Spectral networks craft hour

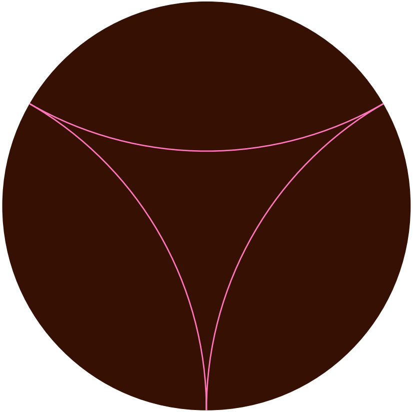
Aaron Fenyes
University of Toronto

Joint Math Meetings 2017

Build a hyperbolic surface with an ideal triangulation

Materials

- Ideal triangles cut from the hyperbolic plane.



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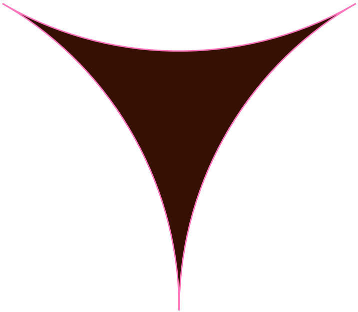

Build a hyperbolic surface with an ideal triangulation

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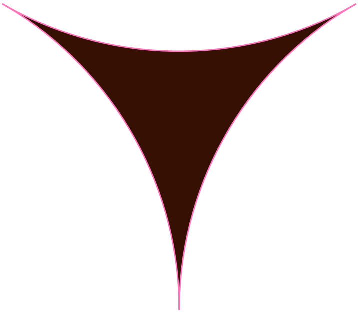


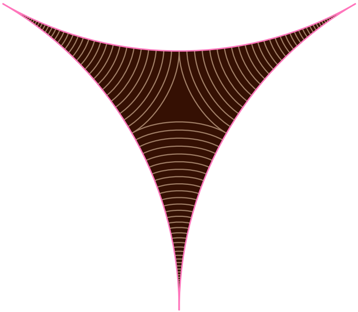
Build a hyperbolic surface with an ideal triangulation

Materials

- Ideal triangles cut from the hyperbolic plane.

Features

- Horocycle foliation.



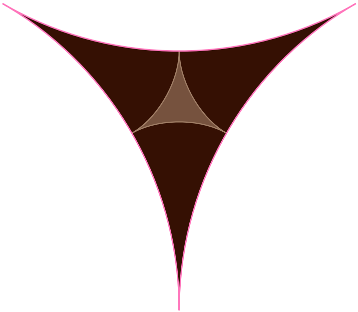
Build a hyperbolic surface with an ideal triangulation

Materials

- Ideal triangles cut from the hyperbolic plane.

Features

- Horocycle foliation.
- Contact triangle.



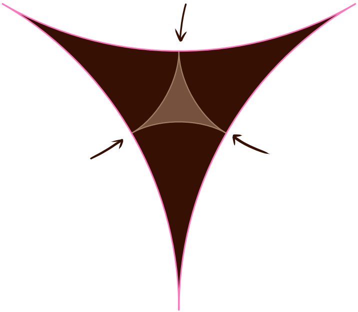
Build a hyperbolic surface with an ideal triangulation

Materials

- Ideal triangles cut from the hyperbolic plane.

Features

- Horocycle foliation.
- Contact triangle.
- Contact points.



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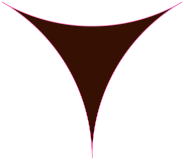
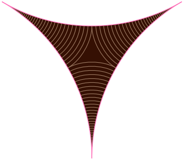
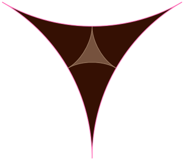


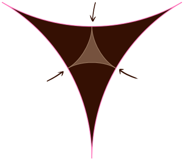
ideal triangle



horocycle foliation

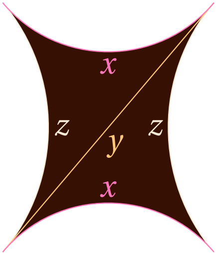


contact triangle



contact points

Build a hyperbolic surface with an ideal triangulation



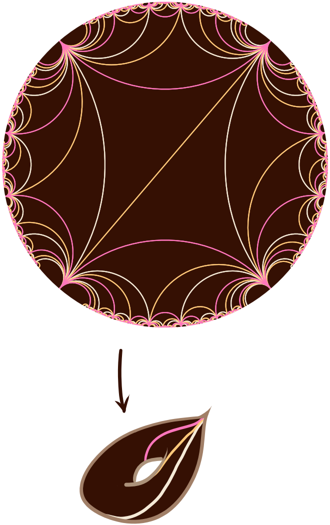
Instructions

- Take a bunch of ideal triangles.
- Glue them along their edges.
- Stop when you run out of edges.

Build a hyperbolic surface with an ideal triangulation

Results

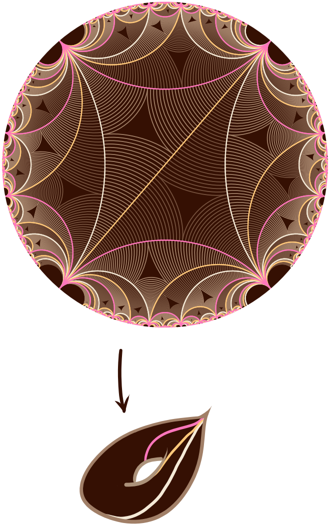
- A hyperbolic surface S with cusps.



Build a hyperbolic surface with an ideal triangulation

Results

- A hyperbolic surface S with cusps.
- Signed distances between contact points, called *shear parameters*, describe hyperbolic structure.



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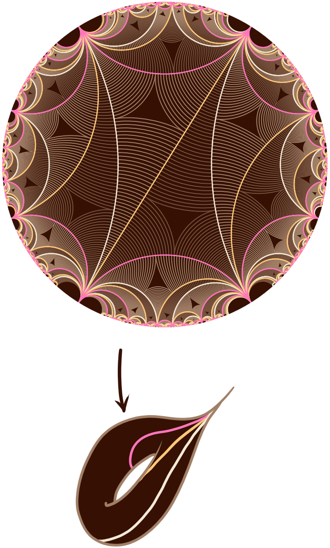

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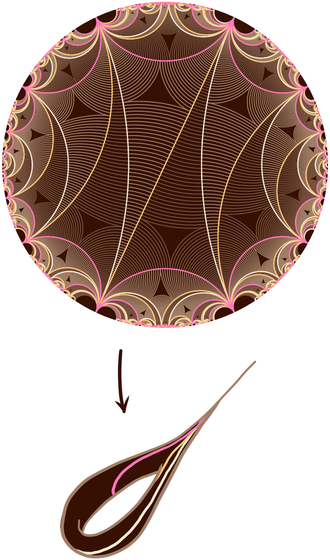

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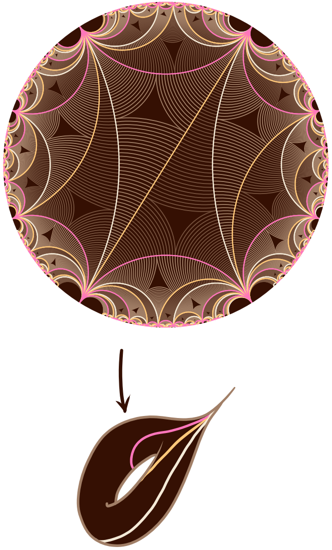

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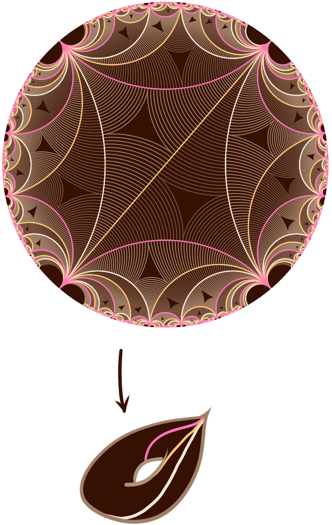
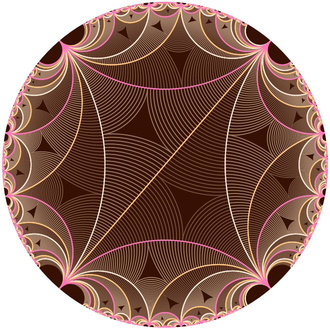
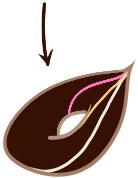


Build a hyperbolic surface with an ideal triangulation

Results

- A hyperbolic surface S with cusps.
- Signed distances between contact points, called *shear parameters*, describe hyperbolic structure.
- A representation $\pi_1 S \to \operatorname{Isom} \mathbb{H}^2$, called the *holonomy representation*, says how charts continue around loops.



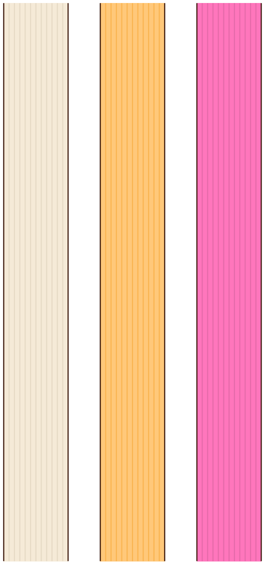


Deflate a hyperbolic surface to a half-translation surface

Like Gupta does in “Asymptoticity of grafting and Teichmüller rays.”

Materials

- Vertical strips cut from $\mathbb{R}^2$.
- One for each edge of the ideal triangulation.



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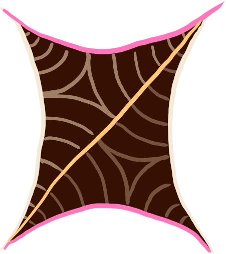
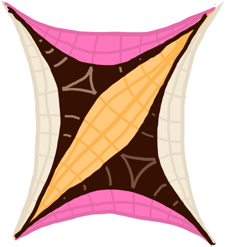
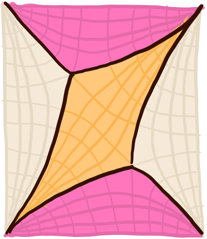


Cut the surface into triangles again.



Glue in a strip along each edge.

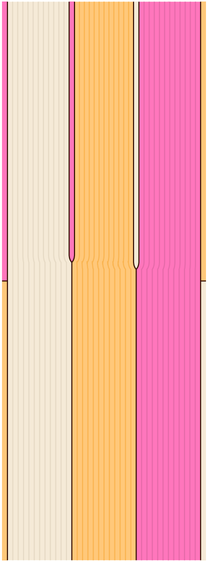


Collapse each triangle along its horocycle foliation, leaving the strips glued along a *spectral network*.

Deflate a hyperbolic surface to a half-translation surface

Results

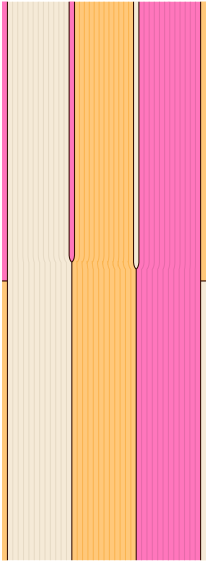
- A singular flat surface $\hat{S}$ with cylindrical ends.
- Signed distances between singularities encode the shear parameters.
- The holonomy representation $\pi_1 \hat{S} \to \operatorname{Isom} \mathbb{R}^2$ lands in the "nearly abelian" subgroup generated by translations and half-rotations.



