Potentially cluster-like coordinates from dense spectral networks

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Old coordinates

The space of twisted $\mathrm{SL}_2\,\mathbb{R}$ local systems (think hyperbolic structures) on a punctured surface has a cluster coordinate system, first described by Fock and Goncharov.

Charts are parameterized **discretely** by ideal triangulations of the surface. They can also be parameterized by spectral networks, which in this context are the star-triangle duals of ideal triangulations.

When you deform a spectral network, its topology can change in discrete “flips,” equivalent to flipping edges of an ideal triangulation. Flipping a spectral network makes the associated coordinates **jump** by a cluster transformation.

New coordinates

Gaiotto, Moore, and Neitzke found a new description of the Fock-Goncharov coordinates, and realized that it should extend to surfaces without punctures.

Charts are parameterized **continuously** by singular half-translation structures on the surface. Choosing one of these structures produces a generalized kind of spectral network on the surface, composed of lines that fill the surface densely.

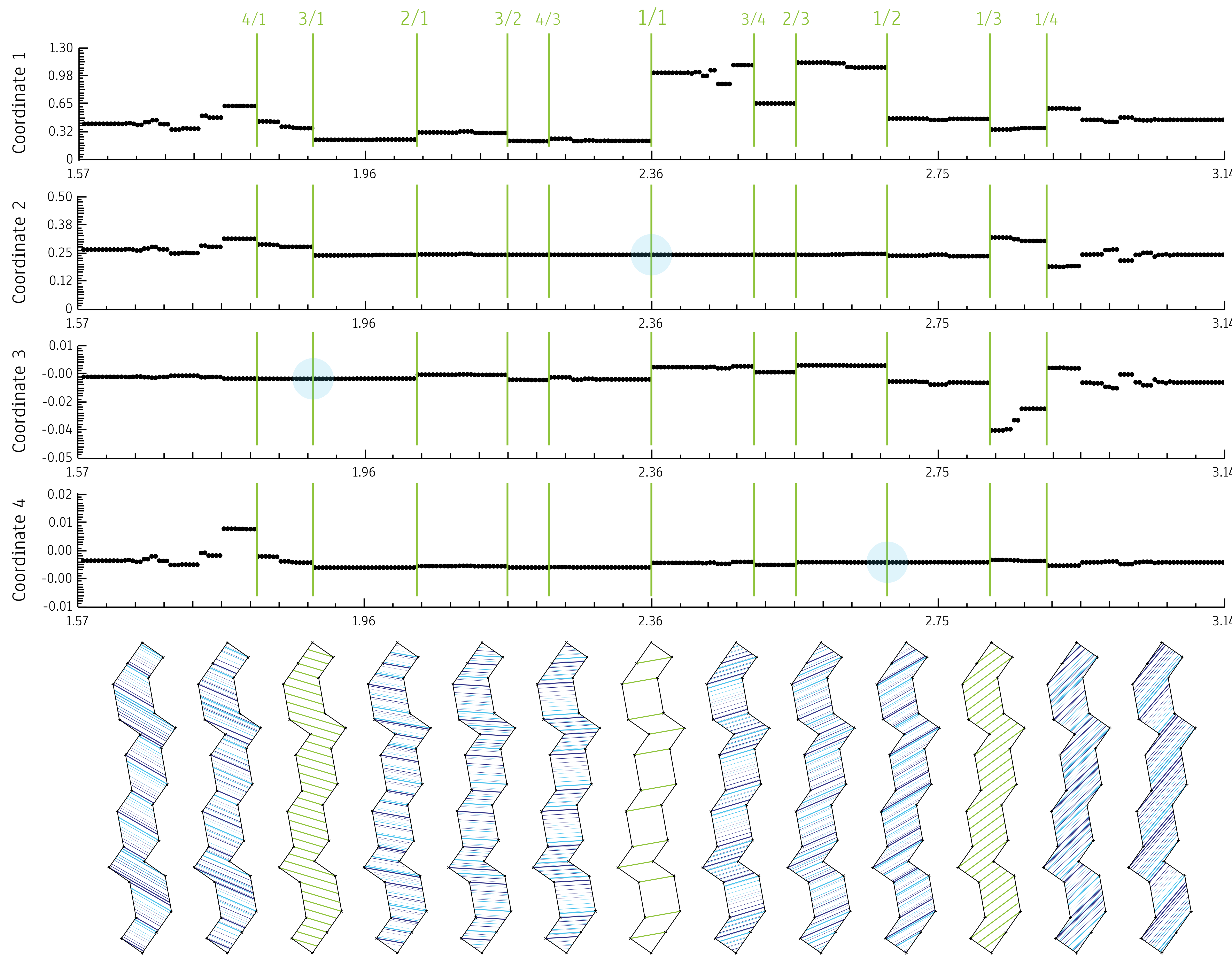
When you deform a half-translation structure, events analogous to the flips of a spectral network happen densely along the deformation path. A rough argument suggests that the associated coordinates should **jump** in the same way the Fock-Goncharov coordinates at each flip.

Related work and future directions

Just as cluster algebras capture the essential algebraic features of Fock-Goncharov coordinates, there may be some cluster-algebra-like structure that reflects the behavior of these new coordinates. One obstacle to finding the right abstraction has been the apparent absence of clusters. On a punctured surface, the cluster variables for a given spectral network are associated with the components of the complement of the spectral network. On an unpunctured surface, the complement of the spectral network is too badly behaved to make sense of this. Maybe the right generalization is some notion of a “cluster algebra without clusters.”

The coordinates described here should be essentially the same as the ones recently described by Bonahon and Dreyer for $\mathrm{SL}_K(\mathbb{R})$. However, Bonahon and Dreyer work in a different geometric setting: their charts are parameterized by geodesic laminations, rather than half-translation structures. Preliminary attempts to fill in the dictionary between their version and this one have been an interesting exercise in analogies between hyperbolic and flat geometry.

Flips and jumps



What you’re looking at

To make the plot above, I fixed an $\mathrm{SL}_2\,\mathbb{R}$ local system on a genus-two surface and computed four of the coordinates associated with a certain half-translation structure. Then I deformed the half-translation structure, by rotating it through an angle of $\pi/2$, and plotted how the values of the coordinates changed.

As the half-translation structure rotates, the generalized spectral network it produces rotates too, as shown below the plot. The surface pictured is not the original genus-two surface, but rather its “translation double cover,” where the half-translation structure lifts to a full translation structure.

Each coordinate is associated with a first homology class on the translation double cover. When two edges of the spectral network flip past each other, the coordinates that jump should be the ones that intersect the loop formed by the colliding edges, just like with Fock-Goncharov coordinates.

Coordinates 2, 3, and 4 come from homology classes that avoid the loops formed by three of the biggest flips. At the circled points on the plot, you can see each of those coordinates standing still while the other three jump.