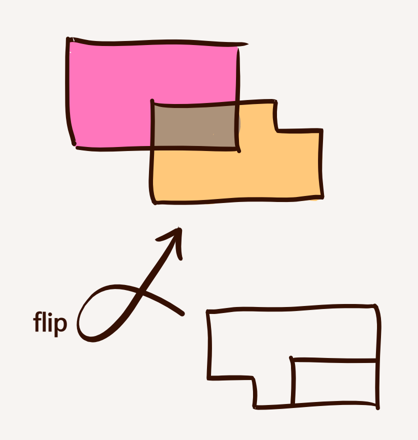


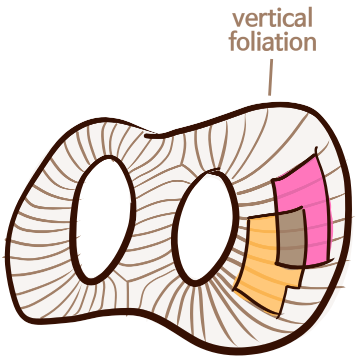


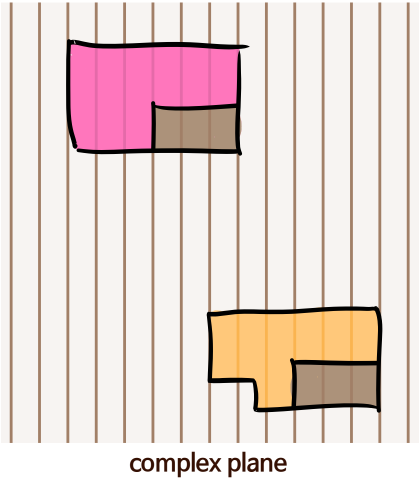
complex plane

Quadratic differentials
as geometric structures

The foliations of the charts by vertical lines fit together into a foliation of the surface.

Horizontal distance gives a local measure on swaths of leaves.





Quadratic differentials
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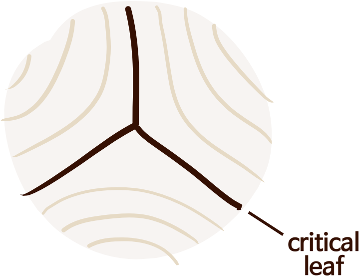
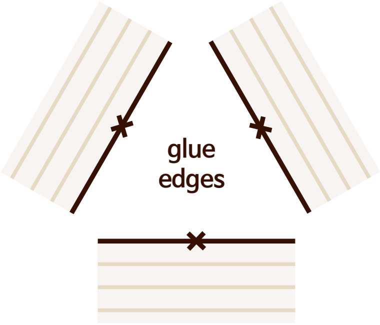
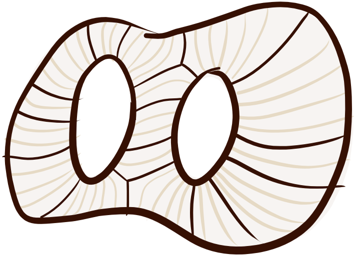
Each zero of ϕ becomes a *singularity* of the atlas.

The vertical leaves that hit the singularity are called *critical*.

Cutting along them splits a neighborhood of the singularity into half-planes.



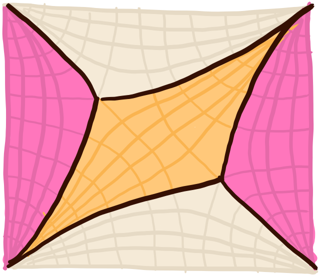




Combinatorics of punctured
half-translation surfaces

Let’s keep cutting along the critical leaves.

For a generic half-translation structure, each end of each vertical leaf falls into a puncture or hits a singularity.



Combinatorics of punctured
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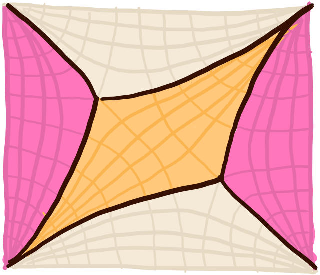
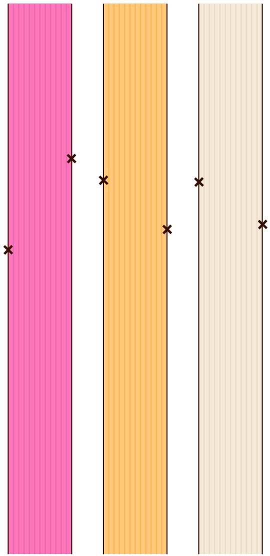
In this case, when you cut, the surface falls apart into strips.





