The basics of homology

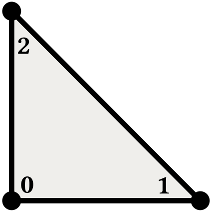
Aaron Fenyes (IHÉS)

Young Data Scientist Seminar
Harvard, November 2020

Building geometric spaces

standard 0-simplex

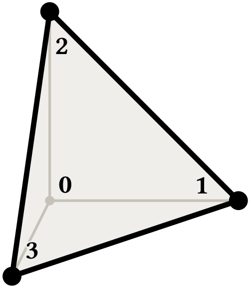
standard 1-simplex



standard 2-simplex

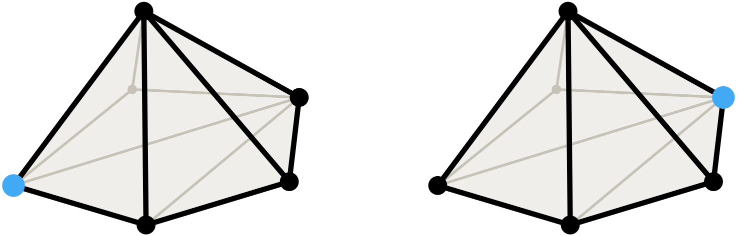
A Δ-*complex* is a space built from simplices, which attach to each other by sharing faces.

(In a *simplicial complex*, no two simplices have the same set of vertices.)



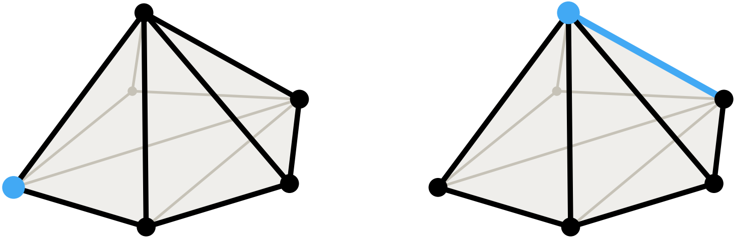
standard 3-simplex

Investigating deformations
of points



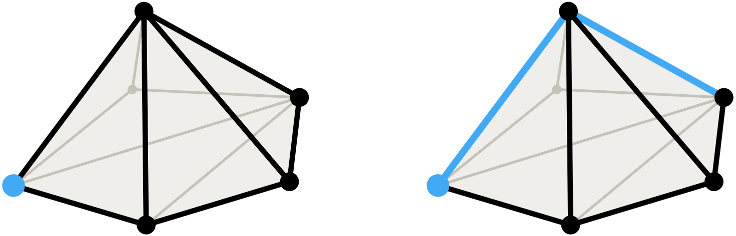
Can these points be deformed into each other?

Investigating deformations
of points



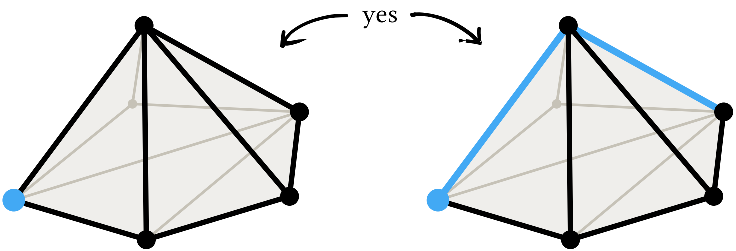
Can these points be deformed into each other?

Investigating deformations
of points



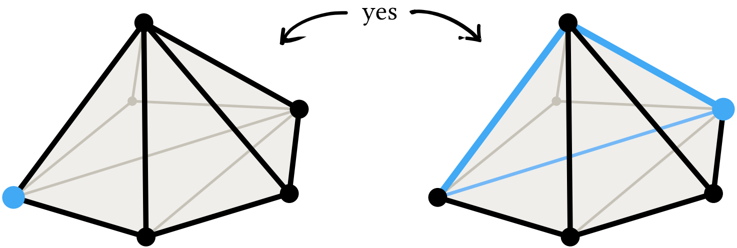
Can these points be deformed into each other?

Investigating deformations
of points



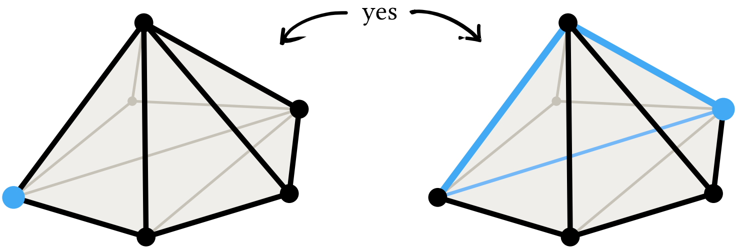
Can these points be deformed into each other?

Investigating deformations
of points



Can these points be deformed into each other?

Investigating deformations
of points

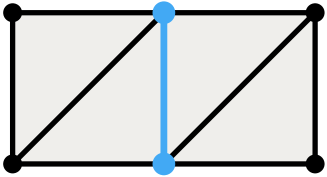


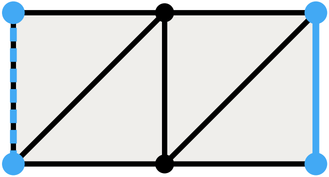
Can these points be deformed into each other?

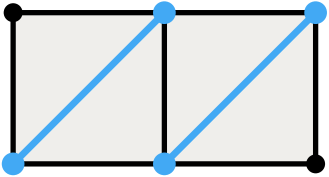
It's easy to decide.

Investigating deformations
of loops

(Identify opposite edges)



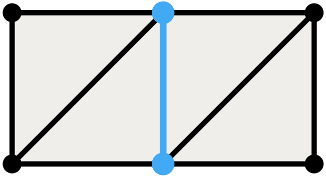


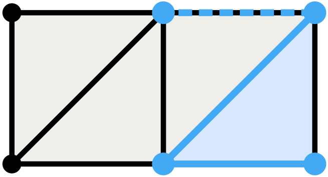


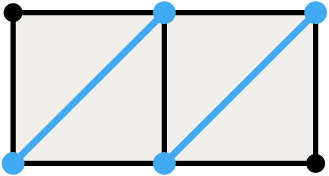
Can these loops be deformed into each other?

Investigating deformations
of loops

(Identify opposite edges)



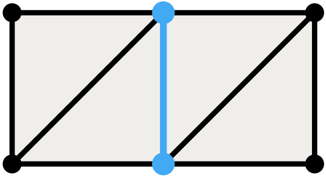


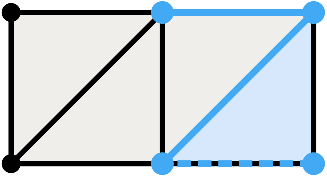


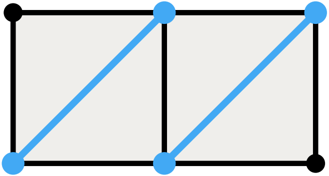
Can these loops be deformed into each other?

