Classification of two-dimensional Frobenius and H^*-algebras

Aaron Fenyes

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Abstract

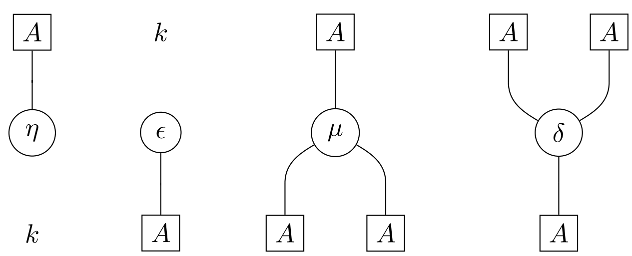
After a quick review of Frobenius and H^*-algebras, I produce explicit constructions of all the two-dimensional algebras of these kinds. With an eye toward higher dimensions, I favor general techniques over elementary ones. Impatient readers can skip straight to the finished constructions in Propositions 1, 2, and 3.

1 Algebra review

1.1 Frobenius algebras

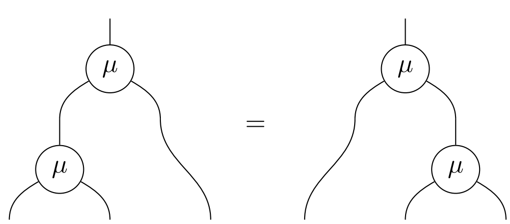
1.1.1 Definition

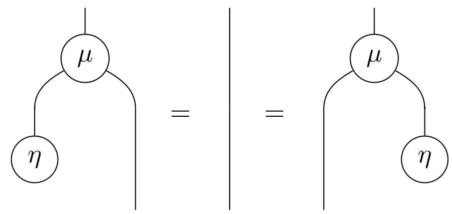
Let k be a field. A *Frobenius algebra* is a k-vector space A equipped with linear maps



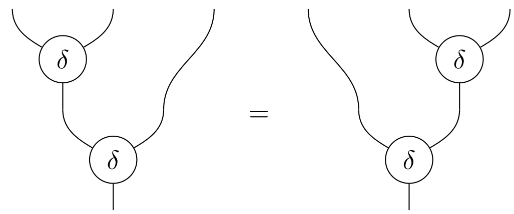
that satisfy the following conditions.

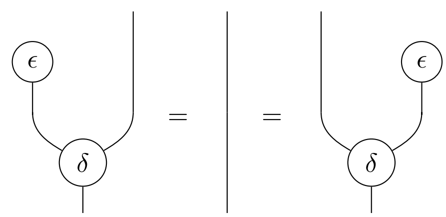
- μ is an associative multiplication with unit η:



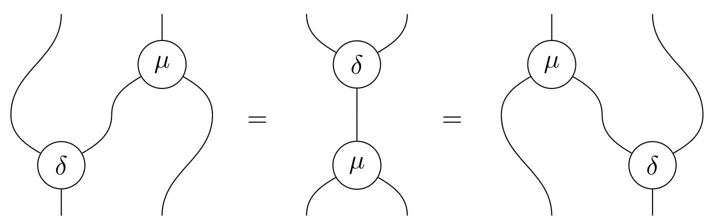


- δ is a coassociative comultiplication with counit ϵ:





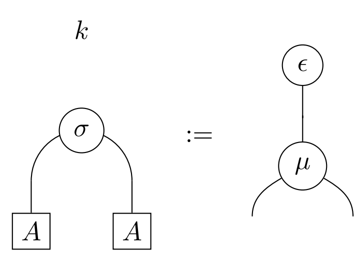
- μ and δ are related by the *Frobenius identity*:



For convenience, define $e := \eta(1)$.