Deflate a hyperbolic surface to a half-translation surface

Geometric interpretation

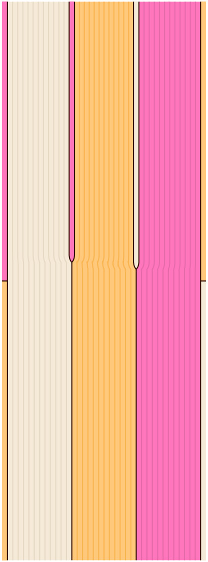
- Deflating turns hyperbolic surfaces into *half-translation surfaces*: singular flat surfaces with transition maps composed of translations and half-rotations.
- It's a geometric expression of the *shear parameterization* of the space of hyperbolic structures.



Deflate a hyperbolic surface to a half-translation surface

Algebraic interpretation

- Deflating turns nonabelian representations $\pi_1 S \to \operatorname{Isom} \mathbb{H}^2$ into nearly abelian representations $\pi_1 \hat{S} \to \mathbb{R}^2 \rtimes \mathbb{Z}/(2)$.
- It's a hands-on construction of Fock and Goncharov's parameterization of the space of representations $\pi_1 S \to \operatorname{Isom} \mathbb{H}^2$. *(See Gaiotto, Moore, and Neitzke's "Spectral networks.")*



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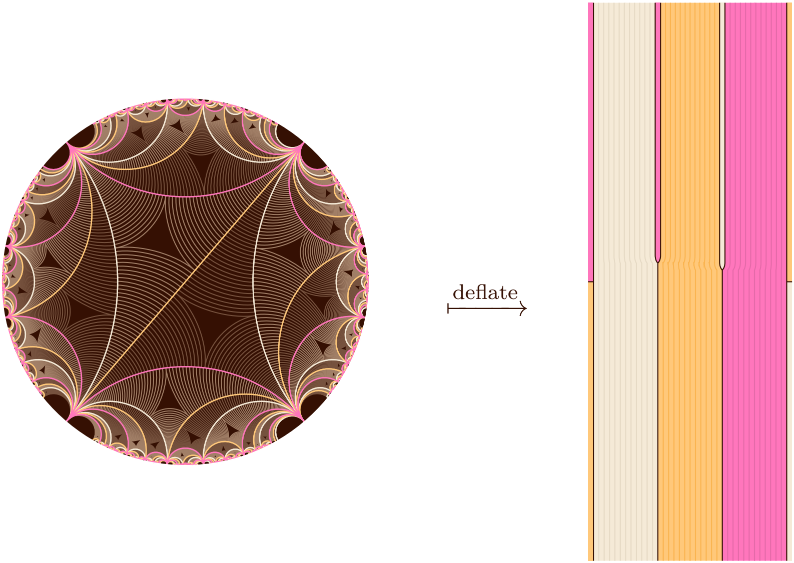

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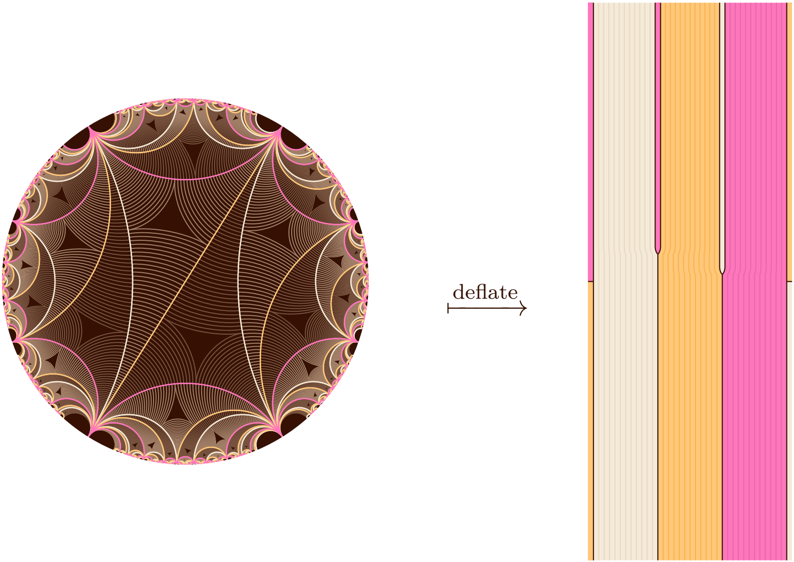

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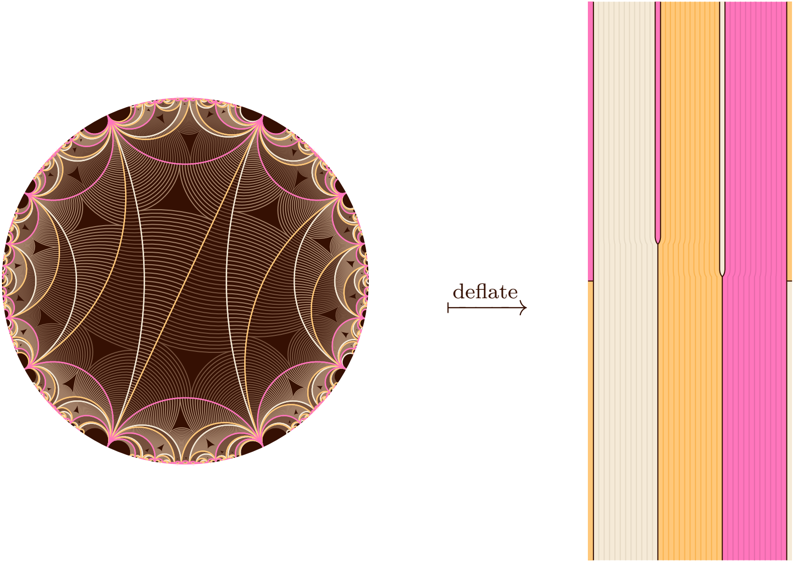

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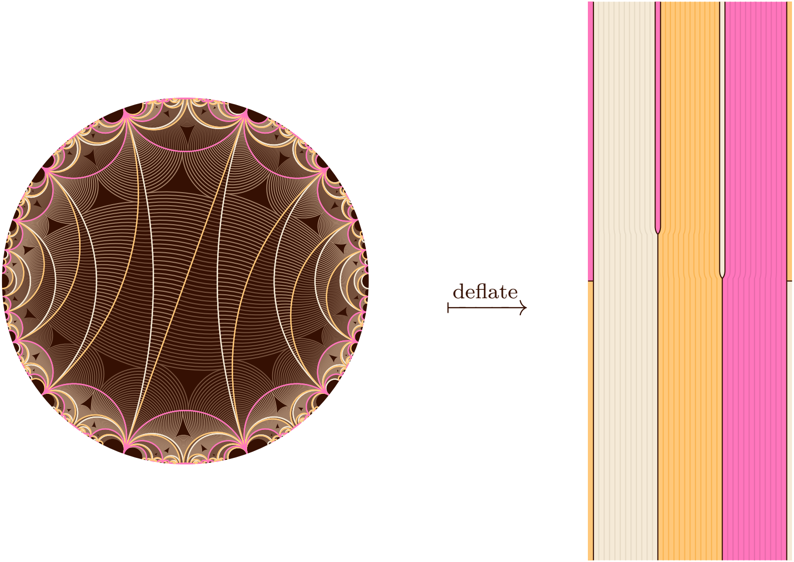

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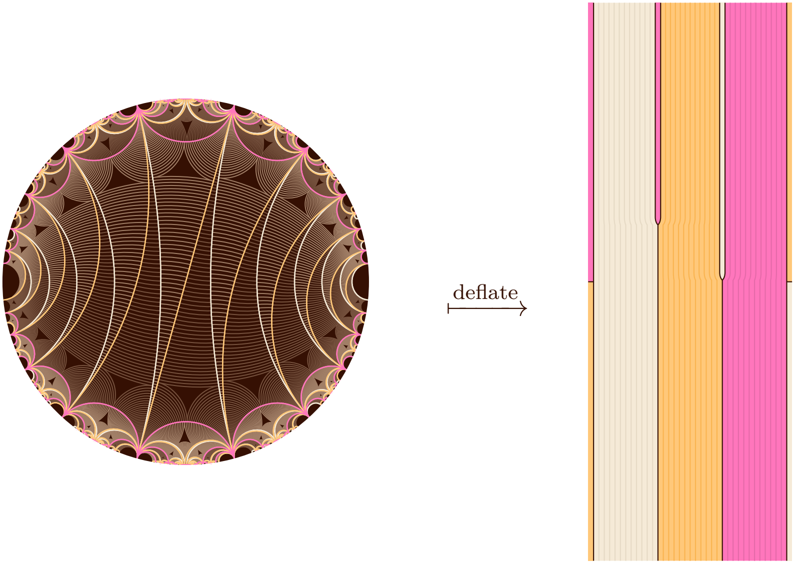

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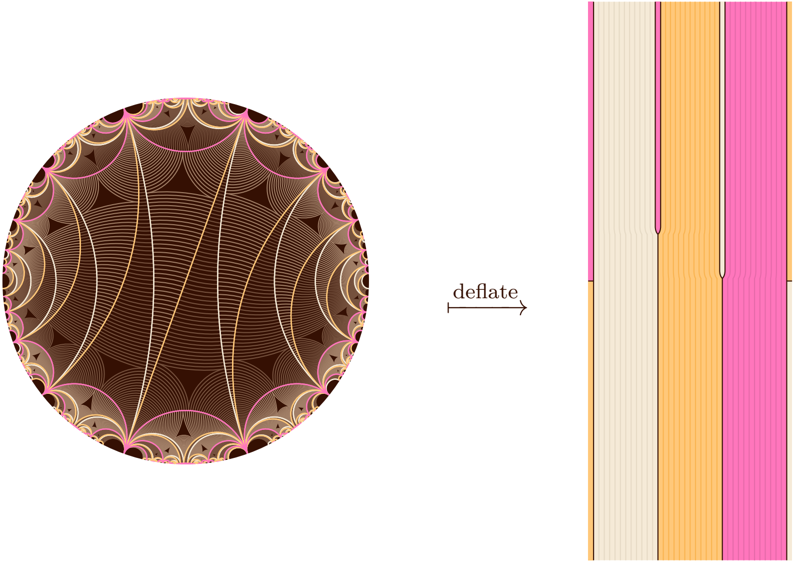

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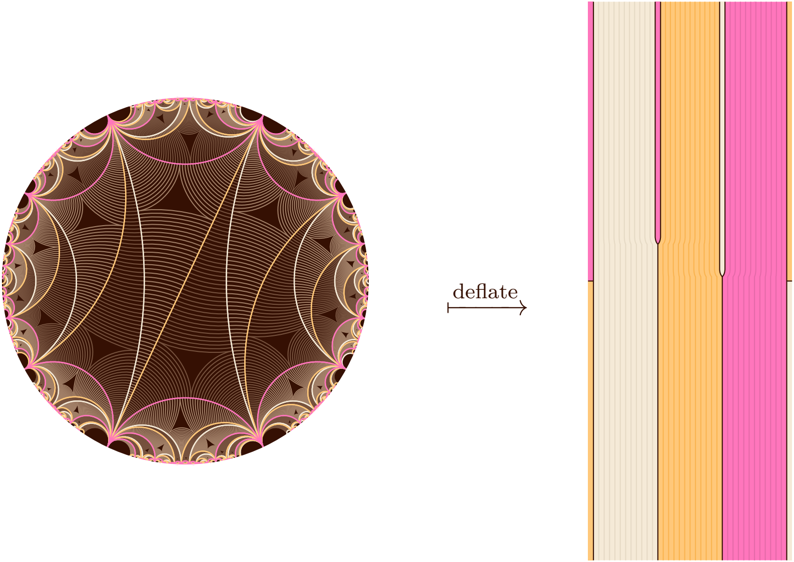

