The geometry of quantum energy levels

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ReNew Quantum Internal Seminar

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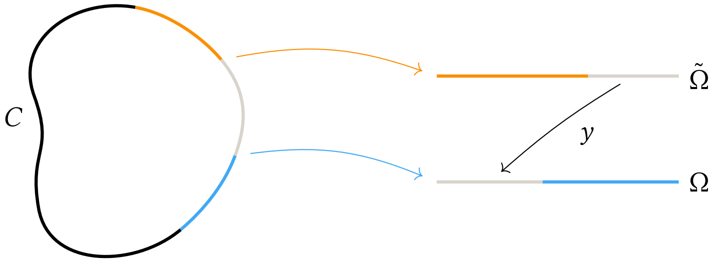
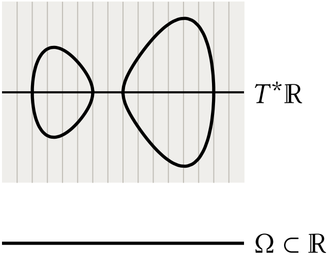
A particle on a curve



Let's study a particle moving on a real curve C.

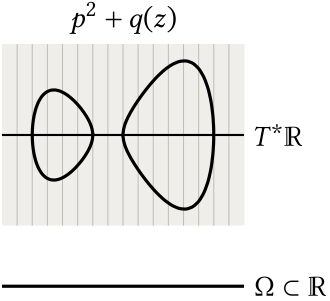
Classical energy levels



An energy level is a curve in T^*C, cut out locally by

$$\frac{1}{2}p^2 + V(z) = E$$

Classical energy levels



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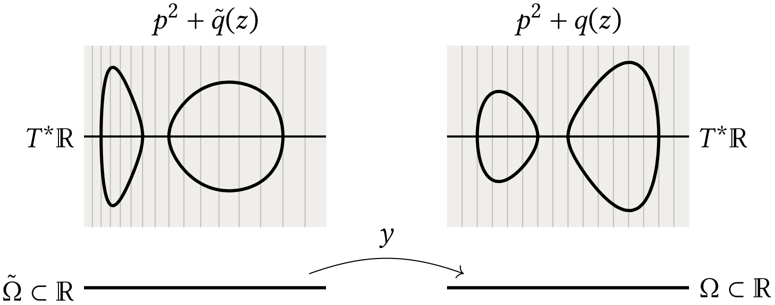

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$$\frac{1}{2}p^2 + V(z) = E \quad \rightsquigarrow \quad p^2 + q(z) = 0$$

For consistency between charts, we need $\tilde{q} = y^* q \; y_z^2$.

Quantization

$\Omega \subset \mathbb{R}$

A quantum energy level should be described locally by

$$\frac{1}{2}\left(i\,\frac{\partial}{\partial z}\right)^2 + [V(z) - E]$$

Quantization

$$\left(\frac{\partial}{\partial z}\right)^2 - q(z)$$

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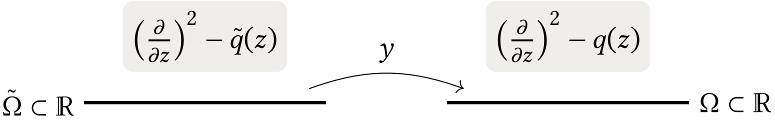
Hill's operator

Quantization



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What condition to impose for consistency between charts?

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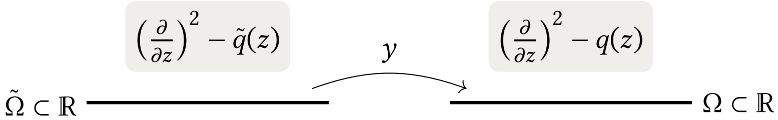

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Hill's operator

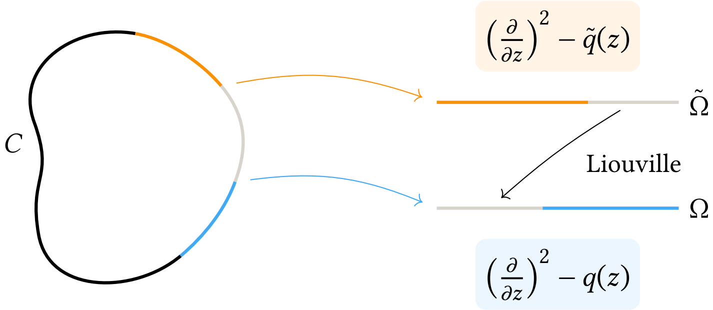
What condition to impose for consistency between charts?