1.1.2 Frobenius form

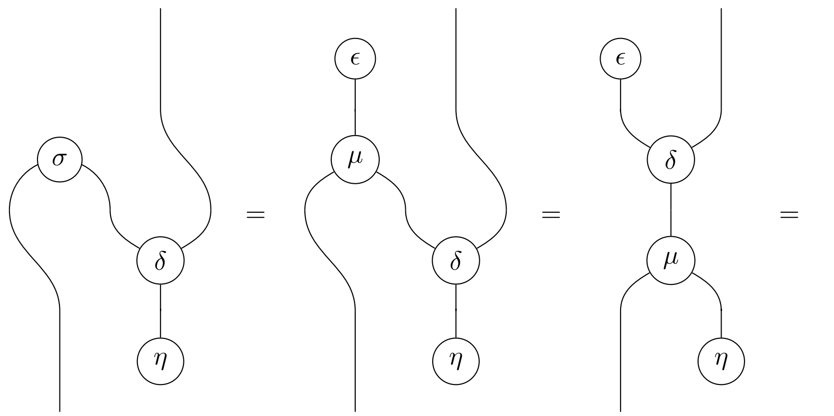
Composing ϵ with μ yields a bilinear form



called the *Frobenius form*. Because μ is associative,

$$\sigma(\mu(x \otimes y), z) = \sigma(x, \mu(y \otimes z))$$

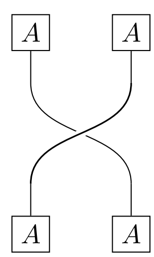
for all $x, y, z \in A$. The identity



implies that the linear functional $\sigma(x, \_)$ is nonzero for every nonzero $x \in A$, because any x for which $\sigma(x, \_) = 0$ would be in the kernel of the map shown above. Combining this argument with its mirror image, we see that σ is non-degenerate.

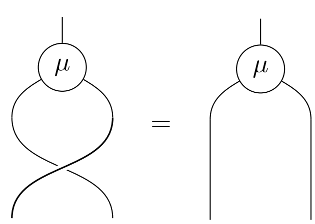
1.1.3 Commutativity and cocommutativity

The *twist operator*

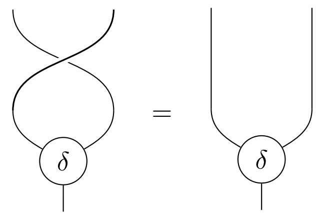


is the map that sends $v \otimes w$ to $w \otimes v$ for all $v, w \in A$.

A Frobenius algebra is said to be *commutative* if



and *cocommutative* if



Because of the way the Frobenius identity relates multiplication and comultiplication, you might suspect that a Frobenius algebra is commutative if and only if it is cocommutative. This turns out to be true. See Appendix A for a proof, based on the one in [2].

1.2 H^*-algebras

1.2.1 Definition

Suppose k is a subfield of $\mathbb{C}$, so a vector space over k can be an inner product space. In this case, an H^*-*algebra* is a Frobenius algebra equipped with an inner product that makes $\delta = \mu^\dagger$ and $\epsilon = \eta^\dagger$.[1]

2 Classification

2.1 Assumptions about the base field

Our classification splits into two cases, which rely on different assumptions about the field k. When $\sigma(e, e) = 0$, we assume that k does not have characteristic two. When $\sigma(e, e) \neq 0$, we assume that every element of k has a square root.

2.2 Frobenius algebras

Say A is a two-dimensional Frobenius algebra. Pick any $v \in A$ outside the span of e. The condition that e is a unit for μ is satisfied if and only if

$$\mu(e \otimes e) = e \qquad \mu(e \otimes v) = v \qquad \mu(v \otimes e) = v.$$

Since $e \otimes e$, $e \otimes v$, $v \otimes e$, and $v \otimes v$ form a basis for $A \otimes A$, it follows that A is commutative.

Because σ is non-degenerate, the kernel of $\sigma(e, \_)$ is one-dimensional. Here, our classificaiton splits into two parts: the case where e is in the kernel of $\sigma(e, \_)$ and the case where it isn't.