Investigating deformations
of loops

(Identify opposite edges)



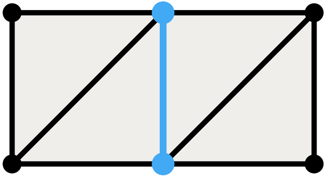


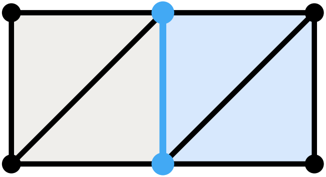


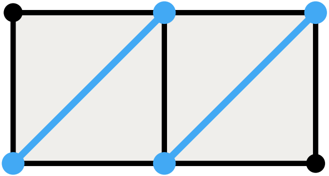
Can these loops be deformed into each other?

Investigating deformations
of loops

(Identify opposite edges)



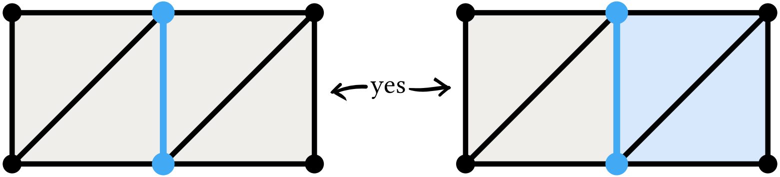


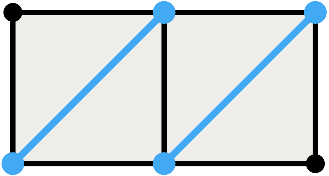


Can these loops be deformed into each other?

Investigating deformations
of loops

(Identify opposite edges)

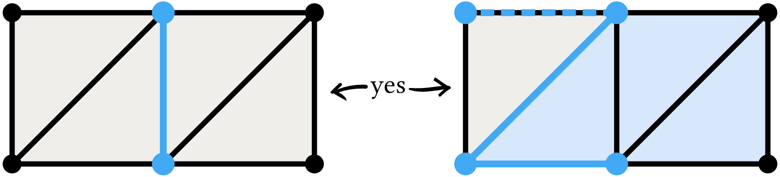


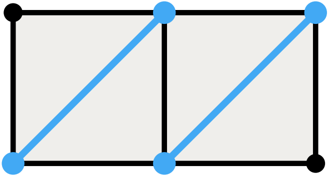


Can these loops be deformed into each other?

Investigating deformations
of loops

(Identify opposite edges)

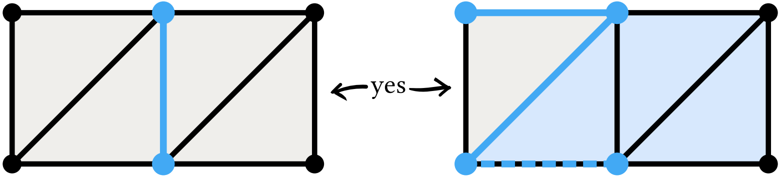


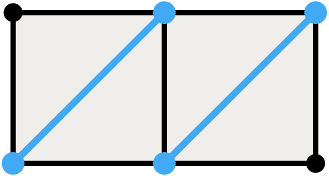


Can these loops be deformed into each other?

Investigating deformations
of loops

(Identify opposite edges)

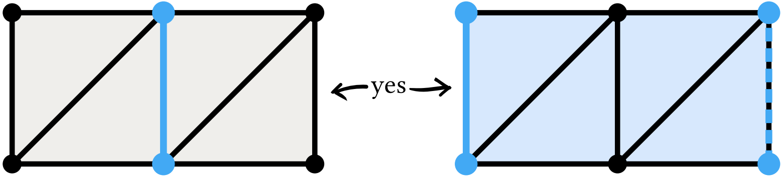


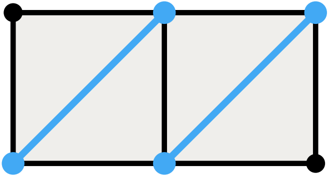


Can these loops be deformed into each other?

Investigating deformations
of loops

(Identify opposite edges)

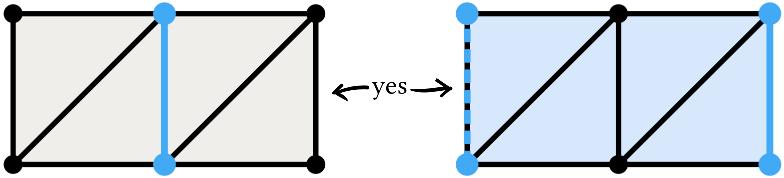


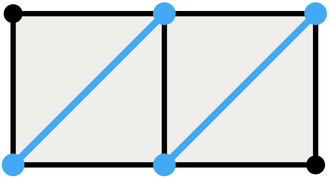


Can these loops be deformed into each other?

Investigating deformations
of loops

(Identify opposite edges)

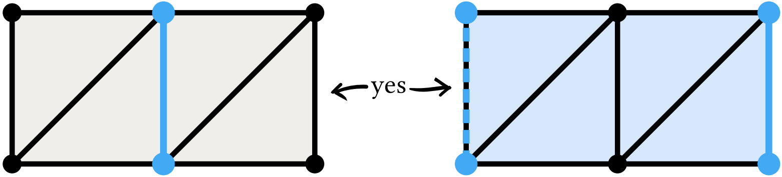


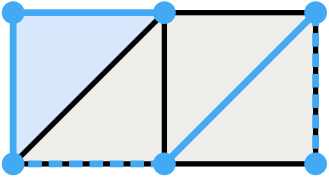


Can these loops be deformed into each other?

Investigating deformations
of loops

(Identify opposite edges)

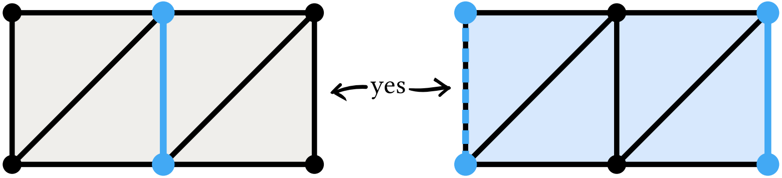


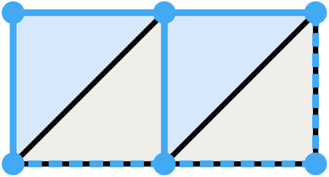


Can these loops be deformed into each other?

Investigating deformations
of loops

(Identify opposite edges)

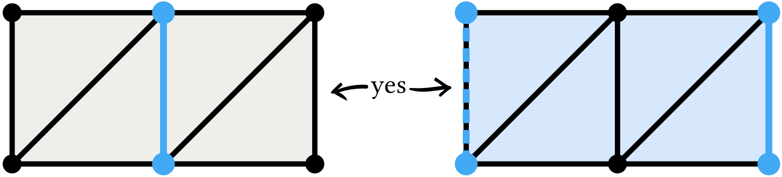


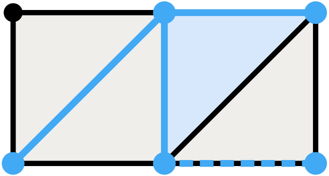


Can these loops be deformed into each other?