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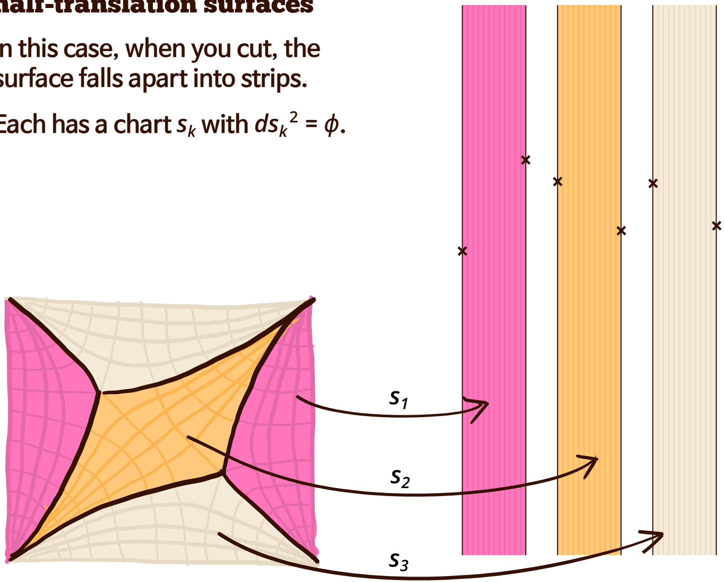
In this case, when you cut, the surface falls apart into strips.

Each has a chart s_k with $ds_k^{\,2} = \phi$.



Combinatorics of punctured
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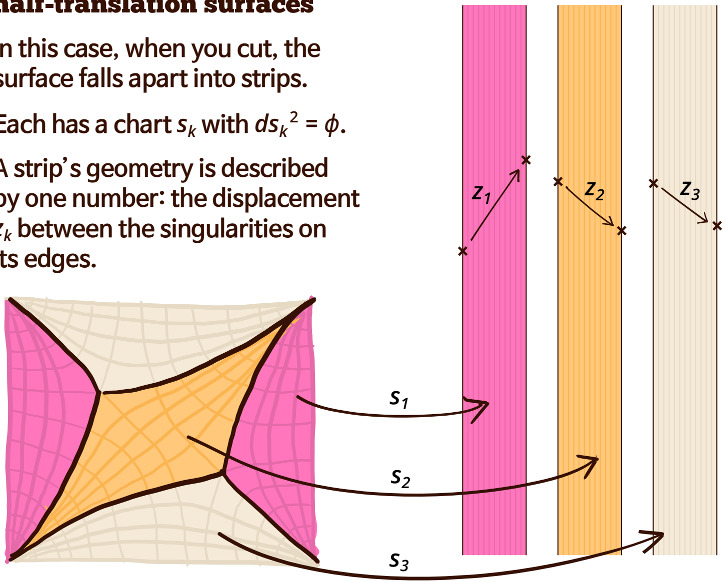
In this case, when you cut, the surface falls apart into strips.

Each has a chart s_k with $ds_k{}^2 = \phi$.

A strip's geometry is described by one number: the displacement z_k between the singularities on its edges.



Combinatorics of punctured
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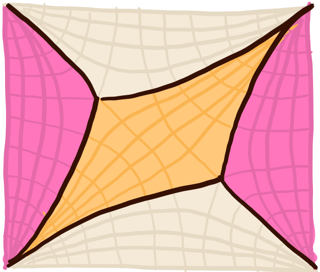
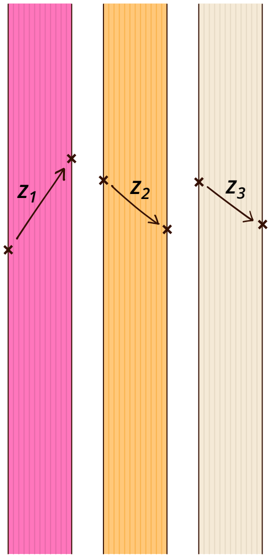
We can reconstruct the half-translation structure from:

- The displacements
  $z_1, \ldots, z_n \in (H_{right})^n$.