[1] To make sense of this definition, remember that the tensor product of two inner product spaces V and W has a canonical inner product, defined by the equation $\langle v \otimes w, \tilde{v} \otimes \tilde{w} \rangle = \langle v, \tilde{v} \rangle \langle w, \tilde{w} \rangle$.

2.2.1 Null unit

Suppose $\sigma(e,e)=0$, and k does not have characteristic two. Since A is commutative, σ is symmetric. The subspace spanned by e is Lagrangian, and k does not have characteristic two, so we can find an element x with $\sigma(x,x)=0$ whose span is complementary to the span of e (see Appendix B for details). We can assume without loss of generality that $\sigma(e,x)=1$. The elements e and x form a basis for A.

Observe that

$$\begin{aligned}\epsilon(e) &= \epsilon\mu(e\otimes e)=\sigma(e,e)=0\\ \epsilon(x) &= \epsilon\mu(e\otimes x)=\sigma(e,x)=1.\end{aligned}$$

Once $\mu(x\otimes x)$ is fixed, the action of μ is uniquely determined by the condition that e is a unit for μ. Write $\mu(x\otimes x)$ as a linear combination $pe+qx$. Observe that

$$\begin{aligned}\sigma(x,x) &= \epsilon\mu(x\otimes x)\\ &= p\epsilon(e)+q\epsilon(x)\\ &= q.\end{aligned}$$

By construction, $\sigma(x,x)=0$, so $q=0$. Thus, the only degree of freedom for μ is the value of p in the equation

$$\mu(x\otimes x)=pe.$$

Write $\delta(e)$ and $\delta(x)$ as linear combinations

$$\begin{aligned}\delta(e) &= a\,e\otimes e+b(e\otimes x+x\otimes e)+c\,x\otimes x\\ \delta(x) &= \tilde{a}\,e\otimes e+\tilde{b}(e\otimes x+x\otimes e)+\tilde{c}\,x\otimes x.\end{aligned}$$

Since

$$\begin{aligned}[\epsilon\otimes \mathrm{id}]\delta(e) &= be+cx\\ [\epsilon\otimes \mathrm{id}]\delta(x) &= \tilde{b}e+\tilde{c}x,\end{aligned}$$

the condition that ϵ is a counit for δ is satisfied if and only if

$$\begin{aligned}b&=1 & c&=0\\ \tilde{b}&=0 & \tilde{c}&=1.\end{aligned}$$

To see when the Frobenius identity is satisfied, let's make tables of values for $[\mathrm{id}\otimes\mu][\delta\otimes\mathrm{id}]$ and $\delta\mu$. Because e is a unit for μ, the Frobenius identity is guaranteed to hold for the inputs $e\otimes e$ and $x\otimes e$, so we can omit these from our tables.

v	$[\delta\otimes\mathrm{id}](v)$	$[\mathrm{id}\otimes\mu][\delta\otimes\mathrm{id}](v)$
$e\otimes x$	$a\,e\otimes e\otimes x+e\otimes x\otimes x+x\otimes e\otimes x$	$a\,e\otimes x+e\otimes pe+x\otimes x$
$x\otimes x$	$\tilde{a}\,e\otimes e\otimes x+x\otimes x\otimes x$	$\tilde{a}\,e\otimes x+x\otimes pe$

v	$\delta\mu(v)$
$e \otimes x$	$\tilde{a}\, e \otimes e + x \otimes x$
$x \otimes x$	$p(a\, e \otimes e + e \otimes x + x \otimes e)$

Comparing the tables, we see that the Frobenius identity is satisfied if and only if $a = 0$ and $\tilde{a} = p$.

It is straightforward to verify that μ and δ are associative and coassociative, respectively, for any value of p. Once that is done, we have proven the following classification result.