Investigating deformations
of loops

(Identify opposite edges)

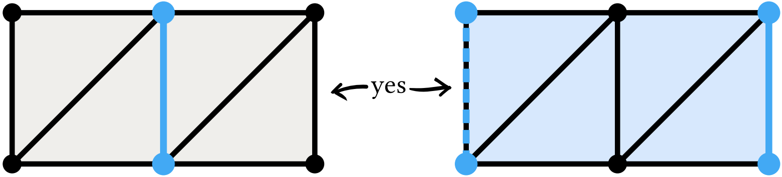


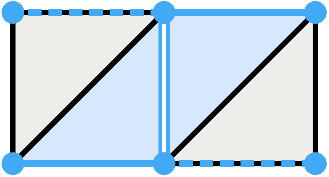


Can these loops be deformed into each other?

Investigating deformations
of loops

(Identify opposite edges)

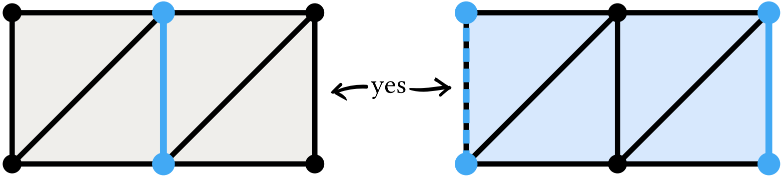


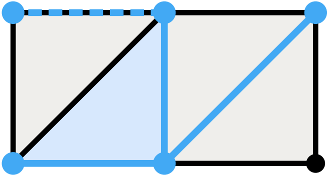


Can these loops be deformed into each other?

Investigating deformations
of loops

(Identify opposite edges)

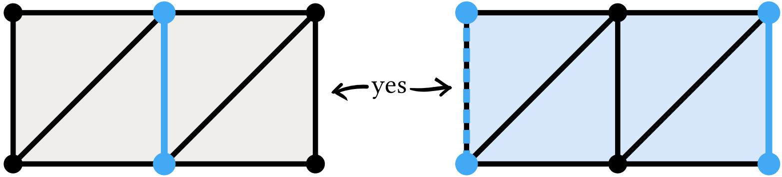


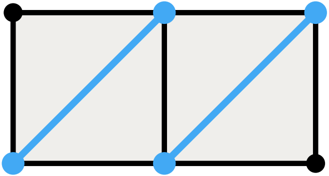


Can these loops be deformed into each other?

Investigating deformations
of loops

(Identify opposite edges)

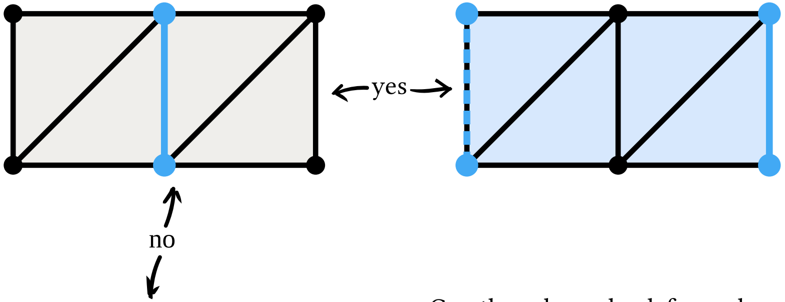


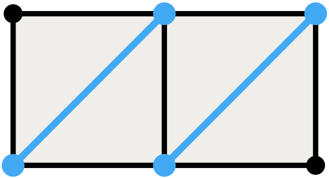


Can these loops be deformed into each other?

Investigating deformations
of loops

(Identify opposite edges)

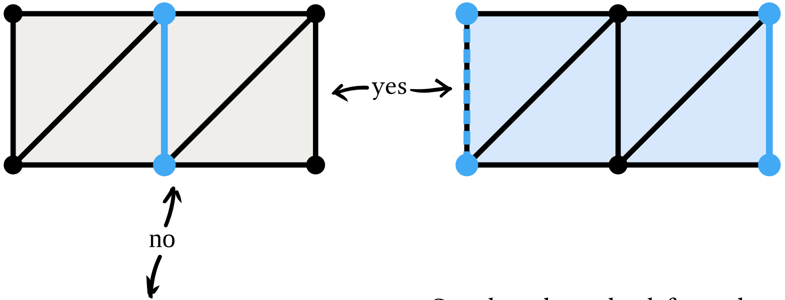


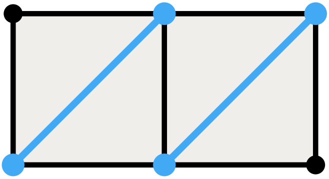


Can these loops be deformed into each other?

Investigating deformations
of loops

(Identify opposite edges)

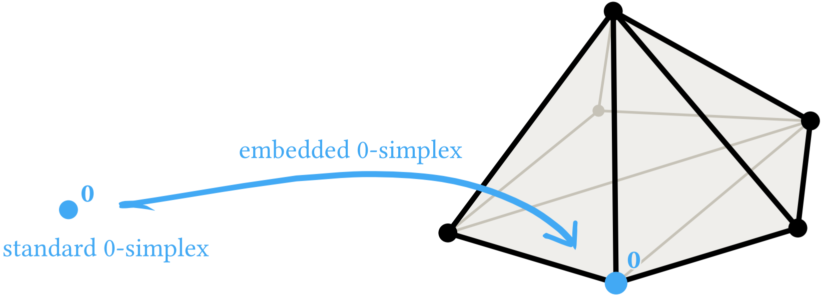




Can these loops be deformed into each other?

Homology gives a systematic way to decide.

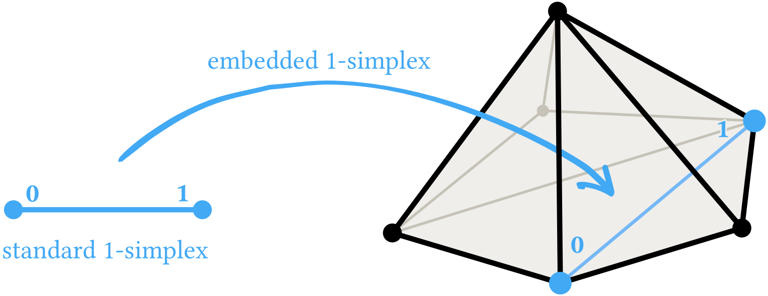
Turning geometry into algebra



An *embedded simplex* is a map that sends the standard n-simplex to one of the n-simplices of a Δ-complex.