Liouville equivalence is a time-honored choice.

$$\left(\frac{\partial}{\partial z}\right)^2 - \tilde{q} \quad = \quad y_z^{3/2} \circ \left[\left(\frac{\partial}{\partial y}\right)^2 - y^* q\right] \circ y_z^{1/2}$$

It can be motivated using an operator-ordering rule.

Quantum energy levels



A *quantum energy level* on C consists of:

Data	An atlas with a Hill's operator on the image of each chart.
Condition	Transitions are Liouville equivalences.

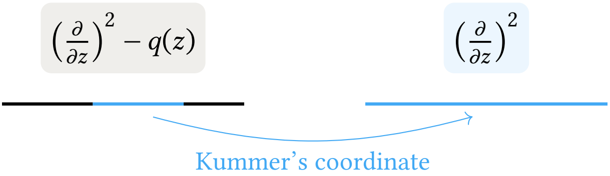
It's a geometric structure!

Simplification

$$\left(\frac{\partial}{\partial z}\right)^2 - q(z) \qquad\qquad \left(\frac{\partial}{\partial z}\right)^2$$

Every Hill's operator is locally Liouville-equivalent to $\left(\frac{\partial}{\partial z}\right)^2$.

Simplification

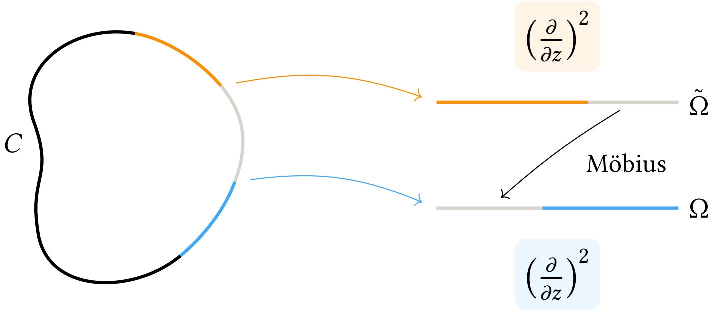


Every Hill's operator is locally Liouville-equivalent to $\left(\frac{\partial}{\partial z}\right)^2$.

Hence, we can describe a quantum energy level using only *projective charts*, which come with $\left(\frac{\partial}{\partial z}\right)^2$ on their images.

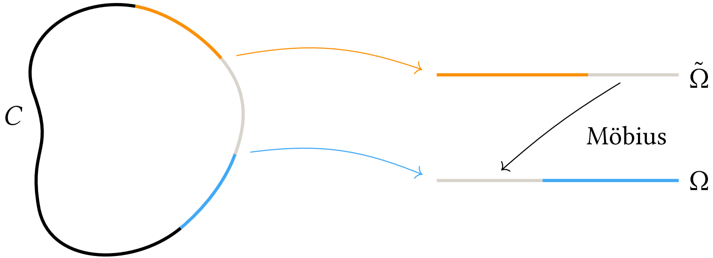
The Liouville self-equivalences of this operator are the Möbius transformations.

Real projective structures



If we stick to projective charts, we can forget about Hill's operators.

Real projective structures



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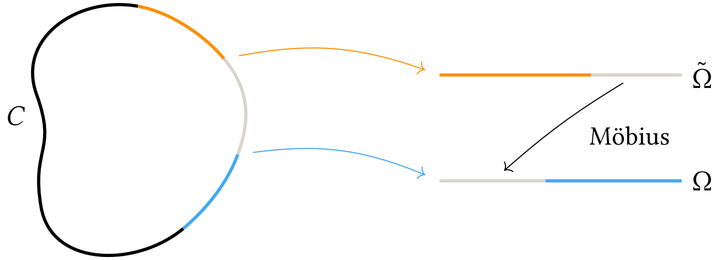


If we stick to projective charts, we can forget about Hill's operators.

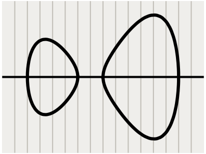
A quantum energy level on C becomes:

Data	An atlas of charts to $\mathbb{RP}^1$.
Condition	Transitions are Möbius transformations.

This is known as a *real projective structure* (see Segal, 1991).

Summary

$$\frac{1}{2}p^2 + q(z)$$



classical
energy level

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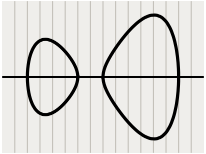

$$\left(\frac{\partial}{\partial z}\right)^2 - q(z)$$

classical
energy level

quantum
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quantize

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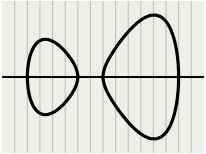

$\left(\frac{\partial}{\partial z}\right)^2 - q(z)$

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classical energy level

quantum energy level

real projective structure

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$\cong$

simplify

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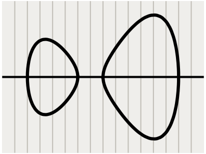

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classical energy level

quantum energy level

real projective structure

quantize

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simplify

A quantum energy level is a real projective structure.