Proposition 1. *Let k be a field that does not have characteristic two. Let A be a two-dimensional k-vector space with basis $\{e, x\}$, and let $\eta\colon k \to A$ be the map $1 \mapsto e$.*

For any $p \in k$, the linear maps defined by the equations

$$\epsilon(e) = 0 \qquad \epsilon(x) = 1$$

$$\mu(e \otimes e) = e \qquad \mu(x \otimes e) = x$$
$$\mu(e \otimes x) = x \qquad \mu(x \otimes x) = pe$$

$$\delta(e) = e \otimes x + x \otimes e \qquad \delta(x) = p\, e \otimes e + x \otimes x$$

make A into a Frobenius algebra. In fact, every two-dimensional Frobenius algebra over k with $\sigma(e, e) = 0$ is of this form.

2.2.2 Non-null unit

Suppose $\sigma(e, e) \neq 0$, and every element of k has a square root. For convenience, define $m := \sigma(e, e)$. Pick an element x that spans the kernel of $\sigma(e, \_)$. Since every element of k has a square root, we can assume without loss of generality that $\sigma(x, x) = m$. The elements e and x form a basis for A.

Since

$$\begin{aligned} \epsilon(e) &= \epsilon\mu(e \otimes e) = \sigma(e, e) = m \\ \epsilon(x) &= \epsilon\mu(e \otimes x) = \sigma(e, x) = 0, \end{aligned}$$

the action of ϵ depends only on m.

Once $\mu(x \otimes x)$ is fixed, the action of μ is uniquely determined by the condition that e is a unit for μ. Write $\mu(x \otimes x)$ as a linear combination $pe + qx$. Observe that

$$\begin{aligned} \sigma(x, x) &= \epsilon\mu(x \otimes x) \\ &= p\epsilon(e) + q\epsilon(x) \\ &= pm. \end{aligned}$$

By construction, $\sigma(x,x) = m$, so $p = 1$. Thus, the only degree of freedom for μ is the value of q in the equation

$$\mu(x \otimes x) = e + qx.$$

Write $\delta(e)$ and $\delta(x)$ as linear combinations

$$\delta(e) = a\, e \otimes e + b(e \otimes x + x \otimes e) + c\, x \otimes x$$
$$\delta(x) = \tilde{a}\, e \otimes e + \tilde{b}(e \otimes x + x \otimes e) + \tilde{c}\, x \otimes x.$$

Since

$$[\epsilon \otimes \mathrm{id}]\delta(e) = ame + bmx$$
$$[\epsilon \otimes \mathrm{id}]\delta(x) = \tilde{a}me + \tilde{b}mx,$$

the condition that ϵ is a counit for δ is satisfied if and only if

$$a = \tfrac{1}{m} \qquad b = 0$$
$$\tilde{a} = 0 \qquad \tilde{b} = \tfrac{1}{m}.$$

To see when the Frobenius identity is satisfied, let's make tables of values for $[\mathrm{id} \otimes \mu][\delta \otimes \mathrm{id}]$ and $\delta\mu$. Because e is a unit for μ, the Frobenius identity is guaranteed to hold for the inputs $e \otimes e$ and $x \otimes e$, so we can omit these from our tables.

v	$[\delta \otimes \mathrm{id}](v)$	$[\mathrm{id} \otimes \mu][\delta \otimes \mathrm{id}](v)$
$e \otimes x$	$\frac{1}{m}\, e \otimes e \otimes x + c\, x \otimes x \otimes x$	$\frac{1}{m}\, e \otimes x + c\, x \otimes (e + qx)$
$x \otimes x$	$\frac{1}{m}(e \otimes x + x \otimes e) \otimes x + \tilde{c}\, x \otimes x \otimes x$	$\frac{1}{m}\, e \otimes (e + qx) + \frac{1}{m}\, x \otimes x + \tilde{c}\, x \otimes (e + qx)$

v	$\delta\mu(v)$
$e \otimes x$	$\frac{1}{m}(e \otimes x + x \otimes e) + \tilde{c}\, x \otimes x$
$x \otimes x$	$\frac{1}{m}\, e \otimes e + \frac{q}{m}(e \otimes x + x \otimes e) + (c + q\tilde{c})\, x \otimes x$

Comparing the tables, we see that the Frobenius identity is satisfied if and only if $c = \frac{1}{m}$ and $\tilde{c} = \frac{q}{m}$.