Ordering the vertices of the standard n-simplex makes it easy to see where the map sends each vertex.

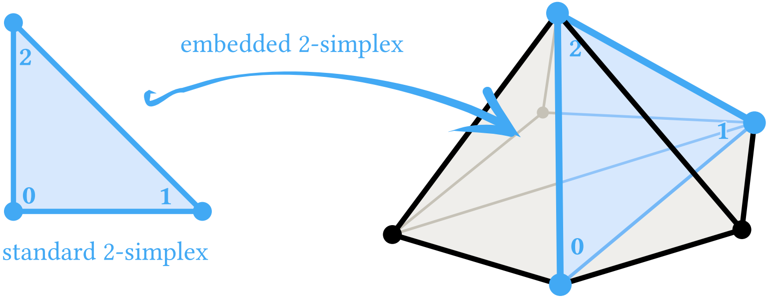
Turning geometry into algebra



An *embedded simplex* is a map that sends the standard n-simplex to one of the n-simplices of a Δ-complex.

Ordering the vertices of the standard n-simplex makes it easy to see where the map sends each vertex.

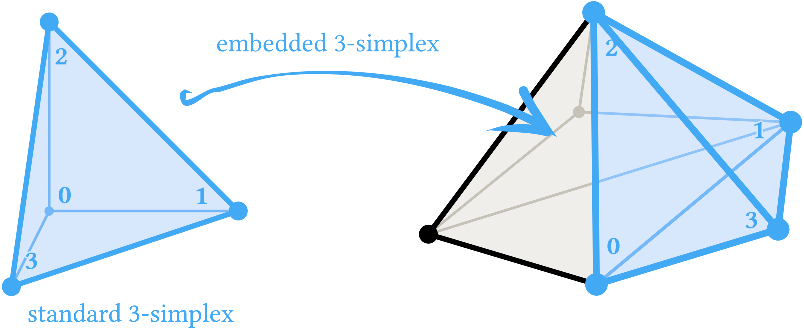
Turning geometry into algebra



An *embedded simplex* is a map that sends the standard n-simplex to one of the n-simplices of a Δ-complex.

Ordering the vertices of the standard n-simplex makes it easy to see where the map sends each vertex.

Turning geometry into algebra

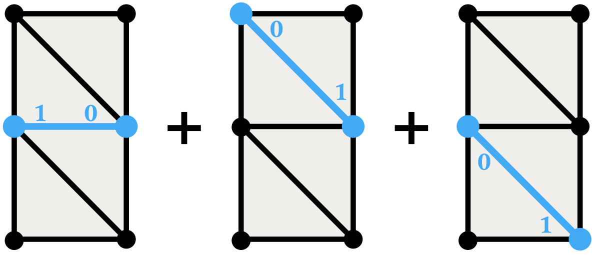


An *embedded simplex* is a map that sends the standard n-simplex to one of the n-simplices of a Δ-complex.

Ordering the vertices of the standard n-simplex makes it easy to see where the map sends each vertex.

Turning geometry into algebra

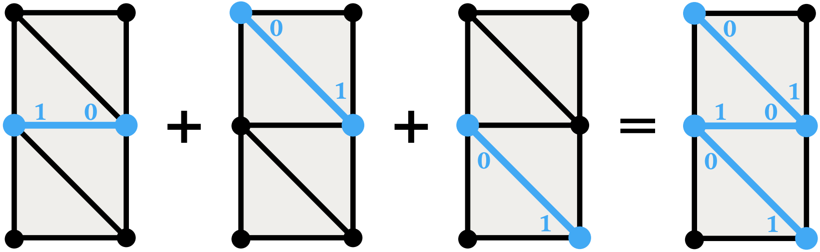
(Identify opposite edges)



An *n-chain* is a “formal sum” of embedded *n*-simplices.

Turning geometry into algebra

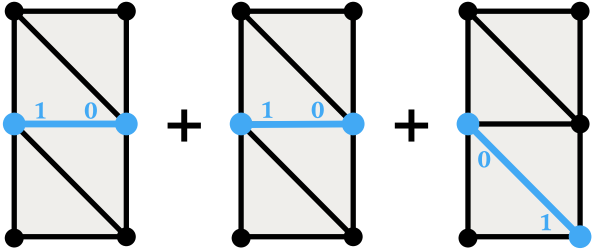
(Identify opposite edges)



An *n-chain* is a “formal sum” of embedded *n*-simplices.

Turning geometry into algebra

(Identify opposite edges)

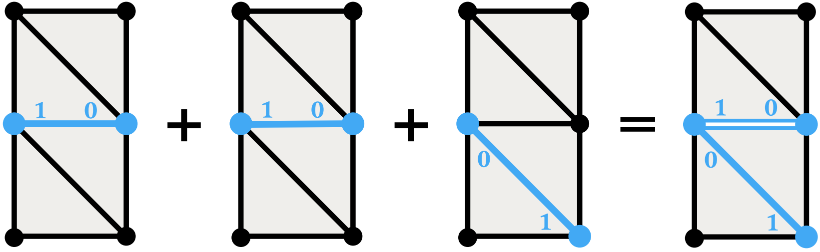


An *n-chain* is a “formal sum” of embedded *n*-simplices.

A simplex can appear more than once in the sum.

Turning geometry into algebra

(Identify opposite edges)

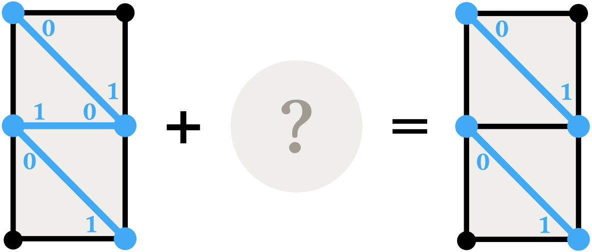


An *n-chain* is a “formal sum” of embedded *n*-simplices.

A simplex can appear more than once in the sum.

Turning geometry into algebra

(Identify opposite edges)



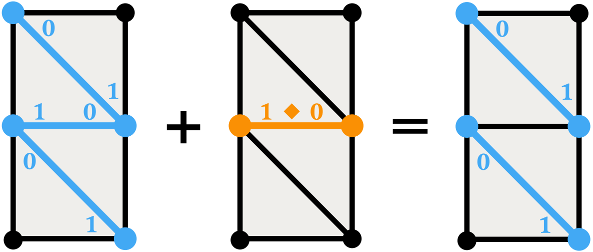
An *n-chain* is a “formal sum” of embedded *n*-simplices.

A simplex can appear more than once in the sum.

For algebraic convenience, we allow negative simplices.

Turning geometry into algebra

(Identify opposite edges)

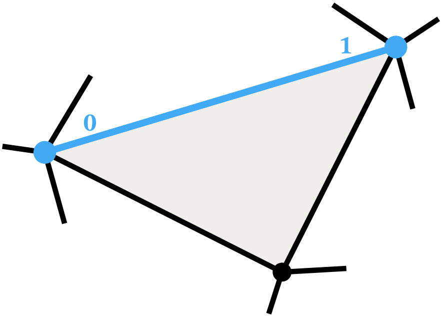


An *n-chain* is a “formal sum” of embedded *n*-simplices.