Classification

Topologically, C is simple: a line or a circle.

Its variety of real projective structures is also pretty simple.

- For example, the real projective structures on a circle are classified, up to isotopy, by conjugacy classes in $\widetilde{\mathrm{PSL}}_2\mathbb{R}$.

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- Vary the energy.
  - We get a family of real projective structures.
  - It encodes all the physics of the system.
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- Vary the energy.
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- Complexify the configuration space.

Complexification

A complex-valued potential models a particle that can escape.

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The potential often extends holomorphically to a complex neighborhood of the path.

Famous examples ($\lambda \in \mathbb{C}$ constant):

$\frac{1}{2}z^2 + \lambda z^4$	perturbation theory (Bender–Wu, 1969)
$-\frac{\lambda}{1+e^{z-1}} + \frac{1}{2z^2}$	nuclear scattering (Knoll–Schaeffer, 1976)

Complexification

$$\left(\frac{\partial}{\partial z}\right)^2 - 2[V(z) - E]$$

When $V(z)$ extends holomorphically, the operators describing quantum energy levels can act on holomorphic functions.

That action should encode all physically relevant information.

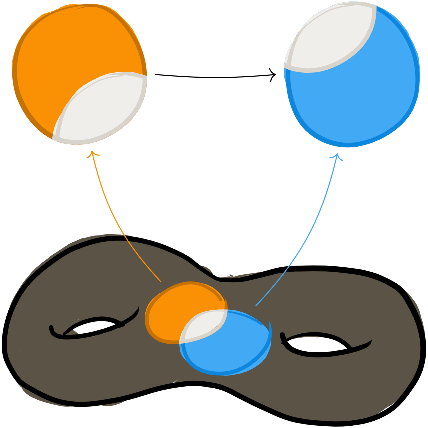
For instance, all generalized eigenfunctions* can be represented holomorphically (see F., 2020).

* For typical choices of rigged Hilbert space.

Complex projective structures

Make C a complex curve.

Carry over definitions:



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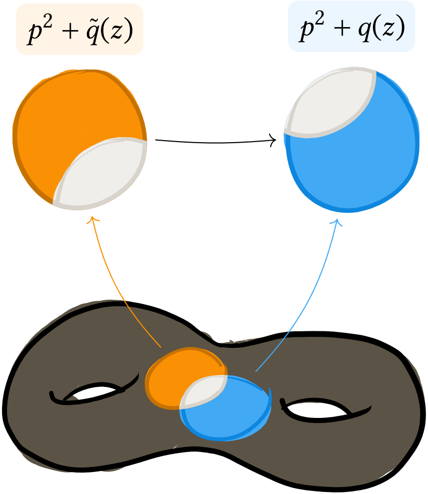
- Classical energy level (spectral curve)



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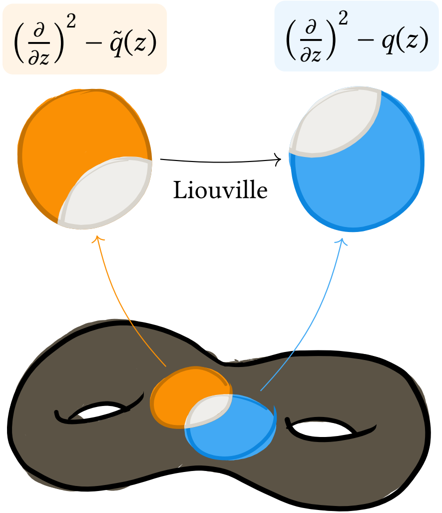

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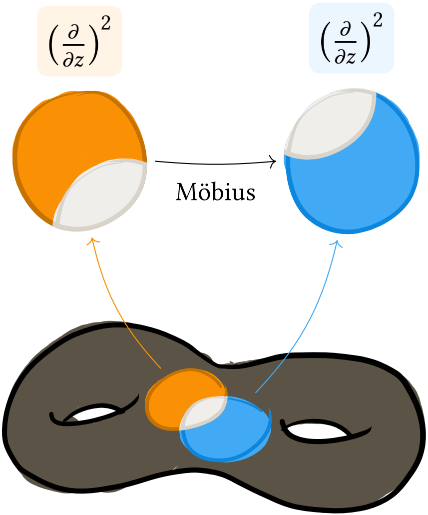