It is straightforward to verify that μ and δ are associative and coassociative, respectively, for any values of m and q. Once that is done, we have proven the following classification result.

Proposition 2. *Let k be a field in which every element has a square root. Let A be a two-dimensional k-vector space with basis $\{e, x\}$, and let $\eta\colon k \to A$ be the map $1 \mapsto e$.*

For any $q \in k$ and any nonzero $m \in k$, the linear maps defined by the equations

$$\epsilon(e) = m \qquad \epsilon(x) = 0$$

$$\mu(e \otimes e) = e \qquad \mu(x \otimes e) = x$$
$$\mu(e \otimes x) = x \qquad \mu(x \otimes x) = e + qx$$

$$\delta(e) = \tfrac{1}{m}(e \otimes e + x \otimes x) \qquad \delta(x) = \tfrac{1}{m}(e \otimes x + x \otimes e + q\, x \otimes x)$$

make A into a Frobenius algebra. In fact, every two-dimensional Frobenius algebra over k with $\sigma(e, e) \neq 0$ is of this form.

2.3 H^*-algebras

Say A is a two-dimensional H^*-algebra. Since

$$\begin{aligned}\langle e, e\rangle &= \langle \eta(1), e\rangle \\ &= \langle 1, \epsilon(e)\rangle \\ &= \epsilon(e) \\ &= \sigma(e, e),\end{aligned}$$

the positive definiteness of the inner product guarantees that $\sigma(e, e) \neq 0$, so A can be presented in the form described by Proposition 2.

We immediately see that $\langle e, e\rangle = m$, implying that m is real and positive. In addition,

$$\begin{aligned}\langle e, x\rangle &= \langle \eta(1), x\rangle \\ &= \langle 1, \epsilon(x)\rangle \\ &= 0.\end{aligned}$$

Furthermore,

$$\begin{aligned}\langle \mu(x \otimes x), e\rangle &= \langle x \otimes x, \delta(e)\rangle \\ \langle e + qx, e\rangle &= \langle x \otimes x, \tfrac{1}{m}(e \otimes e + x \otimes x)\rangle \\ m &= \tfrac{1}{m}\langle x, x\rangle^2 \\ m^2 &= \langle x, x\rangle^2,\end{aligned}$$

so $\langle x, x\rangle = m$ by positive definiteness.

On the other hand, suppose A is a two-dimensional Frobenius algebra in the form described by Proposition 2, equipped with the conjugate-symmetric bilinear form defined by

$$\langle e, e\rangle = m \qquad \langle e, x\rangle = 0 \qquad \langle x, x\rangle = m.$$

Which values of m and q make A into an H^*-algebra?

For $\langle \_, \_ \rangle$ to be an inner product, m must be real and positive. The condition that $\epsilon = \eta^\dagger$ is always satisfied, and it is straightforward to verify that $\delta = \mu^\dagger$ if and only if q is real.[2] Once that is done, we have proven the following classification result.

Proposition 3. *Let A be a two-dimensional Frobenius algebra in the form described by Proposition 2. If m is real and positive, and q is real, the inner product defined by*

$$\langle e, e \rangle = m \qquad \langle e, x \rangle = 0 \qquad \langle x, x \rangle = m$$

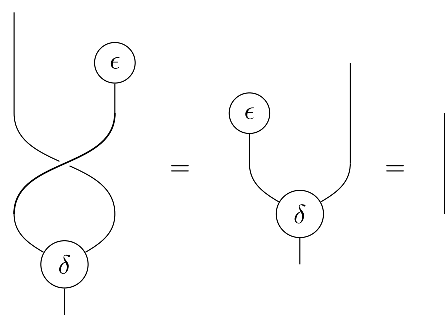
makes A into an H^*-algebra. In fact, every two-dimensional H^*-algebra over k is of this form.