A simplex can appear more than once in the sum.

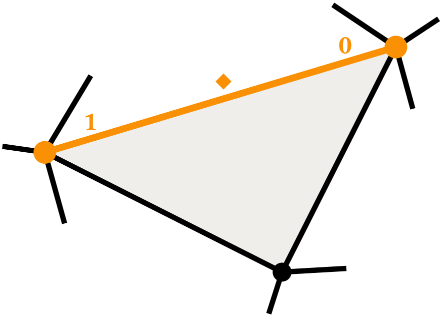
For algebraic convenience, we allow negative simplices.

Turning geometry into algebra



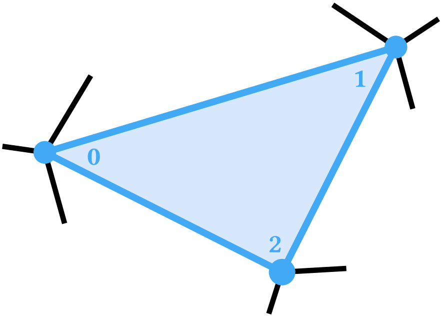
Sign rule: swapping two vertices of an embedded simplex is equivalent to reversing its sign.

Turning geometry into algebra



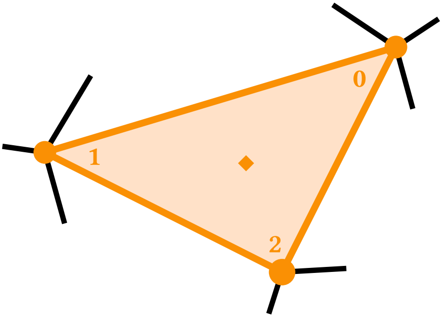
Sign rule: swapping two vertices of an embedded simplex is equivalent to reversing its sign.

Turning geometry into algebra



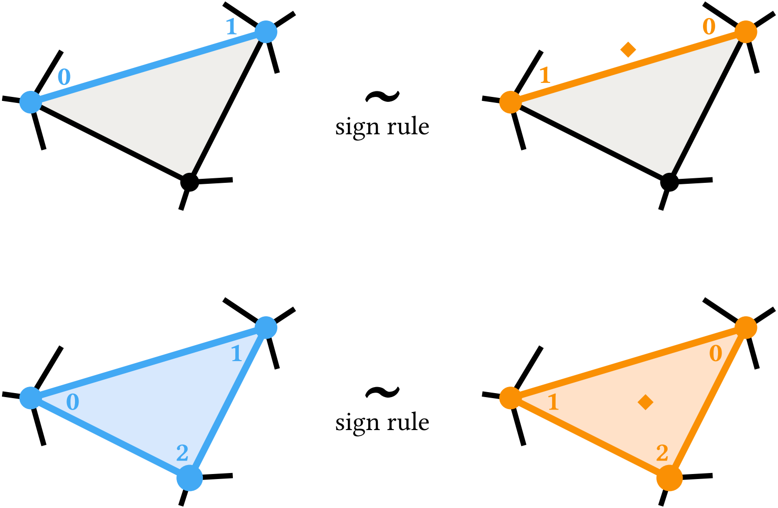
Sign rule: swapping two vertices of an embedded simplex is equivalent to reversing its sign.

Turning geometry into algebra

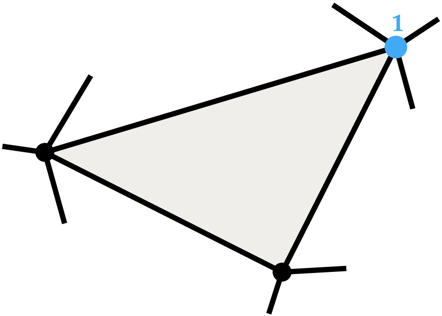


Sign rule: swapping two vertices of an embedded simplex is equivalent to reversing its sign.

Turning geometry into algebra

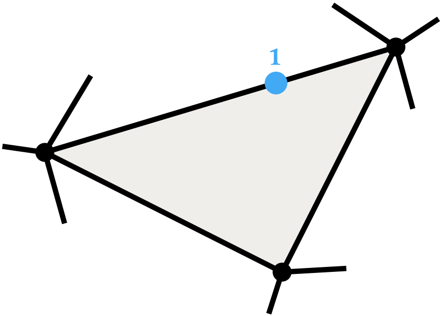


Describing deformations
as substitution rules



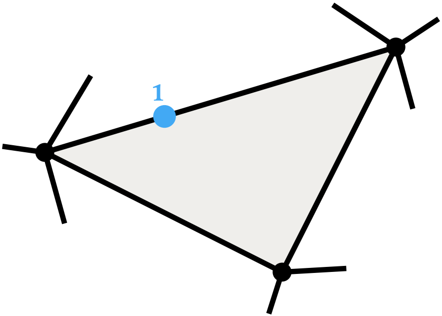
You can deform a 0-simplex by pushing it across a 1-simplex.

Describing deformations
as substitution rules



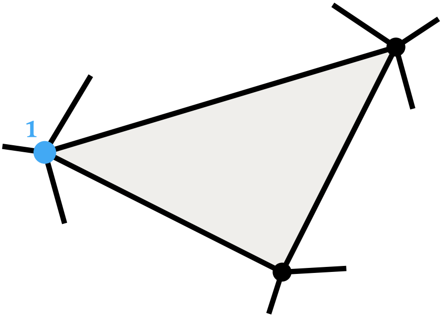
You can deform a 0-simplex by pushing it across a 1-simplex.

Describing deformations
as substitution rules



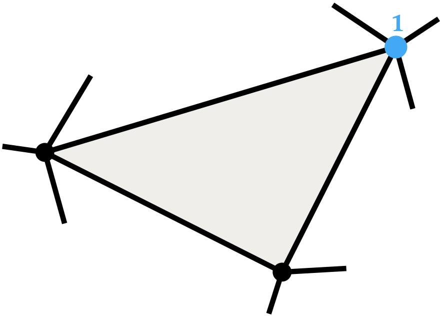
You can deform a 0-simplex by pushing it across a 1-simplex.

Describing deformations
as substitution rules



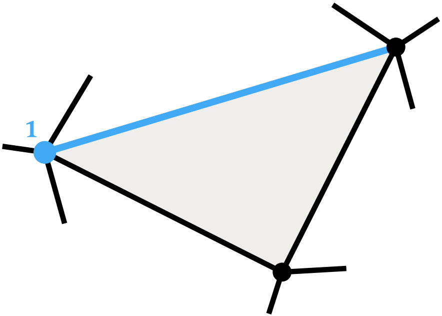
You can deform a 0-simplex by pushing it across a 1-simplex.