Complex projective structures

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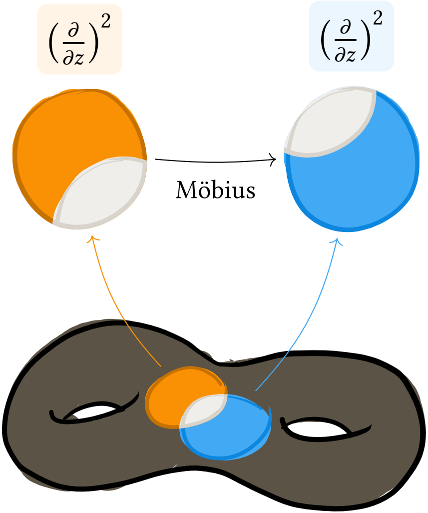
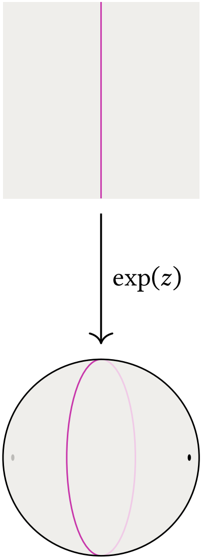
Now even basic systems have cool energy level geometry.



Free particle

$$\left(\frac{\partial}{\partial z}\right)^2 - \frac{1}{4}$$

Free particle



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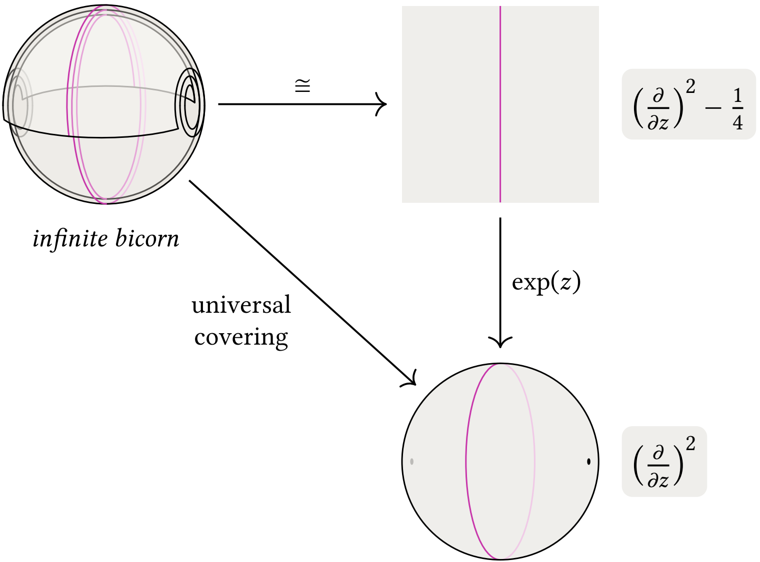
Free particle



Falling particle

$$\left(\frac{\partial}{\partial z}\right)^2 - z$$

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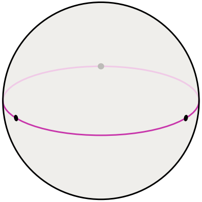
$$\frac{\mathrm{Ai}\left(z\,/\,(-2)^{2/3}\right)}{\mathrm{Ai}\left(z\,/\,(+2)^{2/3}\right)}$$