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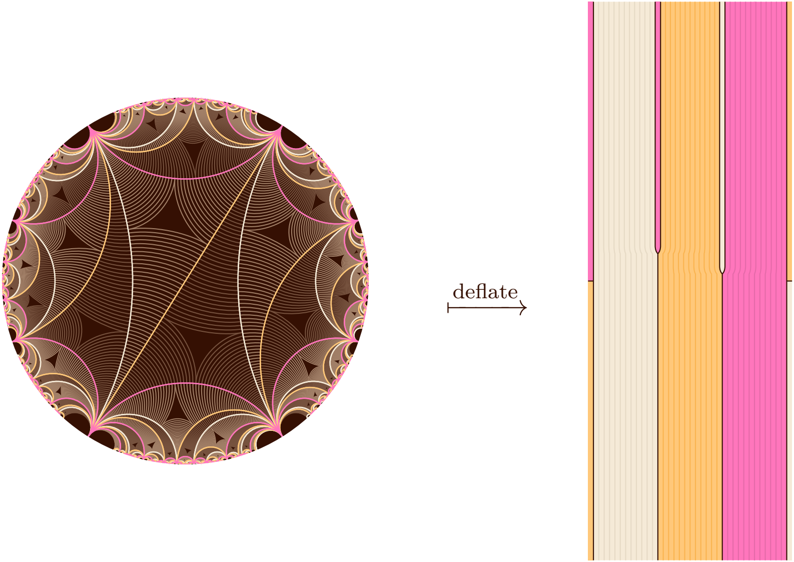

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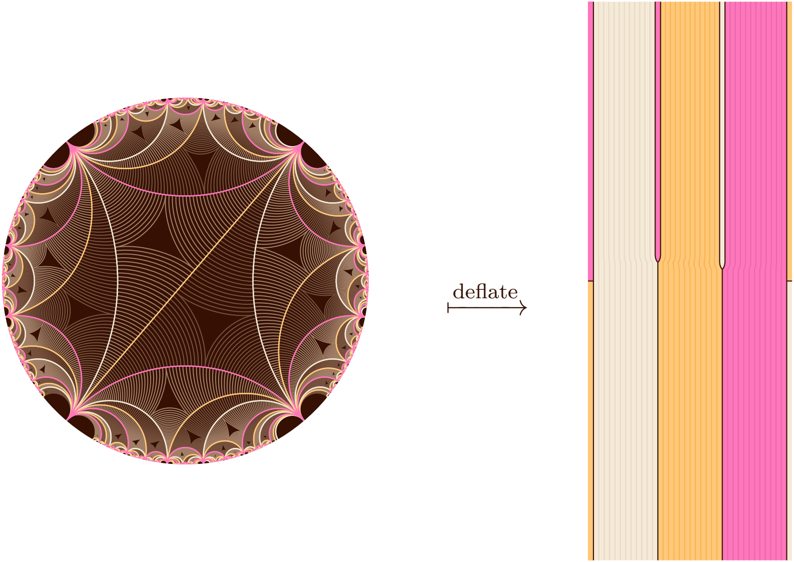
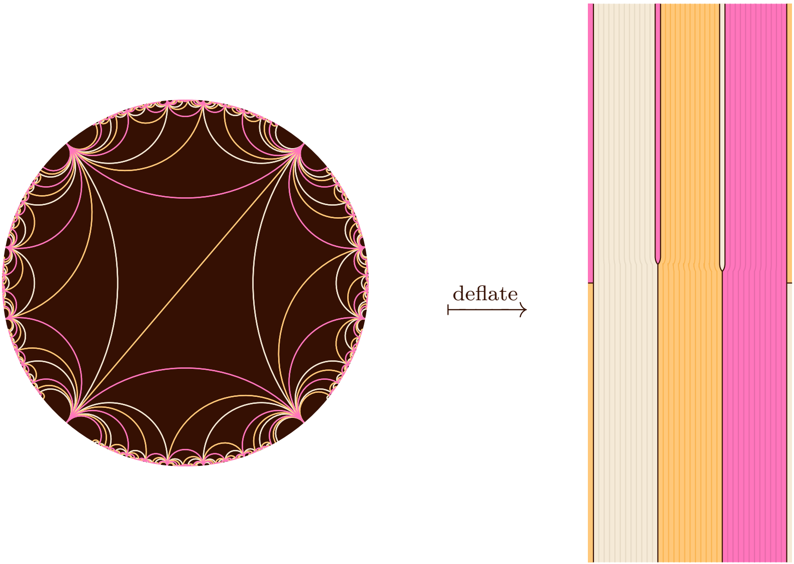


Flip an edge of an ideal triangulation



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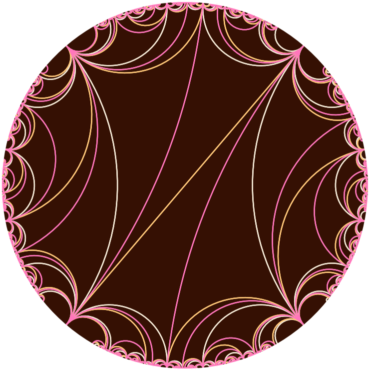

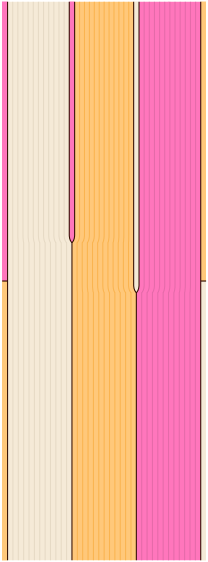

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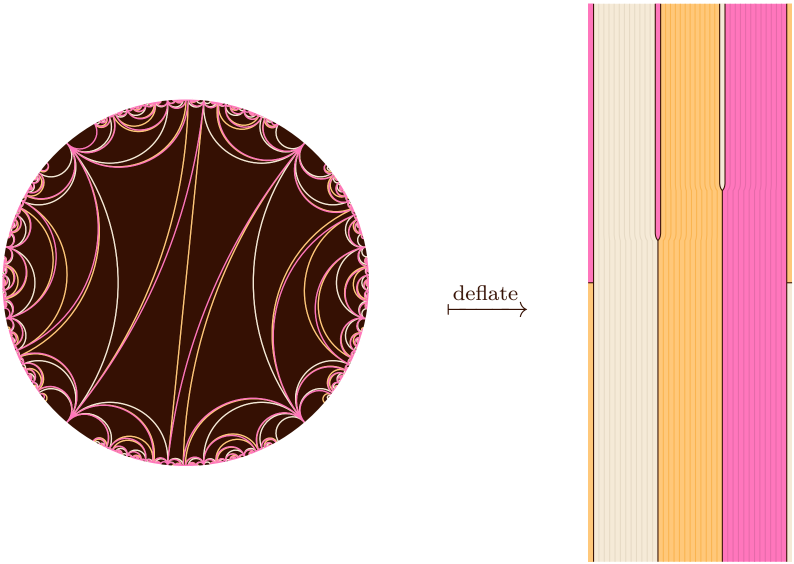

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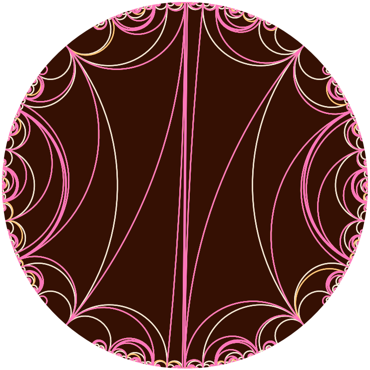
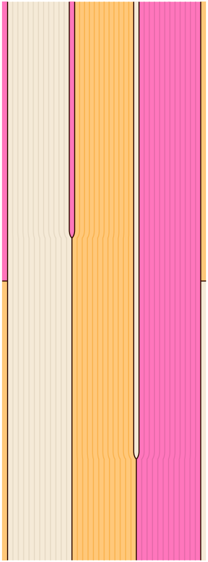




Flip an edge of an ideal triangulation

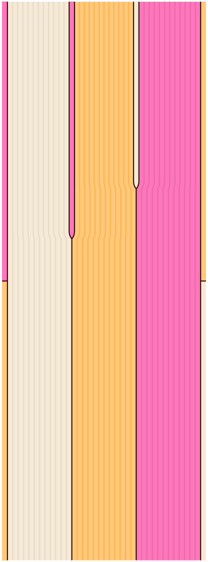
Build a hyperbolic pair of pants with a geodesic lamination

I'll do this live, before your eyes.

Flip a strip of a half-translation surface

Instructions

- Push a singularity horizontally...



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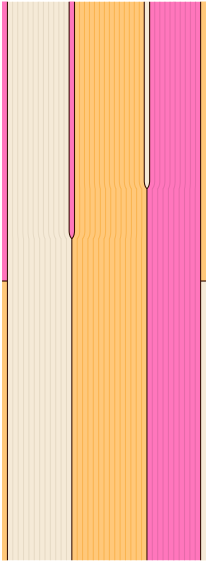

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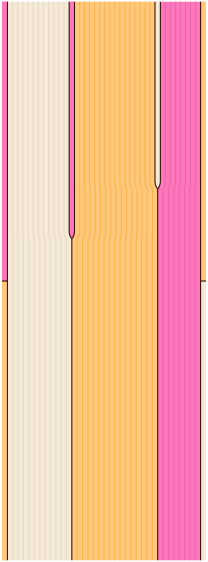

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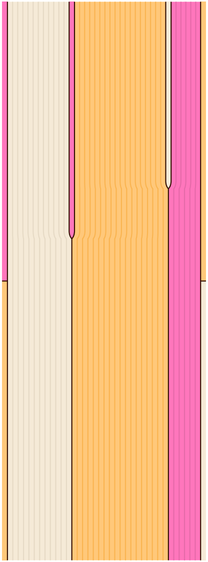

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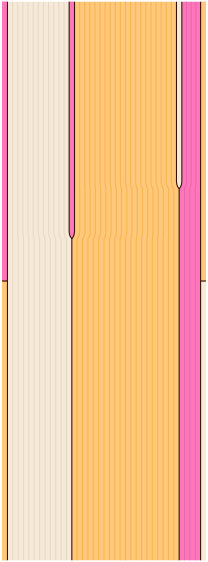
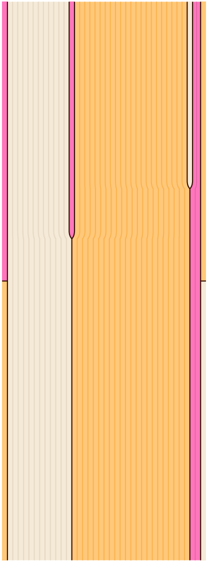


Flip a strip of a half-translation surface

Instructions

- Push a singularity horizontally...
- ... until it goes through the side of a strip. The gluing instructions change!