A Commutativity and cocommutativity

I will show that every commutative Frobenius algebra is cocommutative, following the proof in [2]. To get the converse, turn the argument upside down.

I will assume several identities involving the twist operator. Some of them are rather subtle, so be careful to think about why they are true.

Suppose A is commutative. We get our foot in the door by showing that the "twisted coproduct" satisfies part of the Frobenius identity (Figure 1). Then we observe that



Applying these identities in just the right way, we can untwist the twisted coproduct (Figure 2).

B Lagrangian complements

Assume k does not have characteristic two.

Say V is a $2n$-dimensional vector space equipped with a t-symmetric,[3] nondegenerate bilinear form σ, and $L \subset V$ is a *Lagrangian* subspace—an n-dimensional

[2] The restriction on q comes from the equation $\langle \mu(x \otimes x), x \rangle = \langle x \otimes x, \delta(x) \rangle$.

[3] Symmetric if $t = 1$, skew-symmetric if $t = -1$.

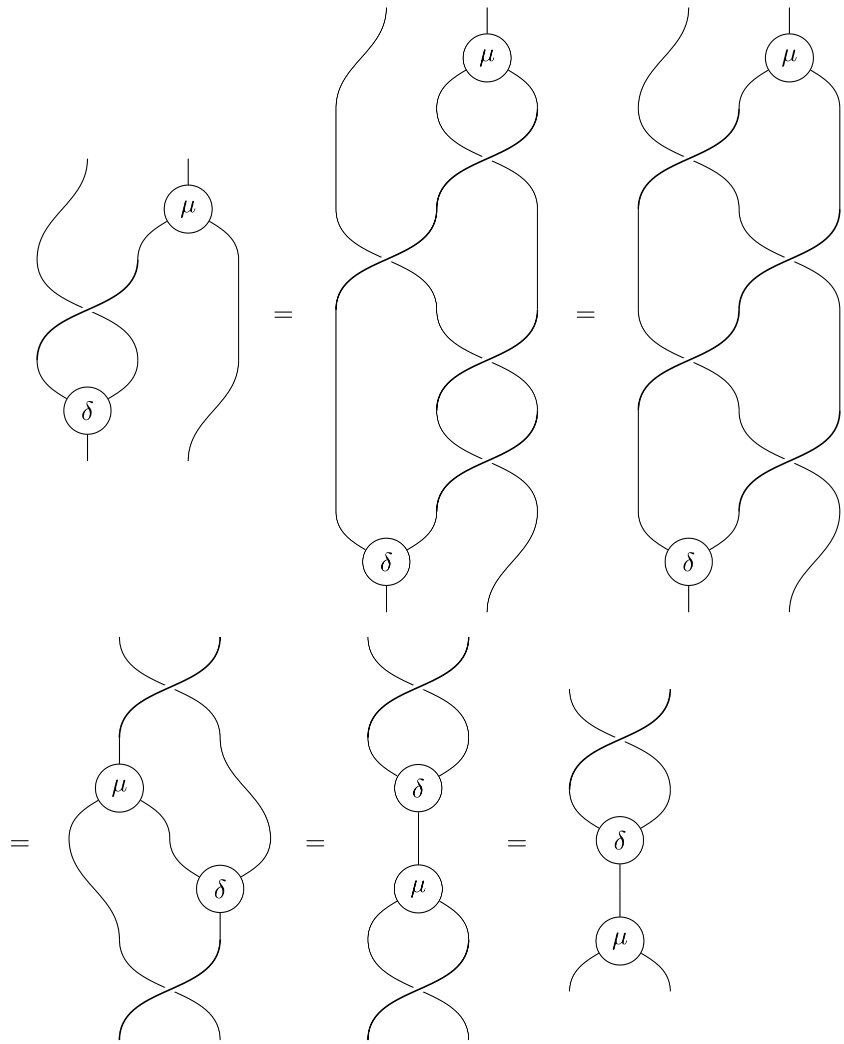


Figure 1: The twisted coproduct satisfies part of the Frobenius identity.