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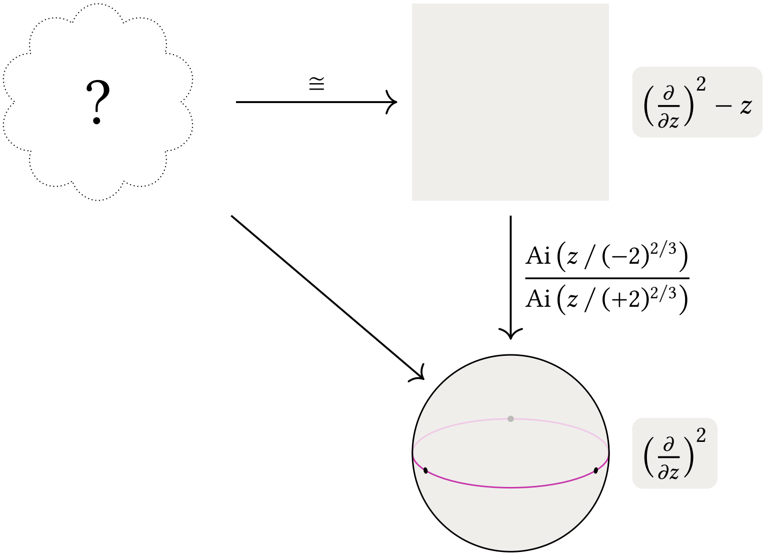
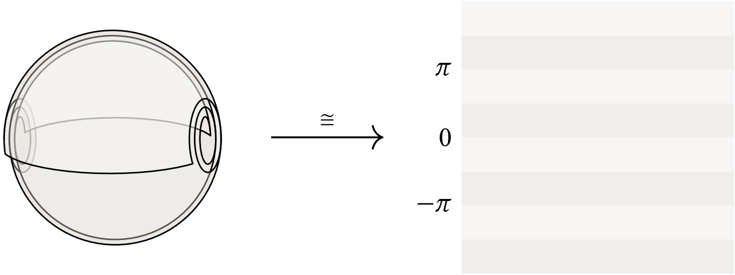
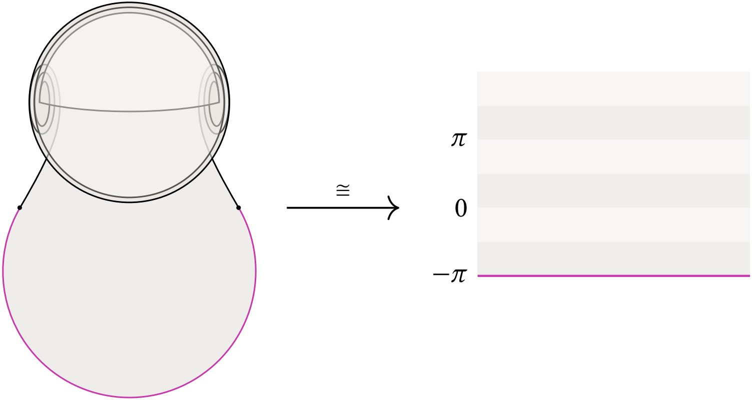


Falling particle: construction



Take an infinite bicorn.

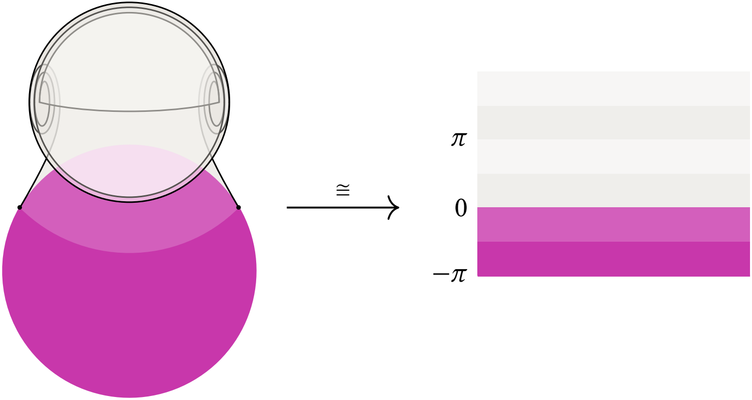
Falling particle: construction



Take an infinite bicorn.

Cut it horizontally, leaving a *half-infinite bicorn*.

Falling particle: construction

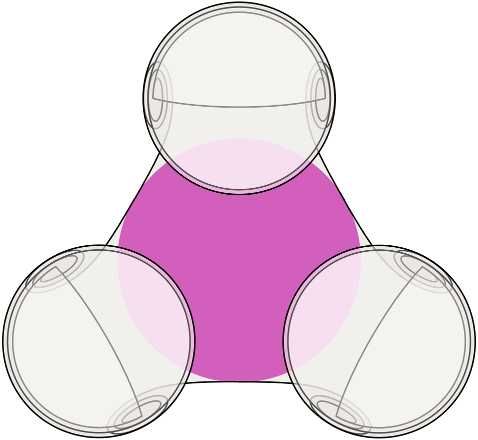


Take an infinite bicorn.

Cut it horizontally, leaving a *half-infinite bicorn*.

At the boundary, there's a tab isomorphic to a disk in $\mathbb{CP}^1$.