Combinatorics of punctured
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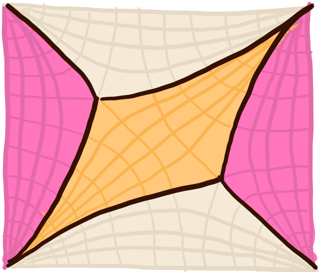
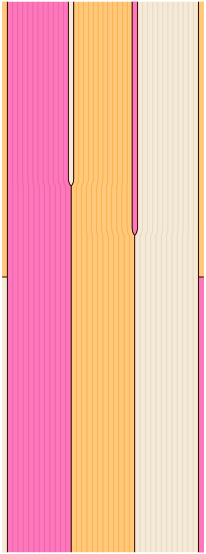
- The combinatorial data of how the strips are glued along their edges.





Combinatorics of punctured
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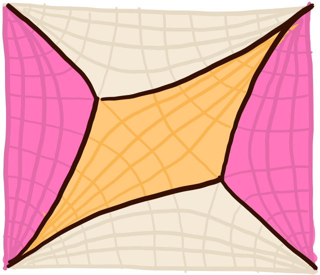
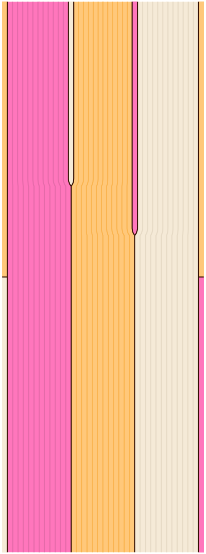
So, the space of half-translation structures is made of cells shaped like $(H_{\mathrm{right}})^n$.

They meet along their facets, where strips shrink to zero width.

Here's a path from one cell to another.





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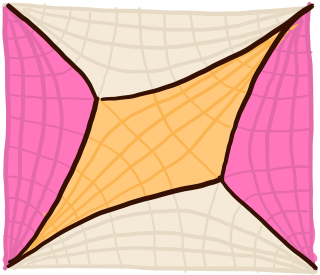

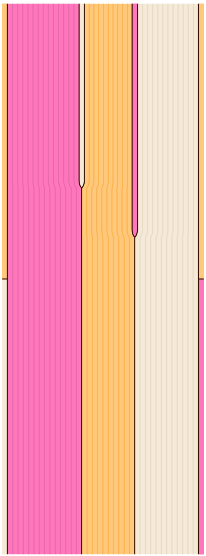

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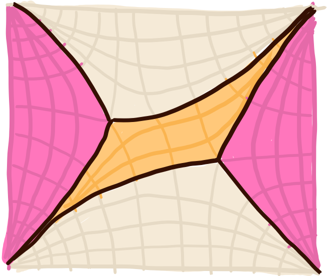

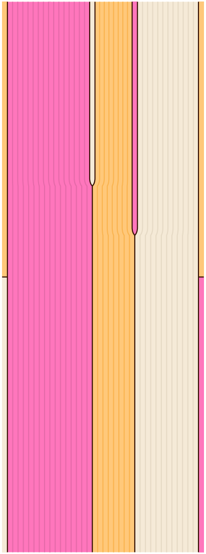

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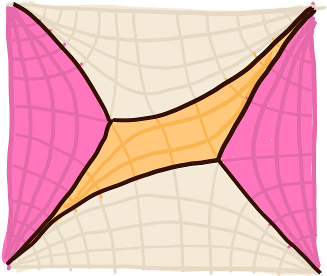

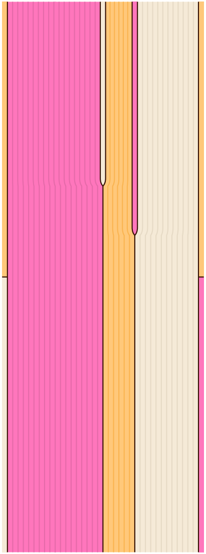

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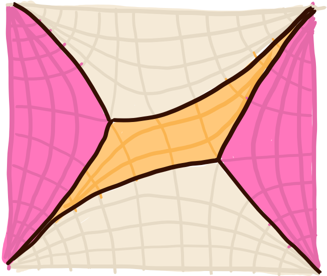

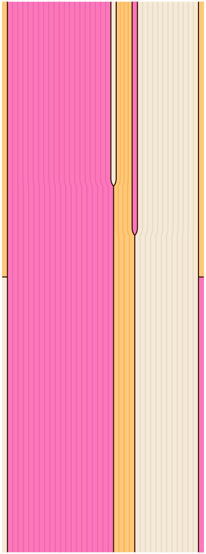

Combinatorics of punctured
half-translation surfaces

So, the space of half-translation structures is made of cells shaped like $(H_{\text{right}})^n$.