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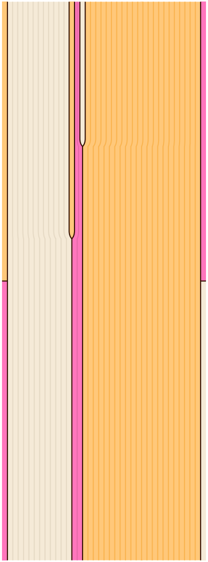

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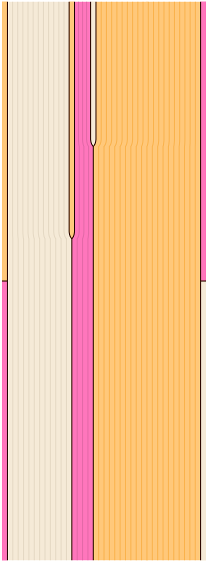

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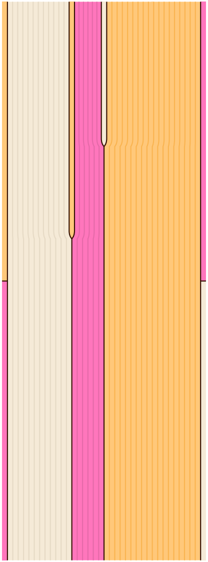

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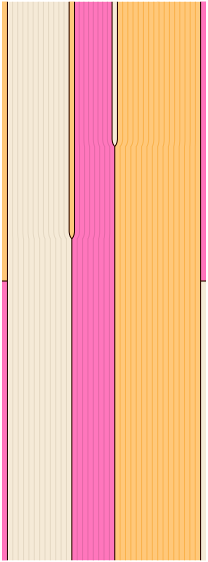

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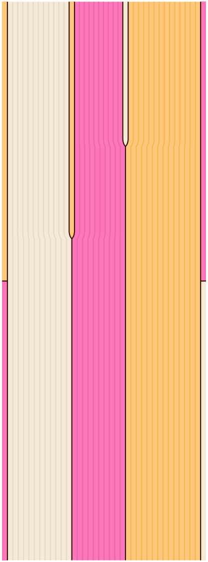
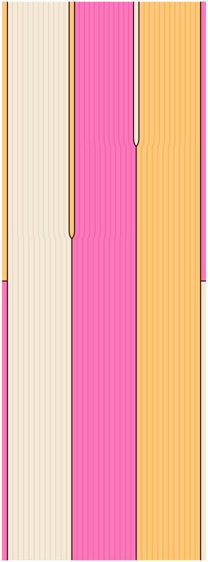


Flip a strip of a half-translation surface

Instructions

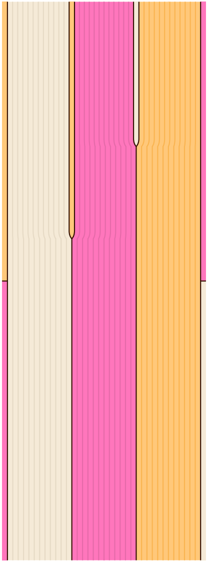
- Push a singularity horizontally...
- ... until it goes through the side of a strip. The gluing instructions change!
- At this point, strips x and y have switched places, and the singularity has moved vertically.



Flip a strip of a half-translation surface

Instructions

- Push a singularity horizontally...
- ... until it goes through the side of a strip. The gluing instructions change!
- At this point, strips x and y have switched places, and the singularity has moved vertically.
- Let's keep going.



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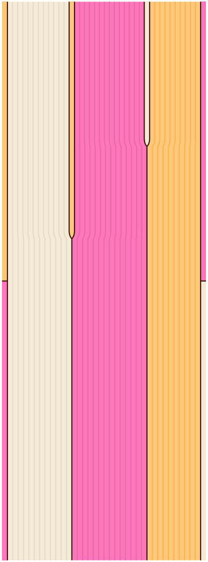

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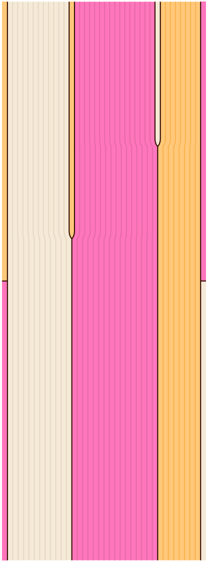

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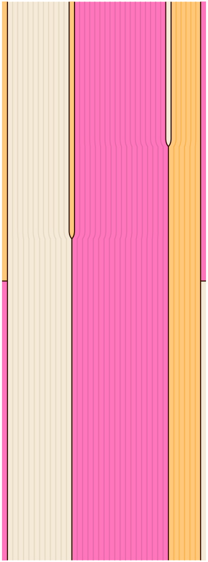

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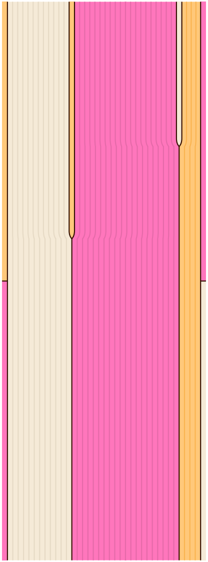

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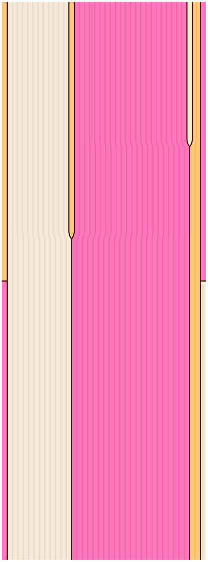

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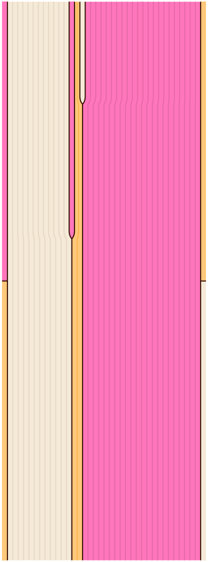
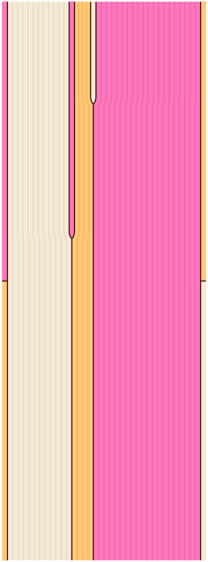
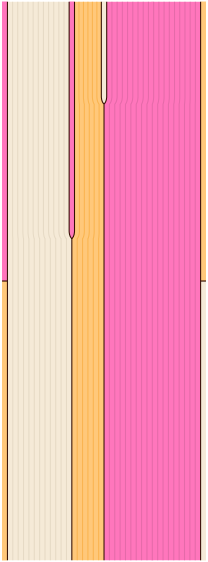


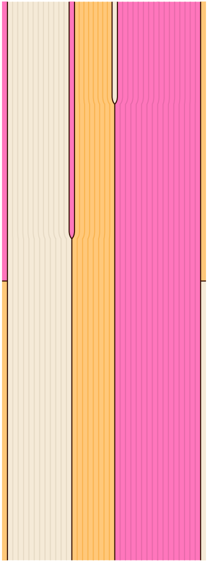
Flip a strip of a half-translation surface



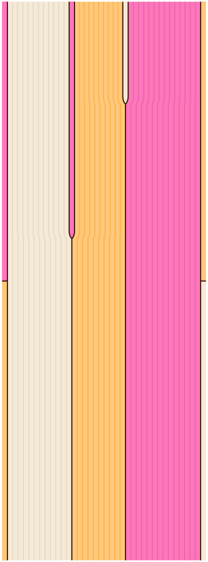
Flip a strip of a half-translation surface



Flip a strip of a half-translation surface

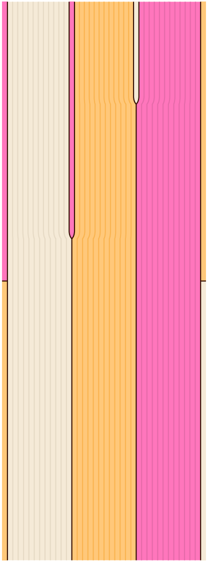


Flip a strip of a half-translation surface



Flip a strip of a half-translation surface

- Now x and y are back in their original places, and the singularity has moved vertically even further.

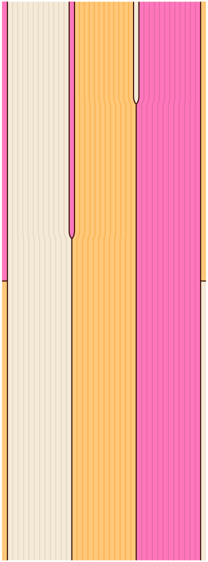


Flip a strip of a half-translation surface

- Now x and y are back in their original places, and the singularity has moved vertically even further.

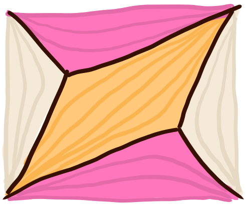
Question

- What does this process look like topologically?



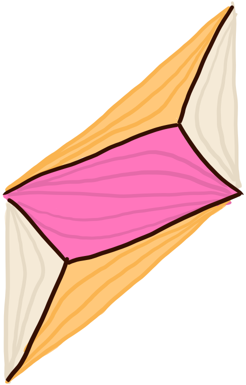
Flip a strip of a half-translation surface

- Rearrange topological picture to center strip x.



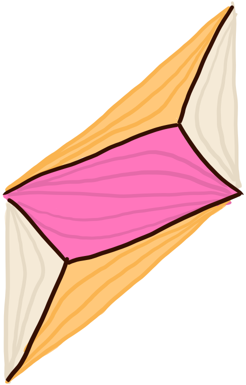
Flip a strip of a half-translation surface

- Rearrange topological picture to center strip x.



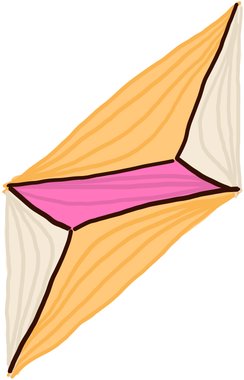
Flip a strip of a half-translation surface

- Rearrange topological picture to center strip x.
- Push the singularity.



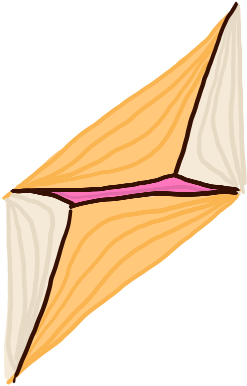
Flip a strip of a half-translation surface

- Rearrange topological picture to center strip x.
- Push the singularity.



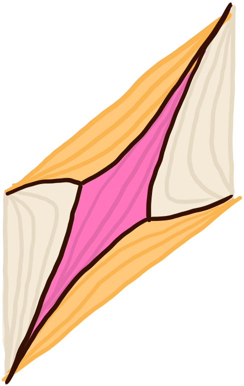
Flip a strip of a half-translation surface

- Rearrange topological picture to center strip x.
- Push the singularity.



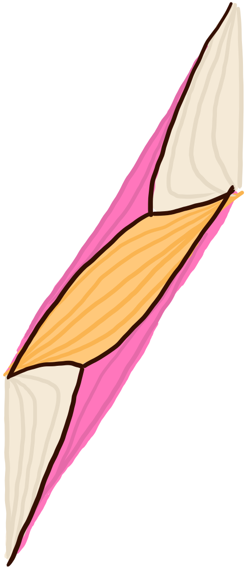
Flip a strip of a half-translation surface

- Rearrange topological picture to center strip x.
- Push the singularity.
- First flip complete.



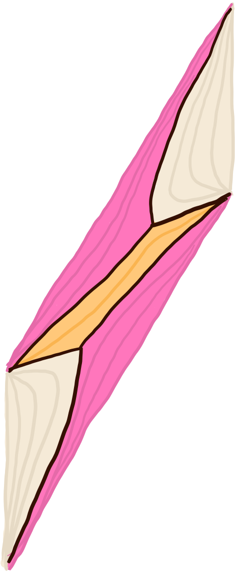
Flip a strip of a half-translation surface

- Rearrange topological picture to center strip x.
- Push the singularity.
- First flip complete.