Falling particle: construction

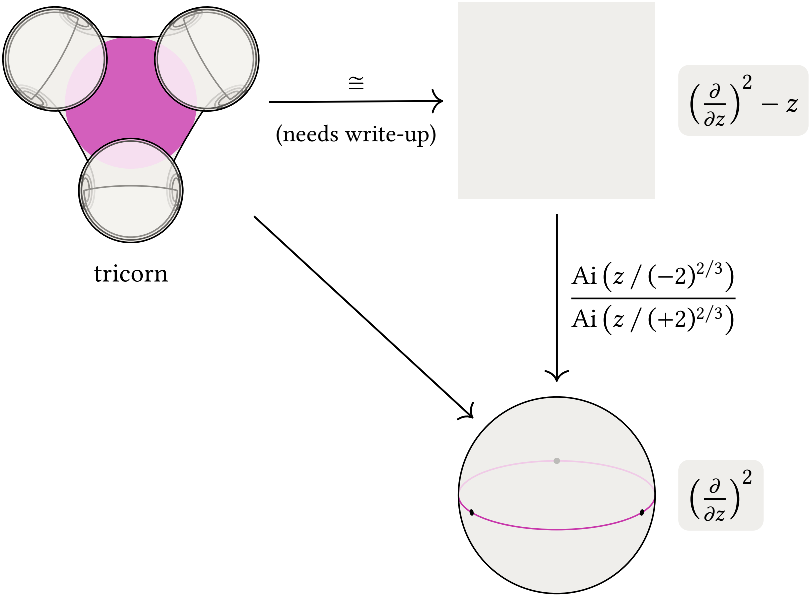


Take three copies of the half-infinite bicorn.

Glue them along the tab, at 120° angles.

We've made a new complex projective curve, the *tricorn*.

Falling particle

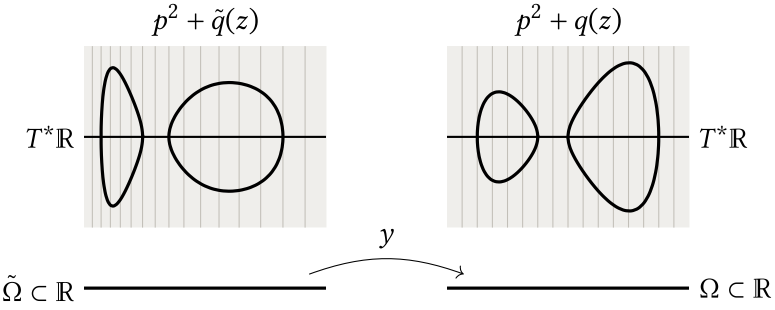


Deformations

Complex projective curves have a rich deformation theory.

Classical energy levels parameterize deformations of quantum energy levels!

Geometry of classical energy levels



Consistency condition:

$$\tilde{q} = y^* q \; y_z^2$$

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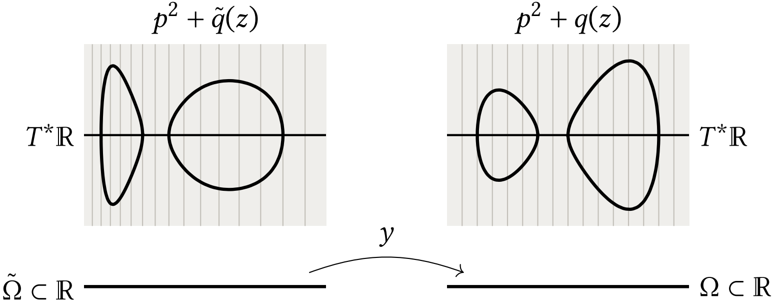
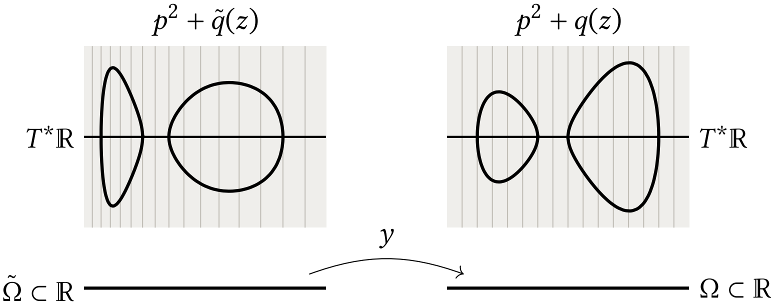


