The Geometry of The Night Sky
or, An Ape Pointing at The Stars

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1 The Celestial Sphere

What do you see when you look into the night sky? I see mostly stars, which appear as points of light. Why do they look that way? Neglecting gravity, the stars and I are all traveling through life along timelike lines in $(1+3)$-dimensional Minkowski spacetime. Every star intersects my current light cone twice: once in the past, when it selflessly gave off the light I'm seeing now, and once in the future, when my thank-you note (if I should send one by radio) will reach it. If the points where two stars intersect my past light cone happen to lie along the same line through my current position, I'll see those stars at the same place in the sky. Stars whose intersections lie along different lines, on the other hand, will appear in different places. Thus, choosing a point in my sky is the same as choosing a line in my light cone.

To me, the sky looks like a sphere. Why? If I were a radiant being of pure energy, it would be because I understand that the Lorentzian metric on my light cone gives a conformal structure to the set of lines in the light cone, making it conformally equivalent to the Riemann sphere. I'm not a being of pure energy, though, so I see the sky as a sphere in a different way: by pointing at a star with my finger. My outstretched arm sweeps out a plane in spacetime, which contains the line along which the star's light reached me. The plane of my arm intersects the spacelike hyperplane orthogonal to my worldline, making a line through what I call space. Being, in the end, just a simple ape, I can't help but imagine that this line points to the place where the star is right now (though this is nowhere near accurate, unless the star's worldline happens to be almost parallel to my own). Thus, instead of seeing points on my sky as lines in my light cone, I see them as lines in space—and I understand that in space, with its Euclidean inner product, the set of lines through the origin is naturally identified with the unit sphere. To me, the sky is not a complex sphere, but a Riemannian one, with a metric I can use to measure angular distances between stars.

2 Lorentz and Möbius

Some time in the past, my species went from being just a bunch of apes to being a bunch of apes on boats. This gave us a newer, more practical appreciation for the night sky, and with it a desire to describe the sky more precisely than we could just by pointing at it. Accordingly, let's imagine I'm on a boat, sailing between the stars. From my pilot's chair, I see the universe in coordinates, describing points in spacetime as linear combinations

$$te_t + xe_x + ye_y + ze_z$$

of four orthonormal vectors:

- e_t, the vector pointing forward in time along my worldline, whose orthogonal complement is what I call space,
- e_x, the space vector pointing up through my canopy,
- e_y, the space vector pointing out to my right, and
- e_z, the space vector pointing forward along the prow of my ship.

When I point at a star whose worldline intersects my past light cone at

$$te_t + xe_x + ye_y + ze_z,$$

the plane of my arm intersects my slice of space along the direction

$$xe_x + ye_y + ze_z.$$

To avoid craning my neck, I look at the sky through a monitor that casts the unit sphere onto a flat screen by stereographic projection from the pole $-e_z$ onto a complex plane through the equator. The correspondence between the monitor coordinate ζ and the sky coordinates x, y, z is given by

$$\zeta = \frac{ix + y}{1 + z} \qquad\qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{1 + |\zeta|^2} \begin{bmatrix} 2\,\mathrm{Im}\,\zeta \\ 2\,\mathrm{Re}\,\zeta \\ 1 - |\zeta|^2 \end{bmatrix}.$$

I have no direct control over my ship's position in spacetime; all I can do is sit back and watch as the flow of time carries it to a new point on its worldline. Using my ship's thrusters and reaction wheels, however, I can adjust its worldline and spatial orientation. Such maneuvers can be composed by doing one after the other, and any maneuver can be undone by carrying it out in reverse. Thus, change-of-heading maneuvers form a group, the Lorentz group.

With every change of heading, the view on the monitor changes as well, giving a map from the Lorentz group to some set of transformations of the complex plane. Composing changes of heading composes the corresponding changes of scene, so this map is a homomorphism, and its image is a group. But what group?

If I rotate my ship, the sky seems to rotate around me, shifting the scene on the monitor by a Möbius transformation. The rigid rotations of the sphere can be distinguished among all angle-preserving maps by the fact that they send great circles to great circles, and the great circles are the ones that intersect the equator at two or more antipodal points. Thus, a Möbius transformation of the scene on my monitor can be realized by a rotation of the ship if and only if it sends circles that intersect the unit circle at two or more opposite points to circles of the same kind.