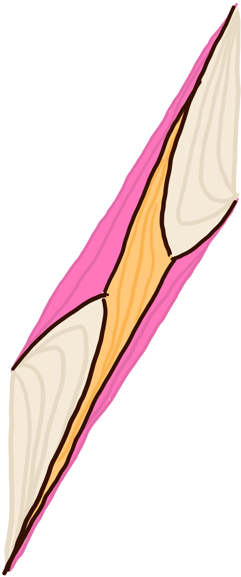
Flip a strip of a half-translation surface

- Rearrange topological picture to center strip x.
- Push the singularity.
- First flip complete.



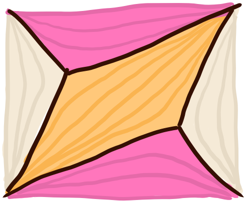
Flip a strip of a half-translation surface

- Rearrange topological picture to center strip x.
- Push the singularity.
- First flip complete.
- Second flip complete.



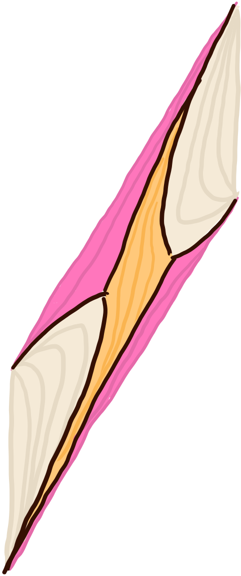
Flip a strip of a half-translation surface

- Rearrange topological picture to center strip x.
- Push the singularity.
- First flip complete.
- Second flip complete.

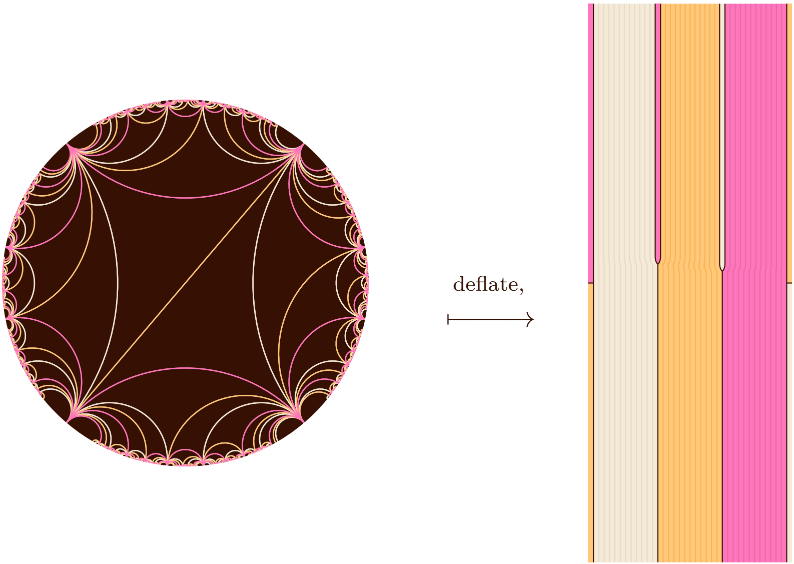


Flip a strip of a half-translation surface

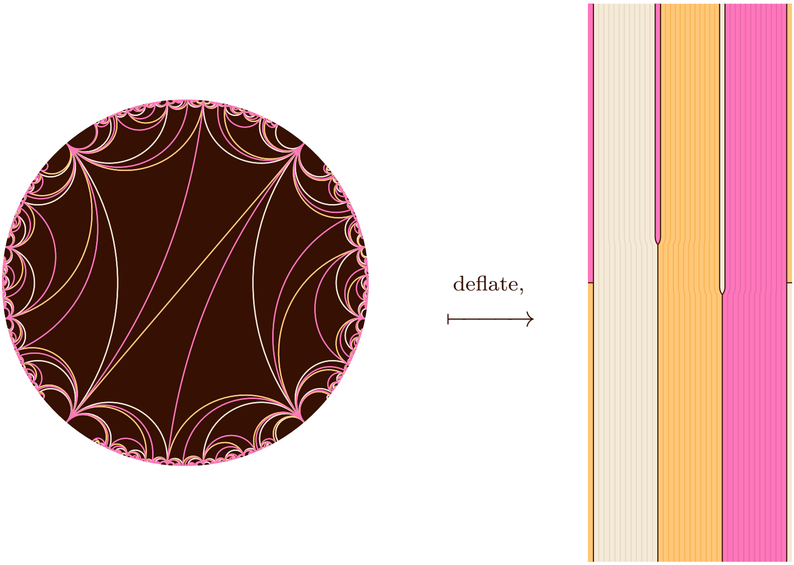
- Rearrange topological picture to center strip x.
- Push the singularity.
- First flip complete.
- Second flip complete.
- Flipping x and then y carries out a double Dehn twist, just like in the hyperbolic version!