Describing deformations
as substitution rules



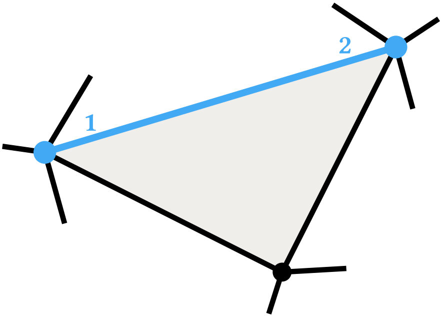
You can deform a 0-simplex by pushing it across a 1-simplex.

Describing deformations
as substitution rules



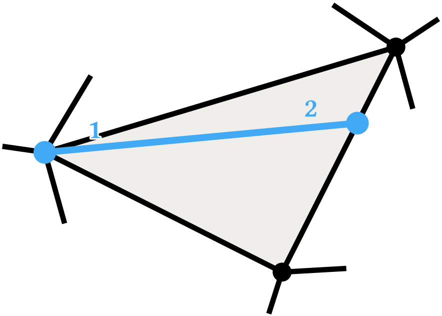
You can deform a 0-simplex by pushing it across a 1-simplex.

Describing deformations
as substitution rules



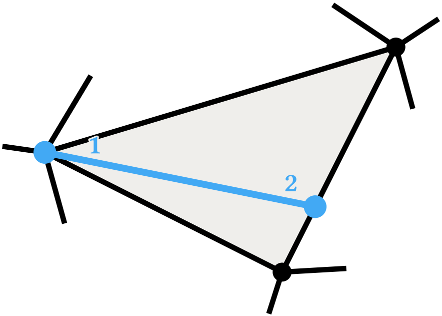
You can deform a 1-simplex by pushing it across a 2-simplex.

Describing deformations
as substitution rules



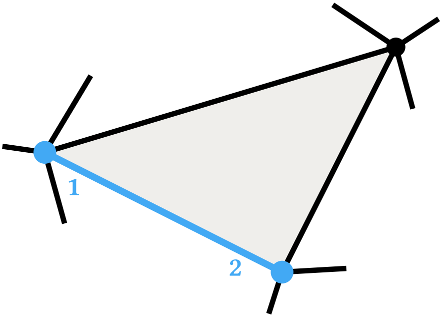
You can deform a 1-simplex by pushing it across a 2-simplex.

Describing deformations
as substitution rules



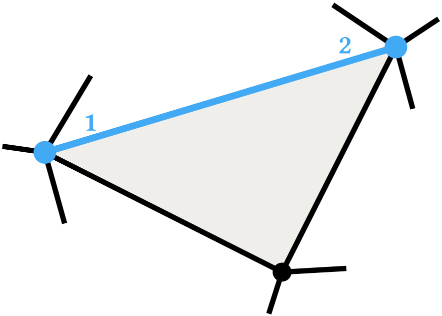
You can deform a 1-simplex by pushing it across a 2-simplex.

Describing deformations
as substitution rules



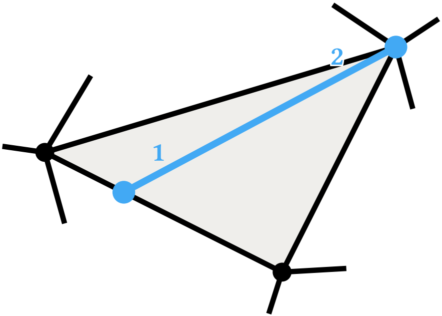
You can deform a 1-simplex by pushing it across a 2-simplex.

Describing deformations
as substitution rules



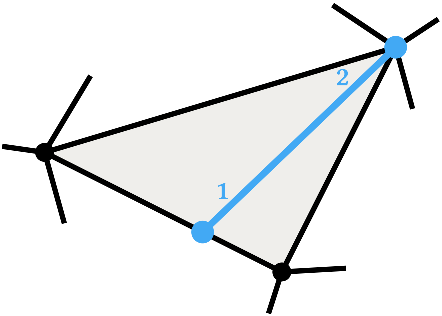
You can deform a 1-simplex by pushing it across a 2-simplex.

Describing deformations
as substitution rules



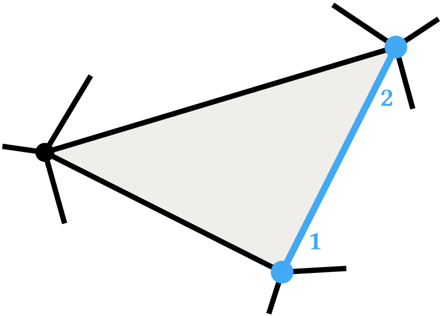
You can deform a 1-simplex by pushing it across a 2-simplex.

Describing deformations
as substitution rules



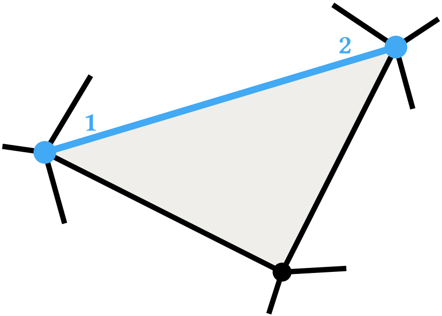
You can deform a 1-simplex by pushing it across a 2-simplex.

Describing deformations
as substitution rules



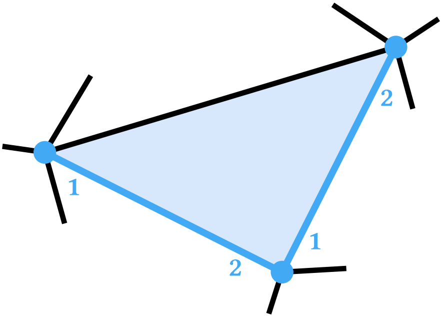
You can deform a 1-simplex by pushing it across a 2-simplex.

Describing deformations
as substitution rules



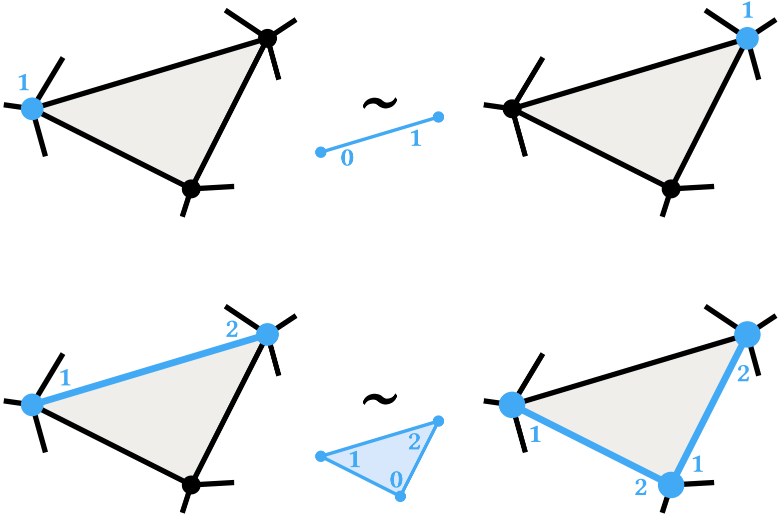
You can deform a 1-simplex by pushing it across a 2-simplex.