Consistency condition:

$$\tilde{q} = y^* q \, y_z^2$$

$$\tilde{q} \, dz^2 = y^*(q \, dz^2)$$

Geometry of classical energy levels

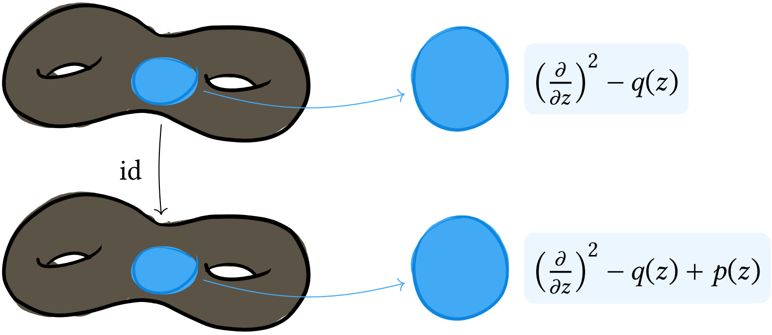


Consistency condition:

$$\tilde{q} = y^* q \; y_z^2$$
$$\tilde{q}\, dz^2 = y^*(q\, dz^2)$$

A classical energy level is a quadratic differential: a section of $\mathrm{Sym}^2\, T^*C$.

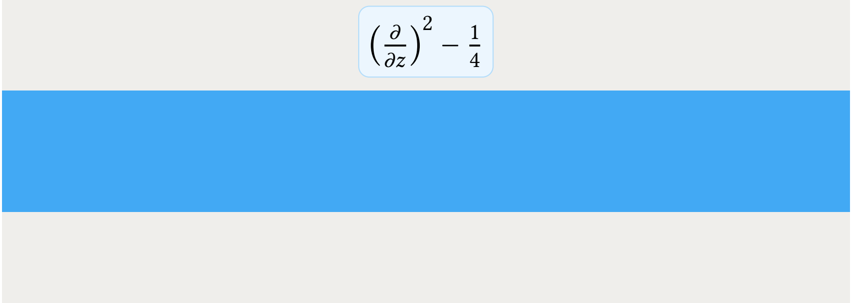
Classical energy levels as deformations



We can deform a complex projective structure locally by adding a function p to the Hill's operator on the image of a chart.

Consistency condition: same as for quadratic differential $p\,dz^2$.

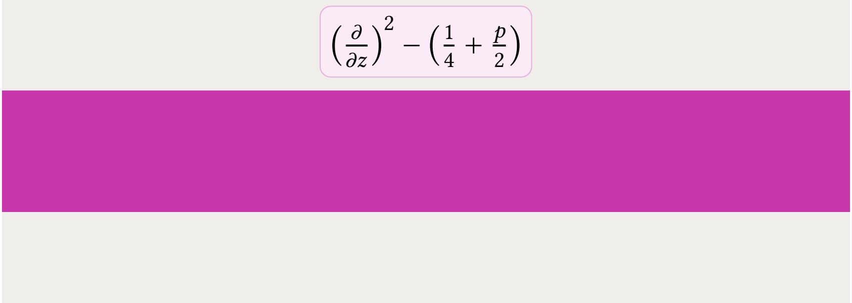
A transformation theory result



Take a horizontal strip $B \subset \mathbb{C}$, of height at least π.

Think of it as a region in the infinite bicorn U.

A transformation theory result



Take a horizontal strip $B \subset \mathbb{C}$, of height at least π.

Think of it as a region in the infinite bicorn U.

Deform its complex projective structure by adding $-\frac{1}{2} p \, dz^2$.

A transformation theory result

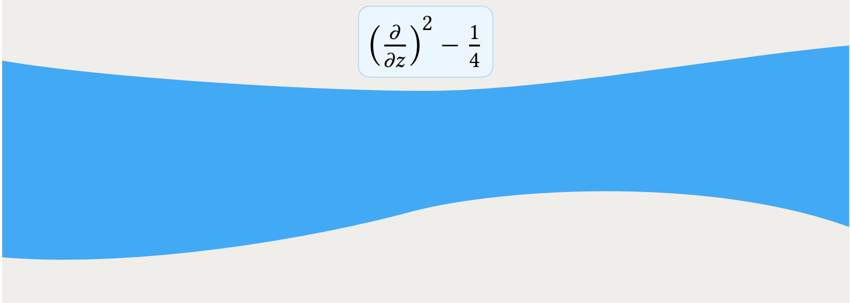
$$\left(\frac{\partial}{\partial z}\right)^2 - \left(\frac{1}{4} + \frac{p}{2}\right)$$

Suppose the perturbation p is small in the sense that

$$\left|\frac{p}{2}\right| < \begin{cases} \left(\frac{1}{2\sin\ell}\right)^2 & \ell \leq \frac{\pi}{2} \\ \frac{1}{4} & \ell \geq \frac{\pi}{2}, \end{cases}$$

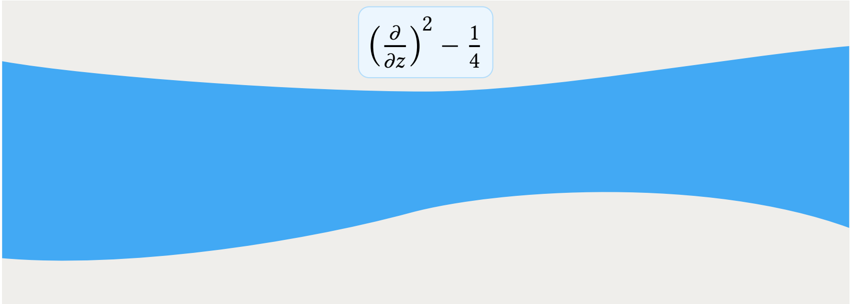
where ℓ is the function that gives the distance to the edge of B.