They meet along their facets, where strips shrink to zero width.

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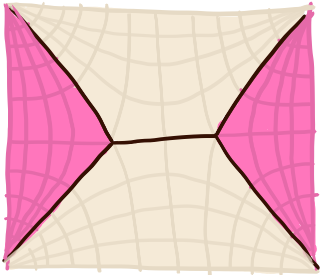

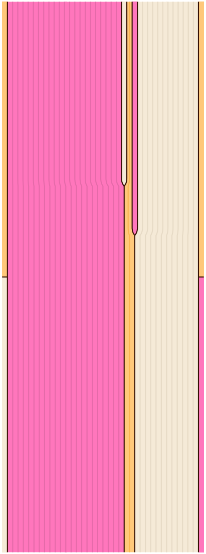

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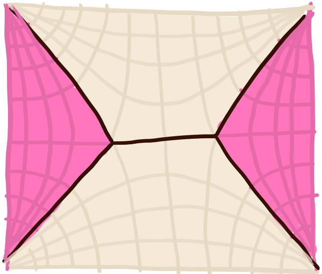

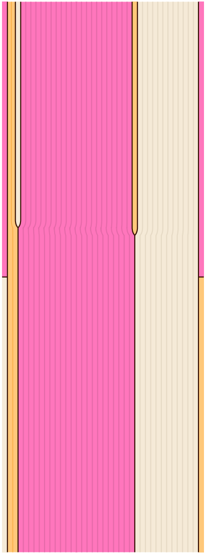

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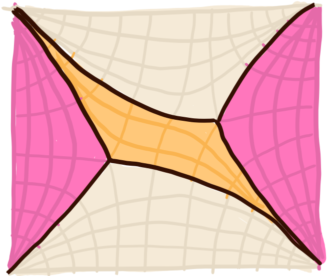

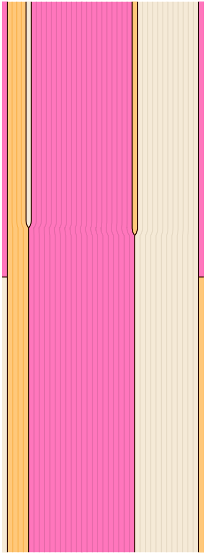

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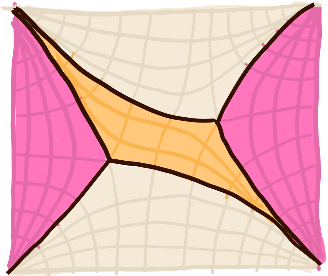

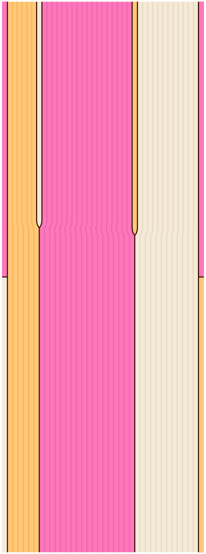

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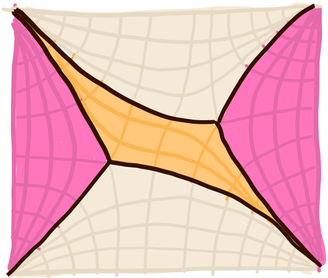

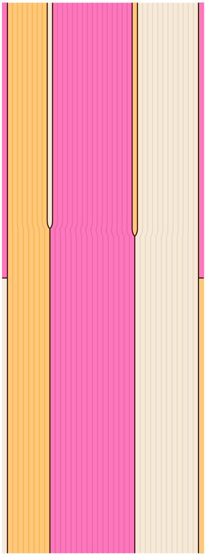

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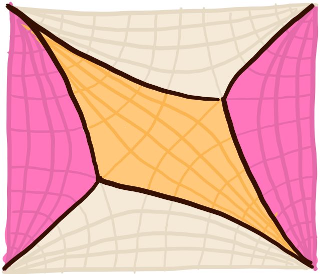