If I fire a brief burst from the ship's engines, the orthonormal vectors e'_t, e'_x, e'_y, e'_z describing my heading after the boost will be related to the ones from before by equations of the form

$$\begin{aligned} e'_t + e'_z &= \lambda(e_t + e_z) \\ e'_t - e'_z &= \tfrac{1}{\lambda}(e_t - e_z) \\ e'_x &= e_x \\ e'_y &= e_y, \end{aligned}$$

where λ is a positive constant determined by the strength and direction of the boost. Firing the rear-facing main engines will deflect my worldline forward, in the e_z direction, so λ will be greater than one. Firing the front-facing braking thrusters, on the other hand, will result in λ less than one.

How is the scene on the monitor affected by a boost? A flash of light emitted at the point

$$-(1+|\zeta|^2)e_t + (2\operatorname{Im}\zeta)e_x + (2\operatorname{Re}\zeta)e_y + (1-|\zeta|^2)e_z$$

will show up on my monitor at the point with coordinate ζ. For convenience, let's rewrite the point above as

$$-|\zeta|^2(e_t+e_z) - (e_t - e_z) + (2\operatorname{Im}\zeta)e_x + (2\operatorname{Re}\zeta)e_y.$$

In my new frame of reference after the boost, this point will be expressed as

$$-|\zeta|^2\tfrac{1}{\lambda}(e'_t+e'_z) - \lambda(e'_t - e'_z) + (2\operatorname{Im}\zeta)e'_x + (2\operatorname{Re}\zeta)e'_y.$$

A flash at this point will appear in the same place in the sky as one at the point

$$-\left|\tfrac{1}{\lambda}\zeta\right|^2(e'_t+e'_z) - (e'_t - e'_z) + \left(2\operatorname{Im}\tfrac{1}{\lambda}\zeta\right)e'_x + \left(2\operatorname{Re}\tfrac{1}{\lambda}\zeta\right)e'_y,$$

which can be rewritten as

$$-\left(1+\left|\tfrac{1}{\lambda}\zeta\right|^2\right)e'_t + \left(2\operatorname{Im}\tfrac{1}{\lambda}\zeta\right)e'_x + \left(2\operatorname{Re}\tfrac{1}{\lambda}\zeta\right)e'_y + \left(1-\left|\tfrac{1}{\lambda}\zeta\right|^2\right)e'_z,$$

corresponding to the point $\frac{1}{\lambda}\zeta$ on the monitor. A star shining on my monitor at coordinate ζ before the boost will therefore appear at coordinate $\frac{1}{\lambda}\zeta$ after the boost. In other words, a boost of strength λ shifts the scene on my monitor by the Möbius transformation $\zeta \mapsto \frac{1}{\lambda}\zeta$.

Any change of heading can be realized through a combination of rotations and boosts along the engine axis, so now we have an answer to the question I asked earlier: the correspondence between changes of heading and changes of scene is a homomorphism from the Lorentz group into the Möbius group. In fact, it's not hard to check that any Möbius transformation of the scene on the monitor can be realized through a combination of rotations and engine-axis boosts, so this homomorphism is surjective.

My goal as a navigator, however, is not to arrange the stars in pretty patterns around me. I'm a busy ape with places to be, and I want to be sure my ship is pointed in the right direction. Can I accurately control my heading by moving the stars to prescribed positions on the monitor? In other words, is the homomorphism from the Lorentz group to the Möbius group injective?

Any two rotations can be distinguished by their effects on the view on the monitor, and any two engine-axis boosts can be distinguished in the same way, so the homomorphism is injective when restricted to either of those subgroups. Now, consider an arbitrary change of heading. Using rotations and engine-axis boosts, this maneuver can be accomplished in two steps. First, send e_t wherever it ought to go by performing a rotation followed by an engine-axis boost. Then do another rotation to move the three space vectors into place. The Möbius transformations corresponding to rotations send great circles to great circles, while the ones corresponding to non-identity engine-axis boosts never do, so the Möbius transformation given by our change of heading can only be the identity if the boost part is the identity. That makes the maneuver a pure rotation, and we already know the homomorphism is injective on those. Hence, the homomorphism is injective everywhere, giving an isomorphism between the Lorentz group and the Möbius group.