Describing deformations
as substitution rules

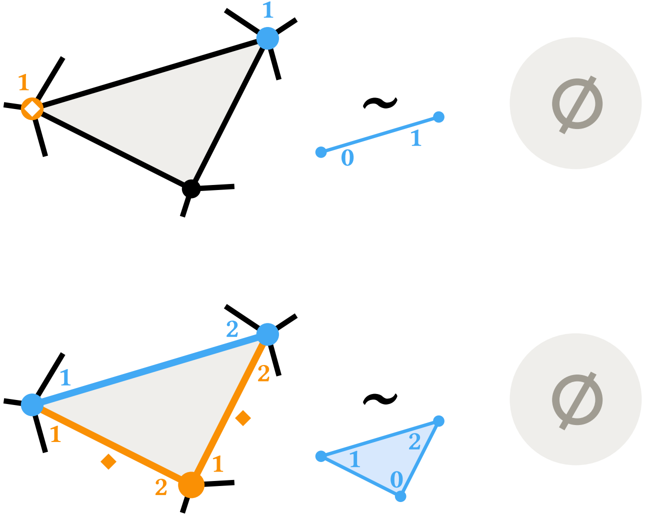


You can deform a 1-simplex by pushing it across a 2-simplex.

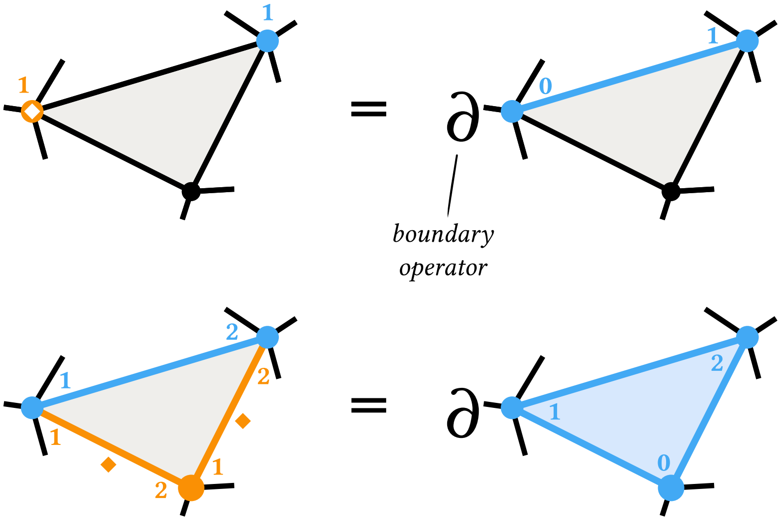
Describing deformations
as substitution rules



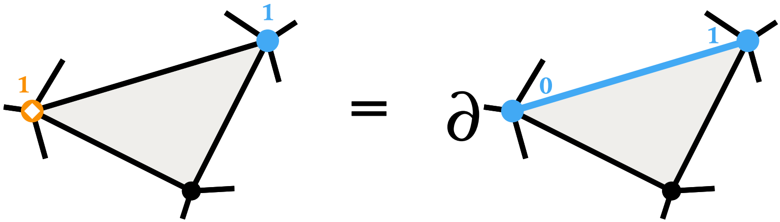
Describing deformations
as substitution rules



Keeping track
of deformation substitutions

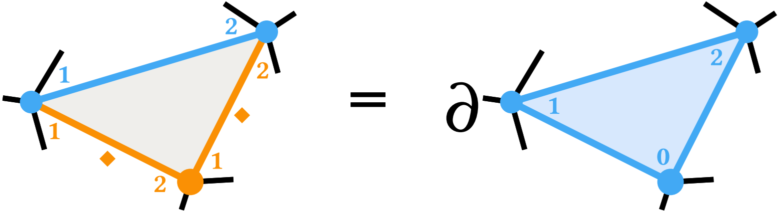


Keeping track
of deformation substitutions



The *boundary operator* sends each embedded $(n+1)$-simplex
to the n-chain that describes deformation across the simplex.

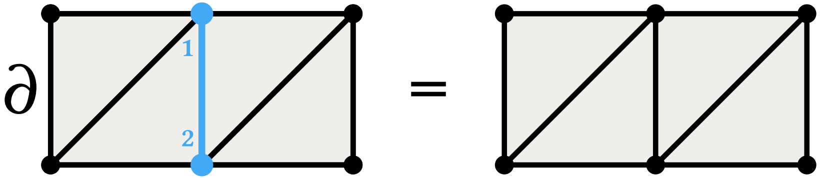
Keeping track
of deformation substitutions



The *boundary operator* sends each embedded (n+1)-simplex to the n-chain that describes deformation across the simplex.

Deformation problems
as algebra problems

(Identify opposite edges)

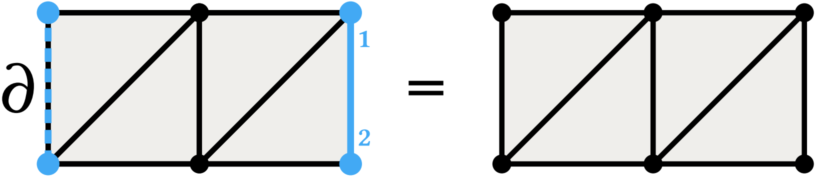


The boundary operator, unexpectedly, characterizes closed loops, and "closed-up shapes" in higher dimensions.

An n-chain C is called an *n-cycle* if $\partial C = 0$.

Deformation problems
as algebra problems

(Identify opposite edges)

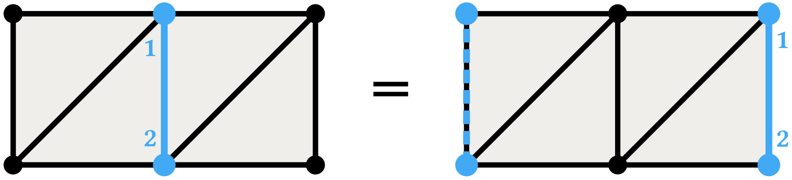


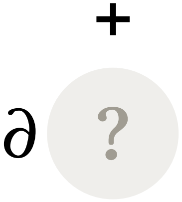
The boundary operator, unexpectedly, characterizes closed loops, and "closed-up shapes" in higher dimensions.

An n-chain C is called an *n-cycle* if $\partial C = 0$.

