Farey Sets in $\mathbb{R}^n$

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The *Farey numbers* of order Q are the fractions between zero and one whose denominators are less than or equal to Q. You can think of these numbers as the intersection of the interval $[0, 1]$ with the set

$$\mathscr{F}_Q = \bigcup_{q=1}^{Q} \tfrac{1}{q}\mathbb{Z},$$

where $\frac{1}{q}\mathbb{Z}$ is shorthand for $\{\frac{p}{q} \mid p \in \mathbb{Z}\}$.

An obvious analogue of $\mathscr{F}_Q$ in $\mathbb{R}^n$ is

$${}^n\mathscr{F}_Q = \bigcup_{q=1}^{Q} \tfrac{1}{q}\mathbb{Z}^n.$$

Look at the plots of ${}^1\mathscr{F}_Q$ and ${}^2\mathscr{F}_Q$ in Figures 1 and 2. What's up with those empty regions? It turns out that in ${}^n\mathscr{F}_Q$, if you pick a lattice point $a \in \mathbb{Z}^n$ and a fraction r/s in lowest terms, the hyperplane

$$a \cdot x = r/s$$

is sandwiched between empty regions of width slightly greater than

$$\frac{\operatorname{gcf}(a)}{Qs\|a\|},$$

with "slightly greater than" going to zero as Q goes to infinity. Here, $\cdot$ is the standard inner product on $\mathbb{R}^n$, $\operatorname{gcf}(a)$ is shorthand for $\operatorname{gcf}(a_1, \ldots, a_n)$, and $\|a\| = \sqrt{a \cdot a}$.

The observation above is a fairly straightforward consequence of the following two facts.

Fact 1. *If you project ${}^n\mathscr{F}_Q$ onto the line generated by $a \in \mathbb{Z}^n$, which is isometric to $\mathbb{R}$, you end up with*

$$\frac{\operatorname{gcf}(a)}{\|a\|}\,{}^1\mathscr{F}_Q.$$

Figure 1: A plot of ${}^1\mathscr{F}_{16}$ on the interval $[-1, 1]$.

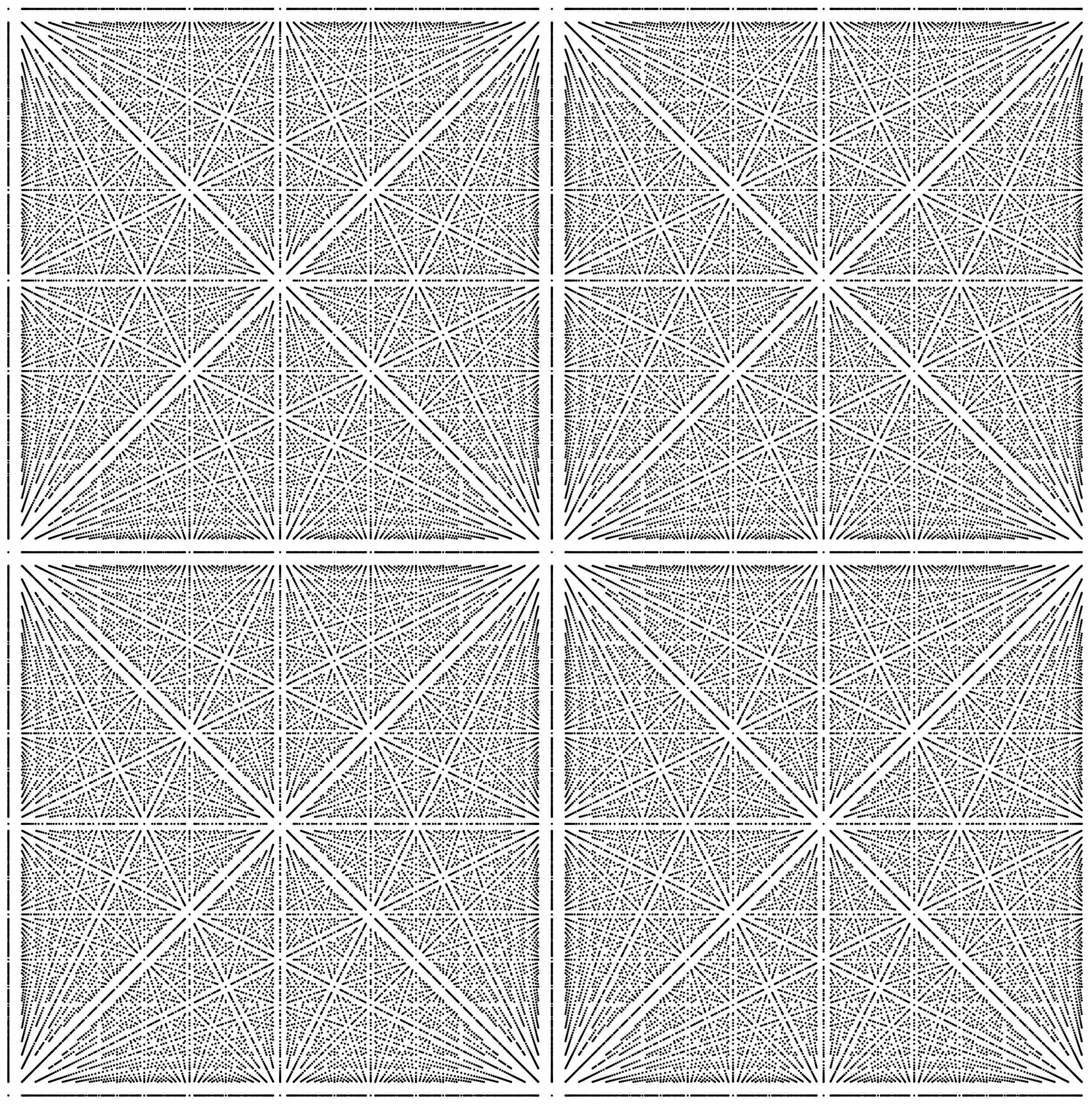

Figure 2: A plot of $^2\mathscr{F}_{40}$ in the box $[-1,1]^2$.