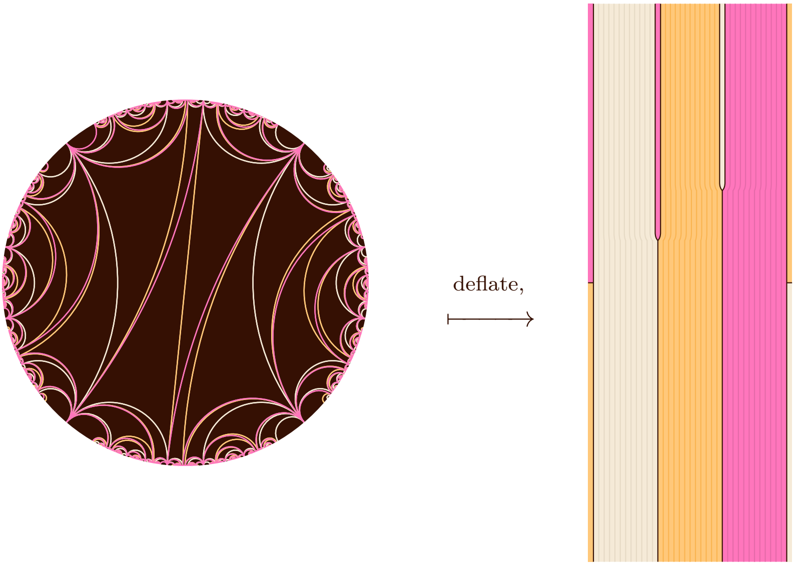
Pit edge flips against strip flips



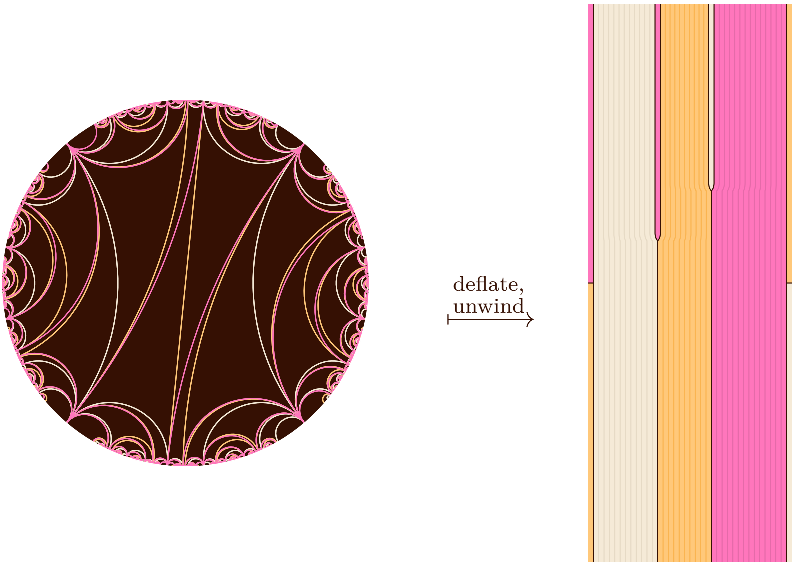
Pit edge flips against strip flips



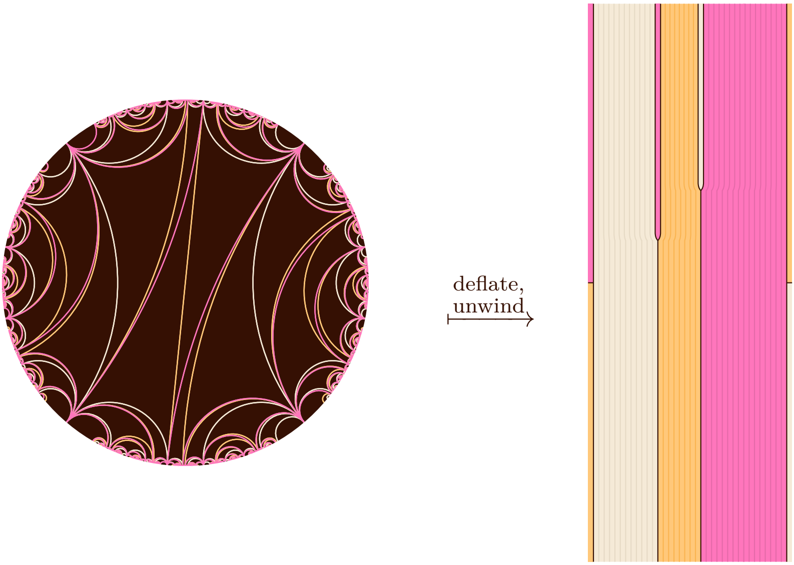
Pit edge flips against strip flips



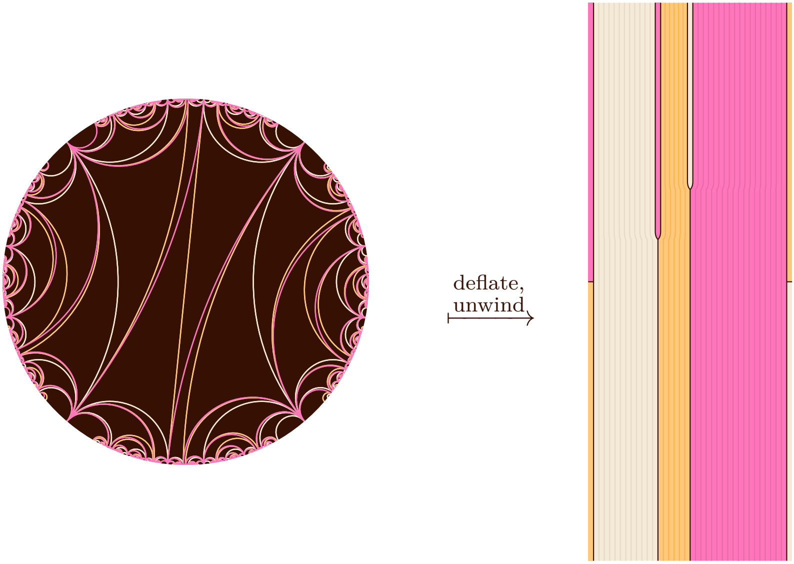
Pit edge flips against strip flips



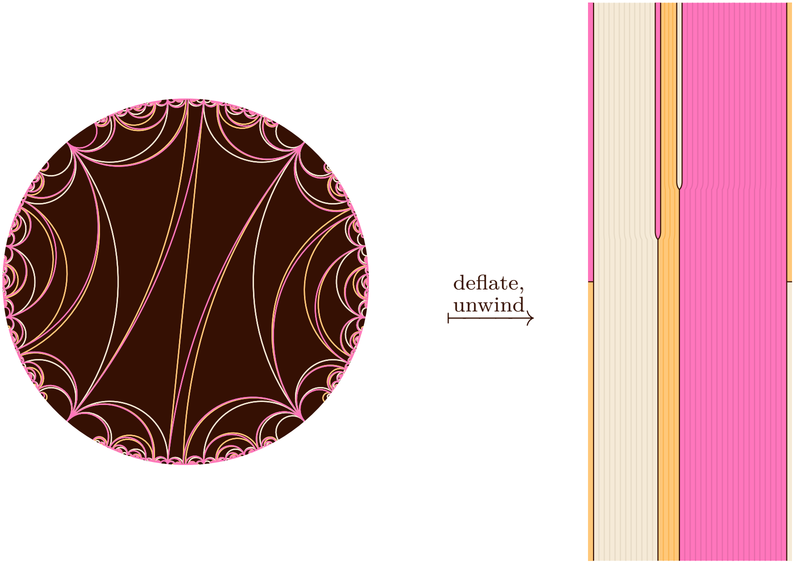
Pit edge flips against strip flips



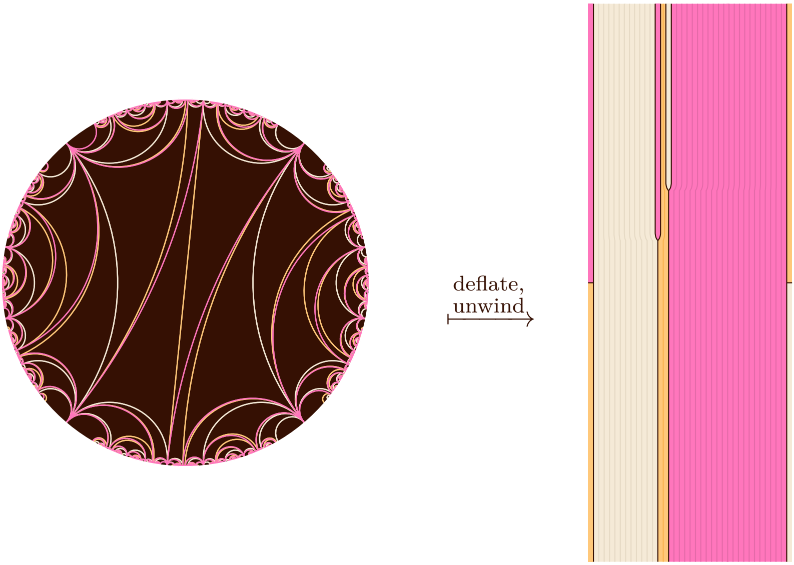
Pit edge flips against strip flips



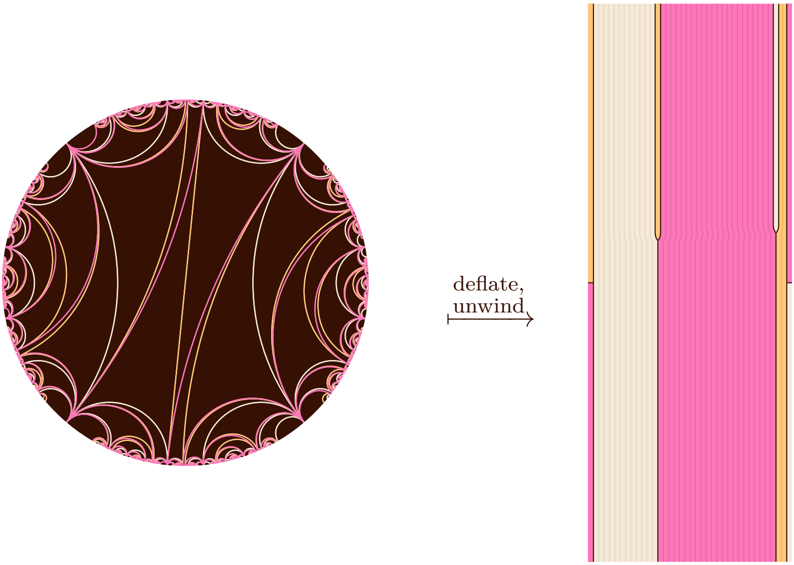
Pit edge flips against strip flips



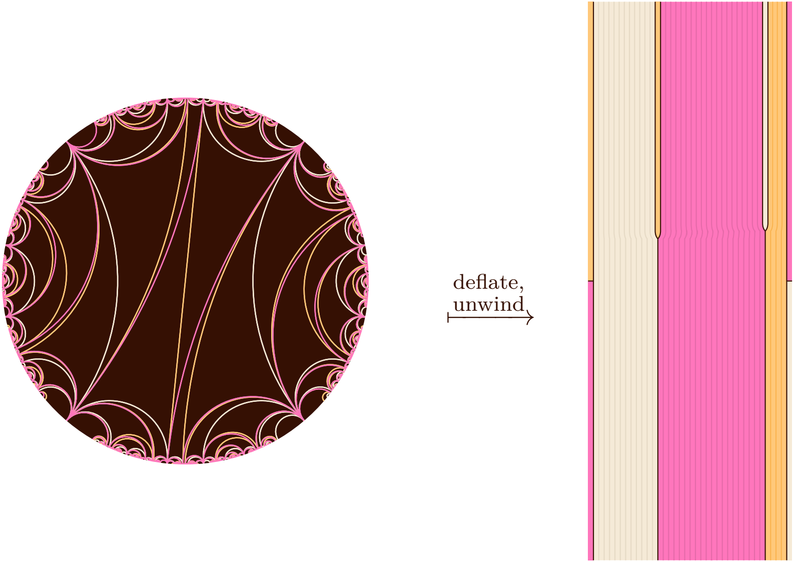
Pit edge flips against strip flips



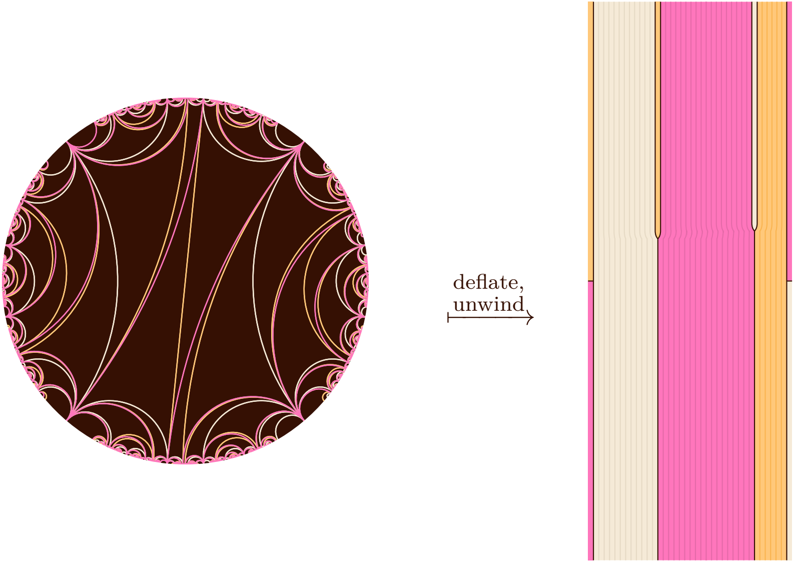
Pit edge flips against strip flips



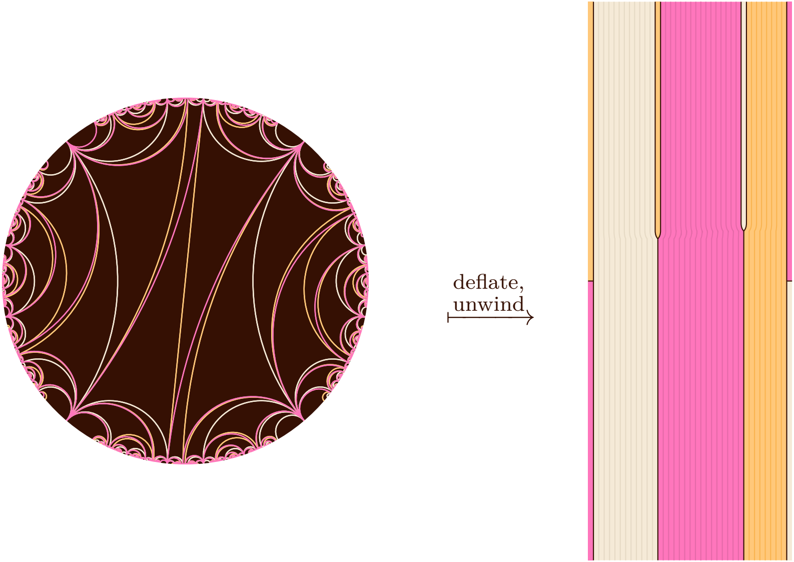
Pit edge flips against strip flips



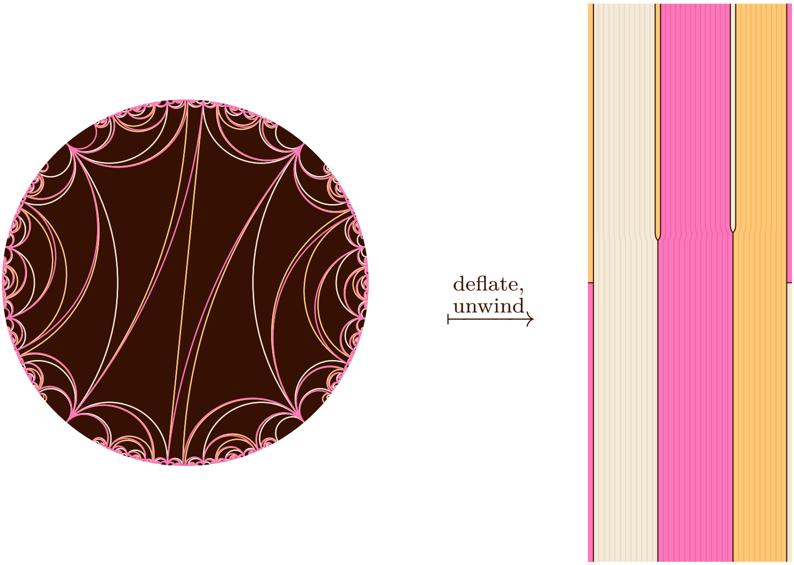
Pit edge flips against strip flips