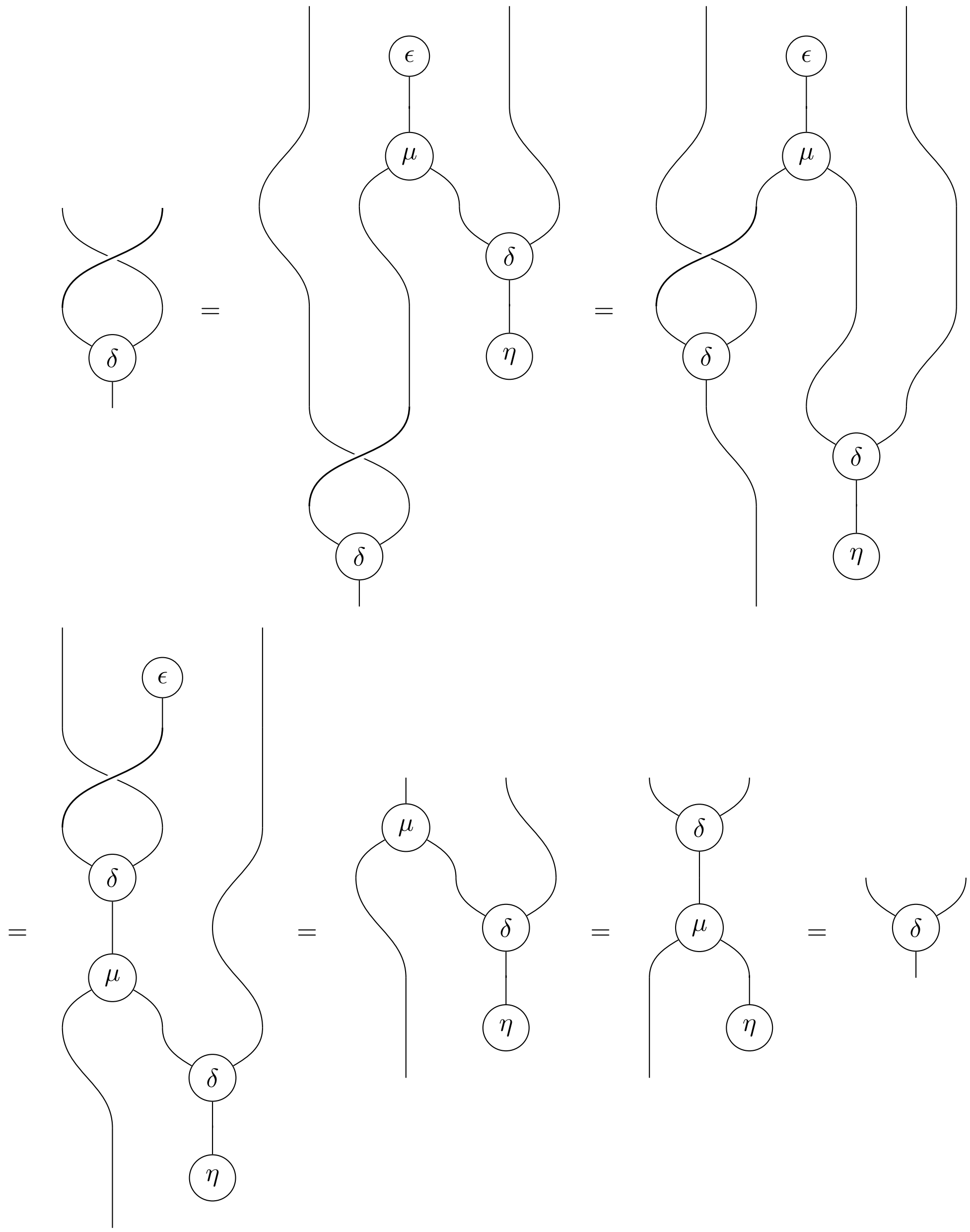


Figure 2: Untwisting the twisted coproduct.