Fact 2. *If the fraction r/s is in lowest terms, the distances between r/s and its neighbors in ${}^1\mathscr{F}_Q$ are equal to or slightly greater than $1/Qs$, with "slightly greater than" going to zero as Q goes to infinity.*

Proof of Fact 1. Since

$$ {}^n\mathscr{F}_Q = \bigcup_{q=1}^{Q} \tfrac{1}{q}\mathbb{Z}^n, $$

the projection of ${}^n\mathscr{F}_Q$ onto the line generated by $a \in \mathbb{Z}^n$ is

$$ \bigcup_{q=1}^{Q} \tfrac{1}{q} \tfrac{a}{\|a\|} \cdot \mathbb{Z}^n, $$

where $a \cdot \mathbb{Z}^n$ is shorthand for

$$ \{a \cdot z \mid z \in \mathbb{Z}^n\} = \{a_1 z_1 + \ldots + a_n z_n \mid z_1, \ldots, z_n \in \mathbb{Z}\}. $$

By Bézout's identity,

$$ \{a_1 z_1 + \ldots + a_n z_n \mid z_1, \ldots, z_n \in \mathbb{Z}\} = \{\text{gcf}(a_1, \ldots, a_n) z \mid z \in \mathbb{Z}\}; $$

in shorthand,

$$ a \cdot \mathbb{Z}^n = \text{gcf}(a)\mathbb{Z}. $$

Therefore, the projection of ${}^n\mathscr{F}_Q$ onto the line generated by $a \in \mathbb{Z}^n$ is

$$ \bigcup_{q=1}^{Q} \tfrac{1}{q} \tfrac{\text{gcf}(a)}{\|a\|} \mathbb{Z} = \frac{\text{gcf}(a)}{\|a\|} \bigcup_{q=1}^{Q} \tfrac{1}{q}\mathbb{Z} = \frac{\text{gcf}(a)}{\|a\|}\, {}^1\mathscr{F}_Q. $$

□

Proof of Fact 2. Since the elements of ${}^1\mathscr{F}_Q$ are rational numbers, we can put them in increasing order, and we can also write them as fractions in lowest terms. In this proof, I'll think of the ${}^1\mathscr{F}_Q$ not as sets of rational numbers, but as increasing sequences of fractions in lowest terms.

We know from the work of Charles Haros, and many others who followed him,[1] that you can turn ${}^1\mathscr{F}_{Q-1}$ into ${}^1\mathscr{F}_Q$ by following a simple rule:

> If you see two adjacent fractions $\frac{a}{b}$ and $\frac{c}{d}$ whose denominators add up to Q, insert their *mediant* $\frac{a+c}{b+d}$ between them.

Starting with ${}^1\mathscr{F}_1$, you can generate ${}^1\mathscr{F}_2, {}^1\mathscr{F}_3, {}^1\mathscr{F}_4 \ldots$ by using this rule over and over. If the fraction r/s is in lowest terms, it first appears in ${}^1\mathscr{F}_s$ as the mediant of two fractions a/b and c/d, with

$$ \frac{a}{b} < \frac{r}{s} < \frac{c}{d}. $$

[1] For details, I recommend the excellent book *A Motif of Mathematics*, by Scott Guthery.

The fraction a/b is the lower neighbor of r/s until you reach ${}^1\mathscr{F}_{b+s}$, where a new fraction appears between a/b and r/s:

$$\frac{a+r}{b+s}.$$

This fraction remains the lower neighbor of r/s until it is displaced, in ${}^1\mathscr{F}_{b+2s}$, by

$$\frac{a+2r}{b+2s}.$$

In general, the lower neighbor of r/s in ${}^1\mathscr{F}_{b+ms}$ is

$$\frac{a+mr}{b+ms}.$$

Similarly, the upper neighbor of r/s in ${}^1\mathscr{F}_{ns+d}$ is

$$\frac{nr+c}{ns+d}.$$

Because a/b is the lower neighbor of r/s in one of the ${}^1\mathscr{F}_Q$, we have the identity $rb - sa = 1$, which you can easily prove by induction. Hence, the distance between r/s and its lower neighbor in ${}^1\mathscr{F}_{b+ms}$ is

$$\frac{r}{s} - \frac{a+mr}{b+ms} = \frac{rb-sa}{s(b+ms)} = \frac{1}{s(b+ms)}.$$

Similarly, from the identity $cs - dr$, we find that the distance between r/s and its upper neighbor in ${}^1\mathscr{F}_{ns+d}$ is

$$\frac{nr+c}{ns+d} - \frac{r}{s} = \frac{1}{(ns+d)s}.$$

Now, for any $Q \geq s$, pick the largest m so that $b + ms \leq Q$, and the largest n so that $ns + d \leq Q$. The distances between r/s and its neighbors in ${}^1\mathscr{F}_Q$ are

$$\frac{1}{s(b+ms)} \text{ and } \frac{1}{(ns+d)s},$$

respectively. Both distances are equal to or slightly greater than $1/Qs$, and as Q goes to infinity, $b + ms$ and $ns + d$ approach Q. □