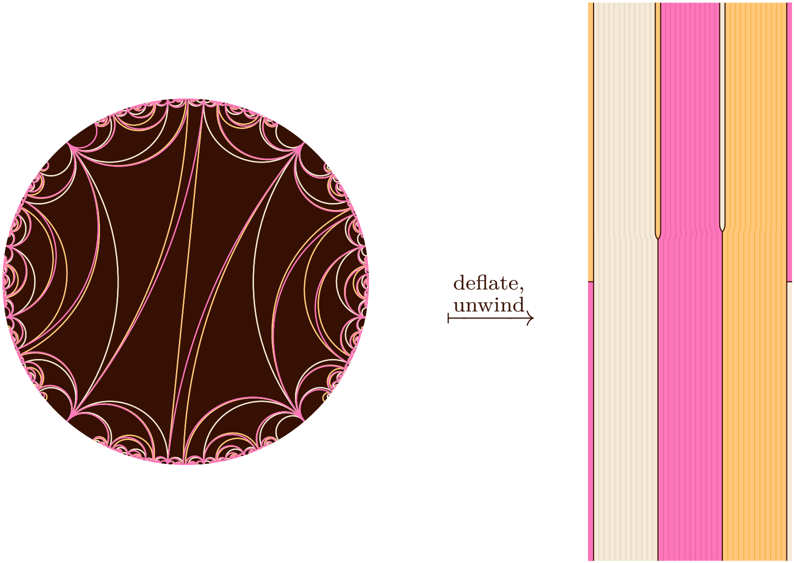
Pit edge flips against strip flips



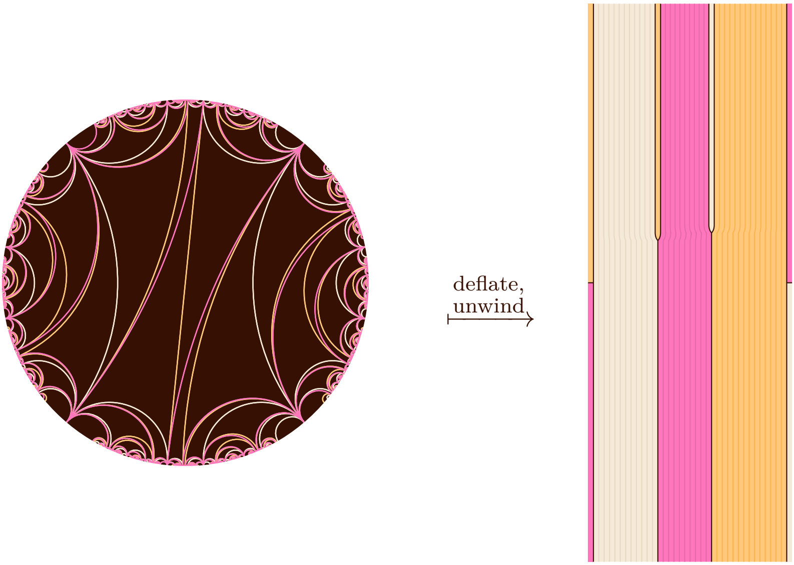
Pit edge flips against strip flips



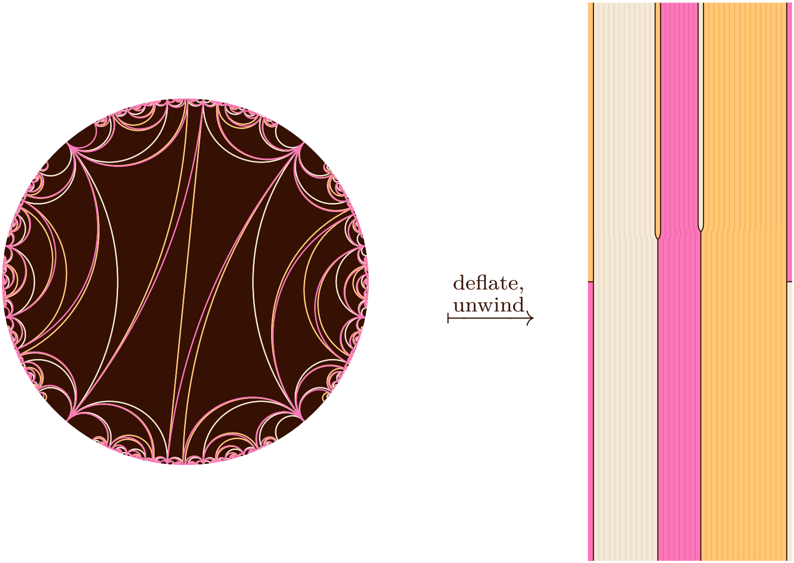
Pit edge flips against strip flips



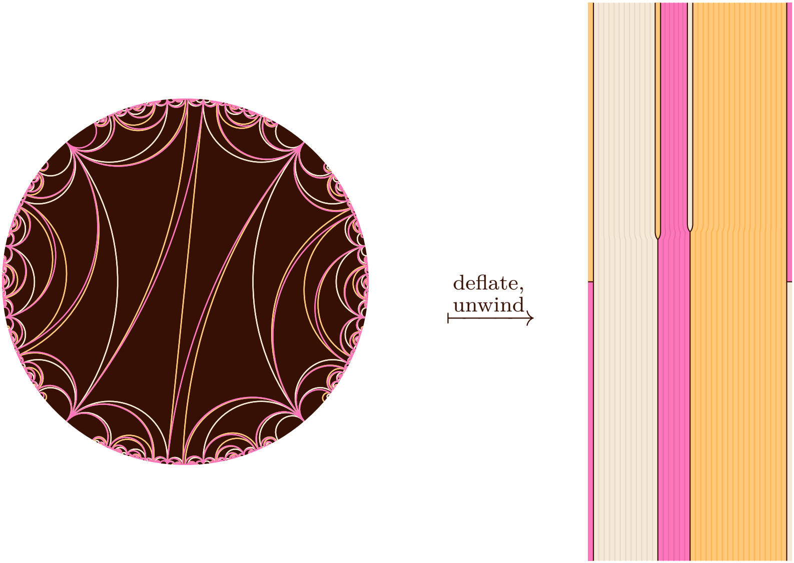
Pit edge flips against strip flips



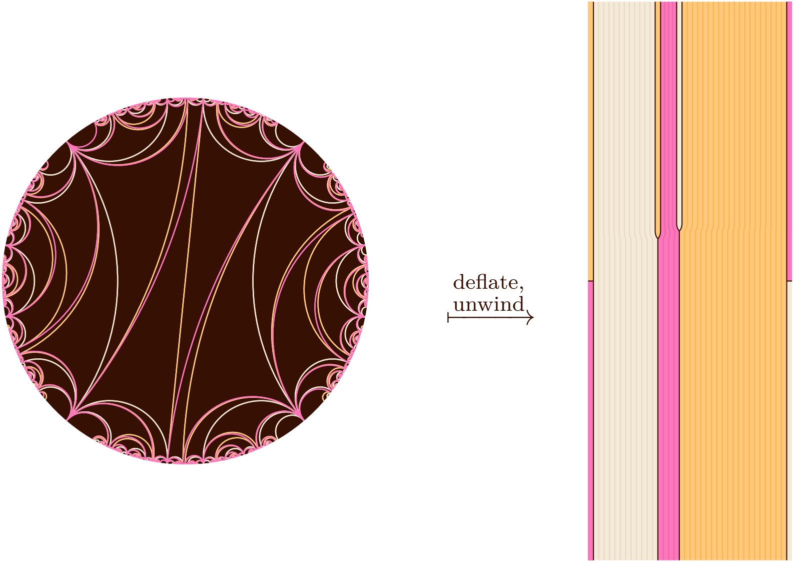
Pit edge flips against strip flips



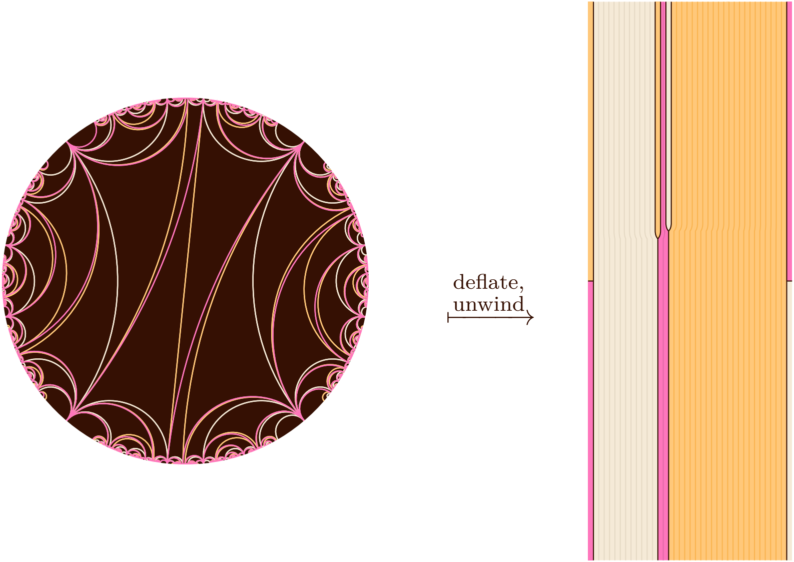
Pit edge flips against strip flips



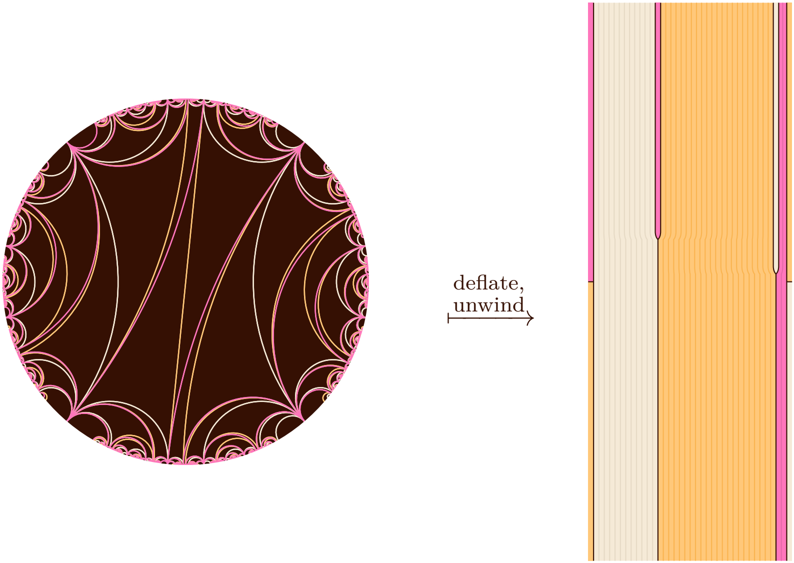
Pit edge flips against strip flips



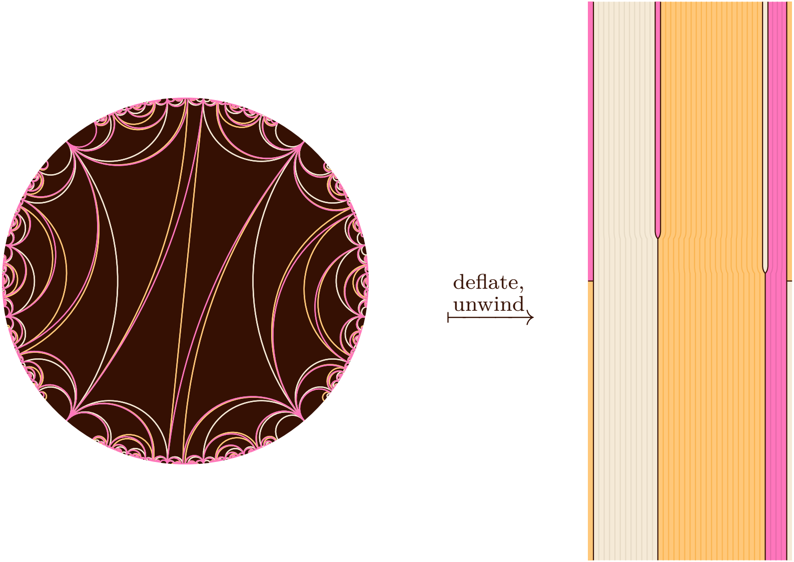
Pit edge flips against strip flips



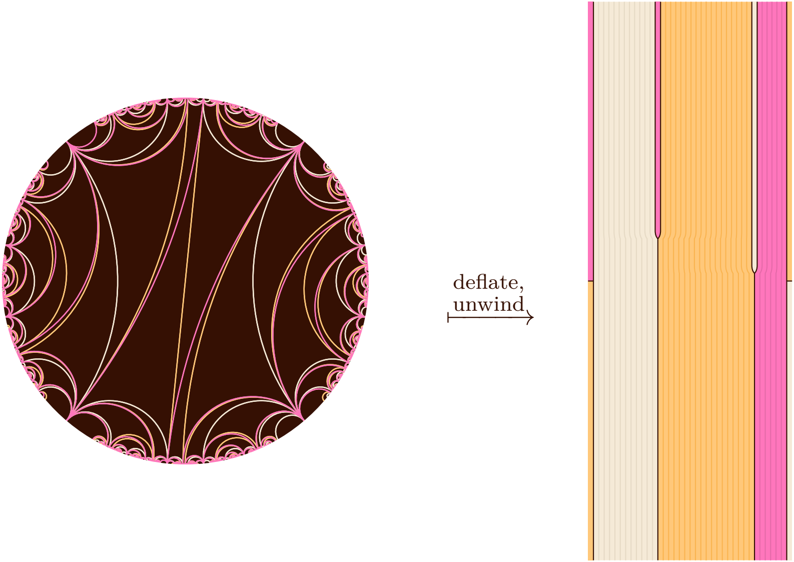
Pit edge flips against strip flips



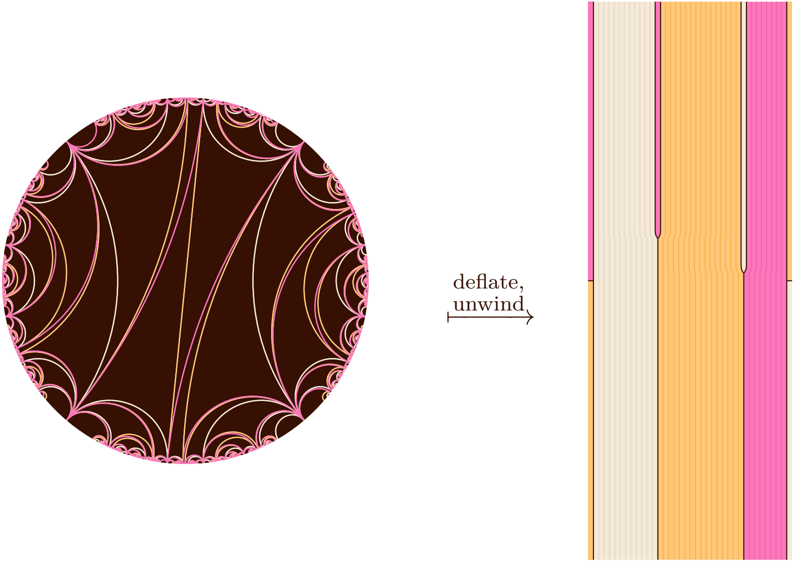
Pit edge flips against strip flips



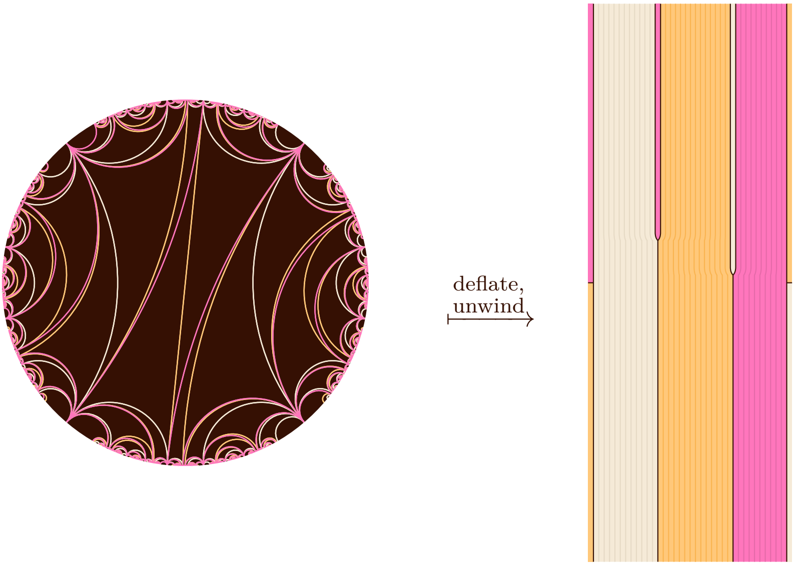
Pit edge flips against strip flips