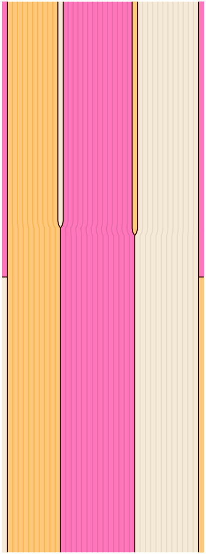

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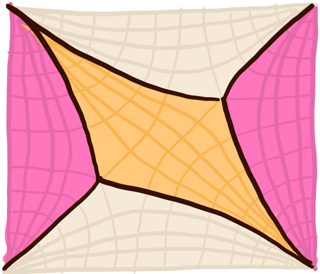
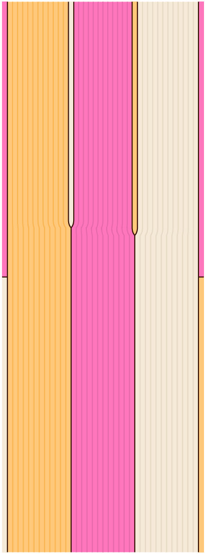
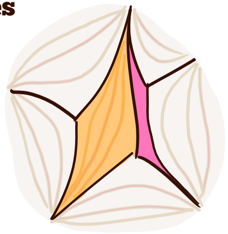
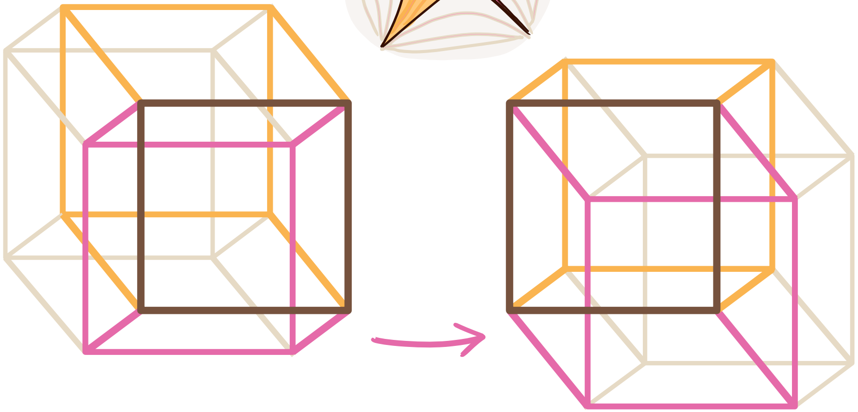




The space of punctured
half-translation structures

Example: two 4d cells
meeting at a 3d face.





The space of punctured
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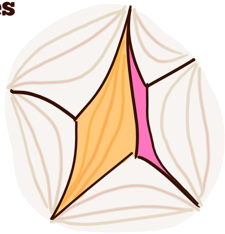
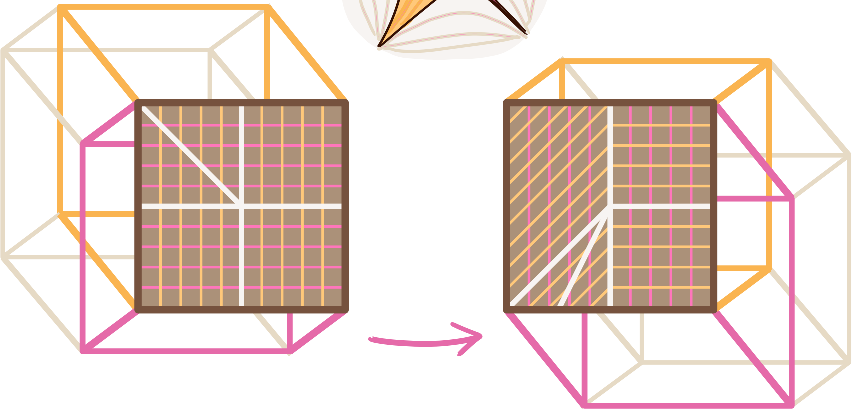
Example: two 4d cells meeting at a 3d face.

Gluing is piecewise linear.





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