A transformation theory result



Then you can remove the perturbation with a Liouville transformation $y : B \to U$.

A transformation theory result

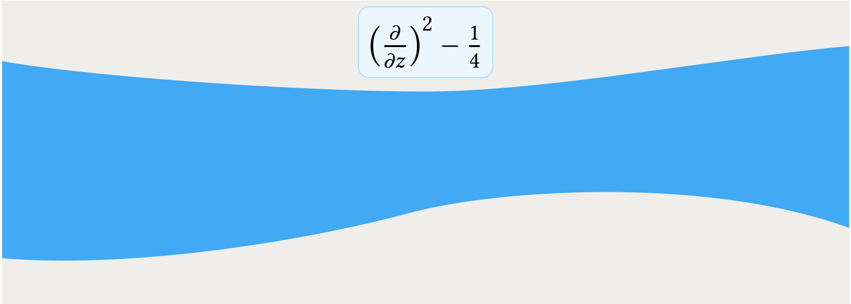


Then you can remove the perturbation with a Liouville transformation $y : B \to U$.

The conformal map y extends to the boundary of B, fixing $\pm\infty$.

The size of p controls how much y alters displacements.

A transformation theory result



Then you can remove the perturbation with a Liouville transformation $y : B \to U$.

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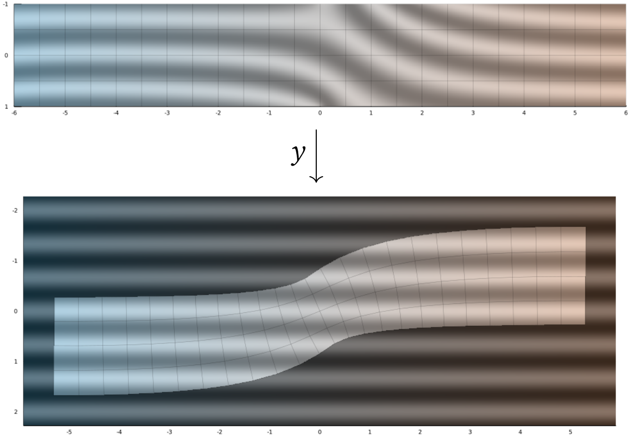
The size of p controls how much y alters displacements.

The perturbed free particle remains free, geometrically.

Example



Free particle perturbed by a “dip in the road”:

$$\left(\frac{\partial}{\partial z}\right)^2 - \left(\frac{1}{4} + \frac{\lambda}{1+(z/2)^2}\right)$$

Sources

Ideas leading to the "quantum energy level" framing:

- Neitzke, "Some new geometric applications of quantum field theory" `https://web.ma.utexas.edu/users/neitzke/talks/html/qft-hausdorff/talk.html`.

Exposition of real projective structures:

- Segal, "The geometry of the KdV equation" `doi:10.1142/S0217751X91001416`.
- Segal, "Unitary Representations of some Infinite Dimensional Groups" (Section 8.b) `doi:10.1007/BF01208274`.

Sources

Holomorphic potential examples:

- Bender and Wu, "Anharmonic Oscillator" `doi:10.1103/PhysRev.184.1231`.
- Knoll and Schaeffer, "Semiclassical Scattering Theory with Complex Trajectories. I. Elastic Waves" `doi:10.1016/0003-4916(76)90040-3`.

Sources

Complex geometry of Hill's operators, infinite bicorn, transformation theory result, and more sources:

- F., "The complex geometry of the free particle, and its perturbations" `arXiv:2008.03836`.

Exact WKB transformation theory results:

- Kawai and Takei, *Algebraic Analysis of Singular Perturbation Theory* (Theorem 2.15).
- Kamimoto and Koike, "On the Borel summability of WKB-theoretic transformation series" (Theorems 2.1 – 2.2) `RIMS:1726`, available at `http://www.kurims.kyoto-u.ac.jp/preprint/preprint_y2011.html`.