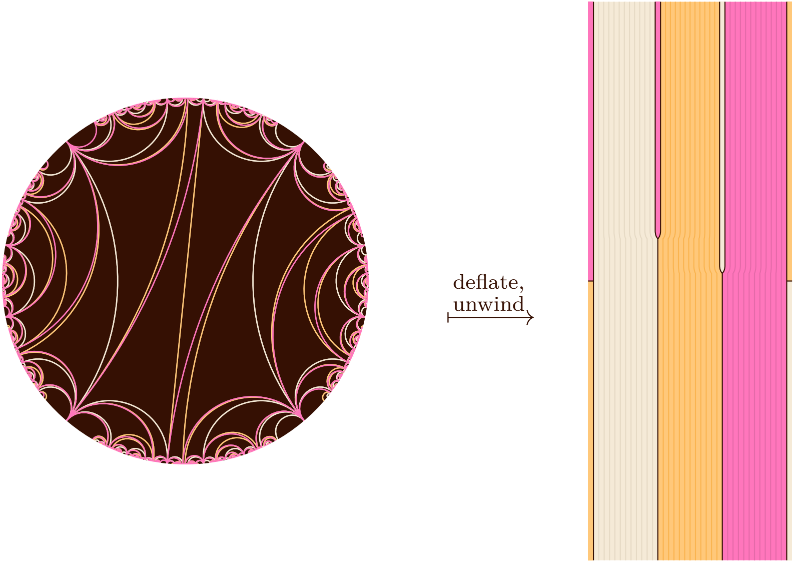
Pit edge flips against strip flips



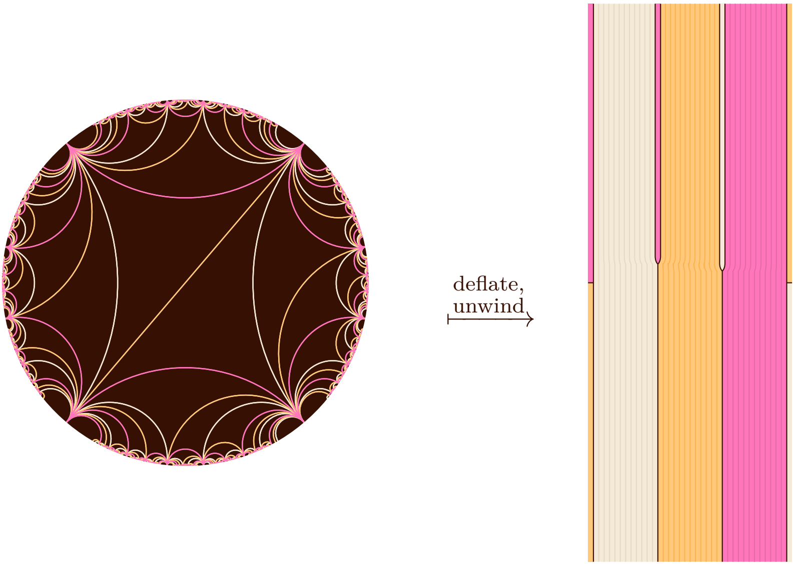
Pit edge flips against strip flips



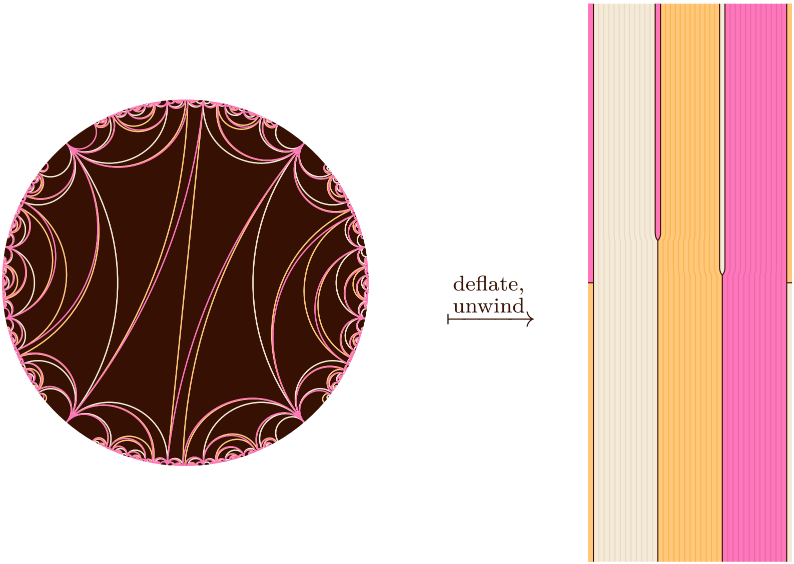
Pit edge flips against strip flips



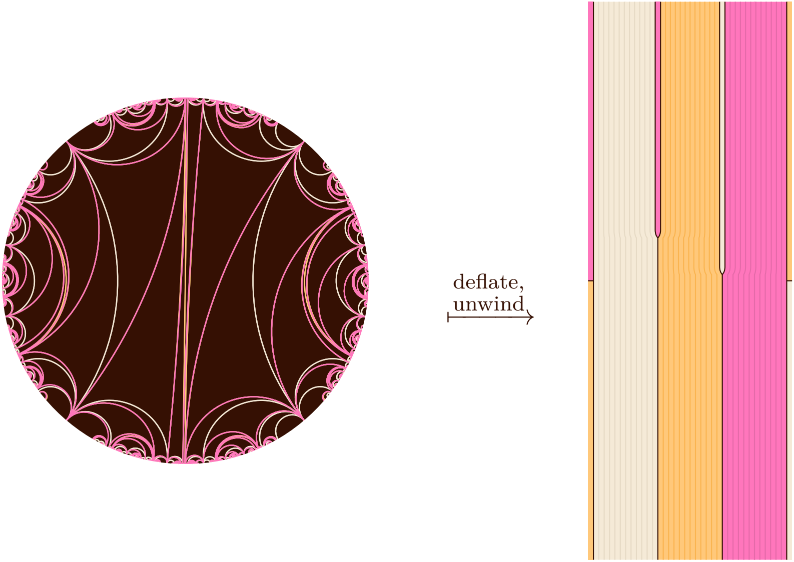
Pit edge flips against strip flips



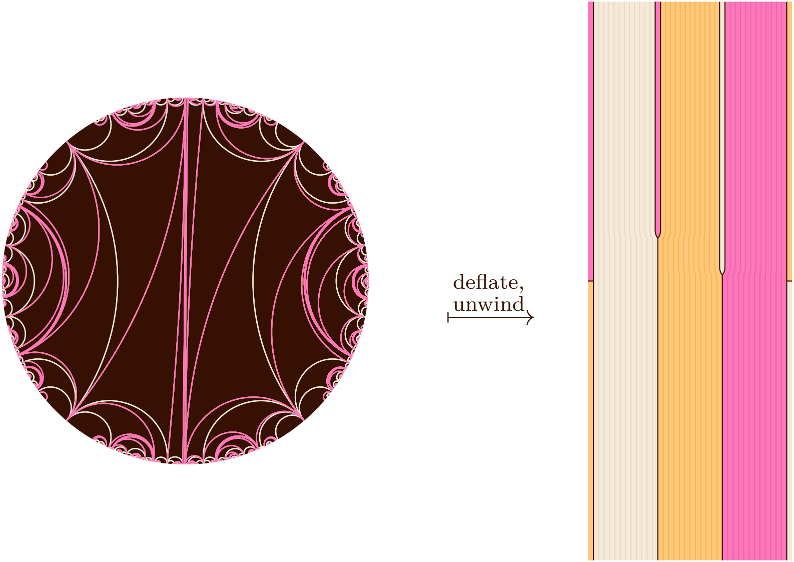
Pit edge flips against strip flips



Pit edge flips against strip flips



Pit edge flips against strip flips



Pit edge flips against strip flips

Geometric interpretation

- Watch the shear parameters z and x of the twisted, deflated, and unwound surface as the number of Dehn twists grows.
- The shear z approaches the cuff length of the pair of pants we built earlier.
- The ratio $2\pi\, y/z$ approaches the Fenchel-Nielsen twist of the cuff, with respect to the marking we picked.