subspace with $\sigma(\ell,\ell) = 0$ for all $\ell \in L$. Following [1, Proposition 8.2], I will give a procedure for turning any subspace $W \subset V$ complementary to L into a Lagrangian subspace of V complementary to L.

Let $S\colon V \to V'$ be the map $v \mapsto \sigma(v, \_)$. Since σ is non-degenerate, S is an isomorphism.

Let $\pi\colon V' \to W'$ be the dual of the inclusion $W \hookrightarrow V$. Observe that

$$\ker(\pi) = \{\xi \in V' : \xi(W) = 0\},$$

so

$$\ker(\pi S) = \{v \in V : S(v)(W) = 0\}.$$

Being the dual of an injection, π is surjective, so

$$\dim \ker(\pi) = \dim V' - \dim W' = n.$$

Since S is an isomorphism, $\ker(\pi S)$ has dimension n as well.

Any vector ℓ in the intersection of $\ker(\pi S)$ and L must have $S(\ell)(W) = 0$ and $S(\ell)(L) = 0$. Since W and L are complements, it follows that $S(\ell) = 0$, which means $\ell = 0$. Therefore, $\ker(\pi S)$ intersects L only at zero.

Think of V as $W \oplus L$. Since $\ker(\pi S)$ intersects L only at zero, it is the graph of a linear map $F\colon W \to L$. Notice that for any $w, \tilde{w} \in W$,

$$S(w + Fw)(\tilde{w}) = 0,$$

because $w + Fw \in \ker(\pi S)$. Therefore,

$$\sigma(Fw, \tilde{w}) = -\sigma(w, \tilde{w})$$

for all $w, \tilde{w} \in W$.

I claim that the graph of $\frac{1}{2}F$ is a Lagrangian subspace of V complementary to L. To see why, pick any $w, \tilde{w} \in W$, and observe that

$$\begin{aligned}
\sigma(w + \tfrac{1}{2}Fw, \tilde{w} + \tfrac{1}{2}F\tilde{w}) &= \sigma(w,\tilde{w}) + \tfrac{1}{2}\sigma(w, F\tilde{w}) + \tfrac{1}{2}\sigma(Fw,\tilde{w}) + \tfrac{1}{4}\sigma(Fw, F\tilde{w}) \\
&= \sigma(w,\tilde{w}) + \tfrac{1}{2}t\sigma(F\tilde{w}, w) + \tfrac{1}{2}\sigma(Fw,\tilde{w}) + 0 \\
&= \sigma(w,\tilde{w}) - \tfrac{1}{2}t\sigma(\tilde{w}, w) - \tfrac{1}{2}\sigma(w,\tilde{w}) \\
&= \sigma(w,\tilde{w}) - \tfrac{1}{2}\sigma(w,\tilde{w}) - \tfrac{1}{2}\sigma(w,\tilde{w}) \\
&= 0.
\end{aligned}$$

The graph of F has dimension n, so the graph of $\frac{1}{2}F$ also has dimension n. Therefore, the graph of $\frac{1}{2}F$ is a Lagrangian subspace of V. Since $\frac{1}{2}F$ is a map to L, its graph is complementary to L.

References

[1] Ana Cannas da Silva. *Lectures on symplectic geometry.* Springer, 2008.

[2] Joachim Kock. *Frobenius Algebras and 2D Topological Quantum Field Theories.* Cambridge University Press, 2003.