Deformation problems
as algebra problems

(Identify opposite edges)



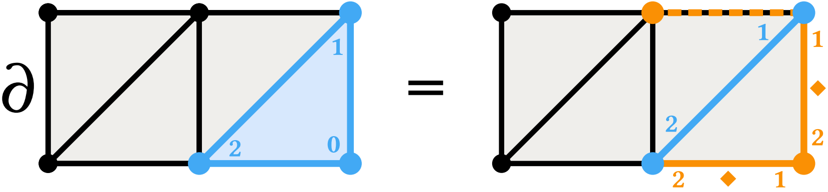


For n-cycles A, B, these are equivalent:

- We can deform B into A.
- We can turn B into A algebrically using deformation substitutions.
- $A = B + \partial F$ for some $(n+1)$-chain F.

Deformation problems
as algebra problems

(Identify opposite edges)

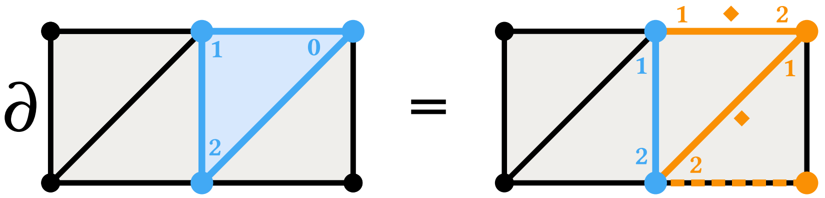


For n-cycles A, B, these are equivalent:

- We can deform B into A.
- We can turn B into A algebrically using deformation substitutions.
- $A = B + \partial F$ for some $(n+1)$-chain F.

Deformation problems
as algebra problems

(Identify opposite edges)

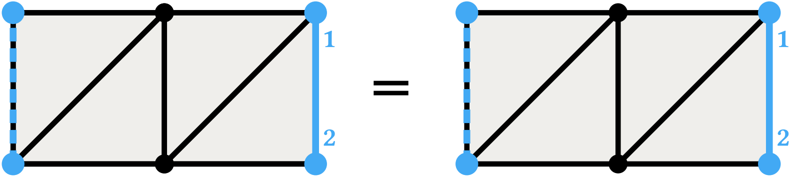


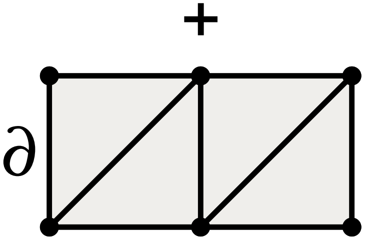
For n-cycles A, B, these are equivalent:

- We can deform B into A.
- We can turn B into A algebrically using deformation substitutions.
- $A = B + \partial F$ for some $(n+1)$-chain F.

Deformation problems
as algebra problems

(Identify opposite edges)



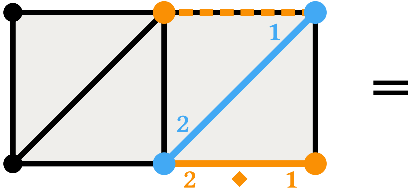


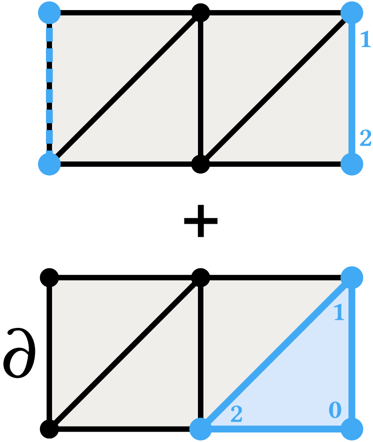
For n-cycles A, B, these are equivalent:

- We can deform B into A.
- We can turn B into A algebrically using deformation substitutions.
- $A = B + \partial F$ for some $(n+1)$-chain F.

Deformation problems
as algebra problems

(Identify opposite edges)



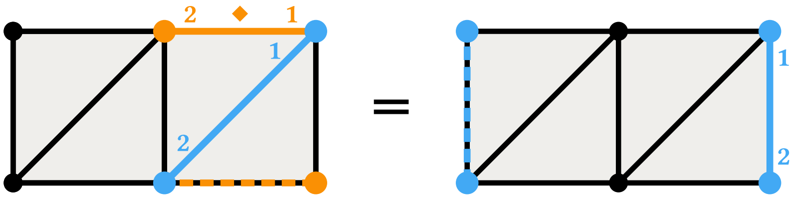


For n-cycles A, B, these are equivalent:

- We can deform B into A.
- We can turn B into A algebrically using deformation substitutions.
- $A = B + \partial F$ for some $(n+1)$-chain F.

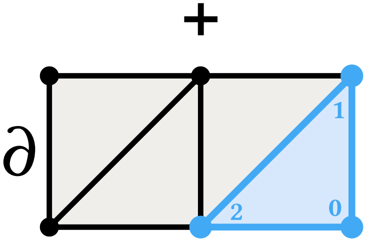
Deformation problems
as algebra problems

(Identify opposite edges)



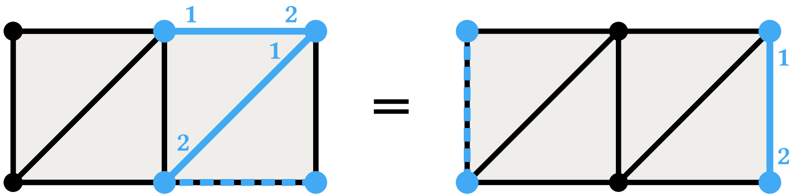
For n-cycles A, B, these are equivalent:

- We can deform B into A.
- We can turn B into A algebrically using deformation substitutions.
- $A = B + \partial F$ for some $(n{+}1)$-chain F.



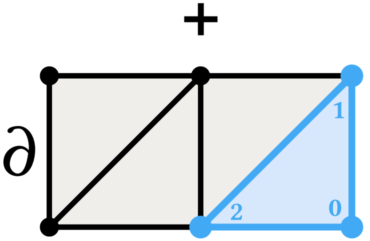
Deformation problems
as algebra problems

(Identify opposite edges)



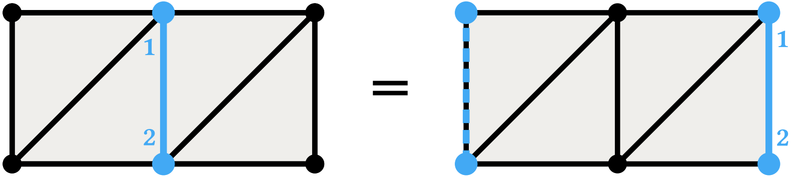
For n-cycles A, B, these are equivalent:

- We can deform B into A.
- We can turn B into A algebrically using deformation substitutions.
- $A = B + \partial F$ for some $(n{+}1)$-chain F.



Deformation problems
as algebra problems

(Identify opposite edges)



For n-cycles A, B, these are equivalent:

- We can deform B into A.
- We can turn B into A algebrically using deformation substitutions.
- $A = B + \partial F$ for some $(n+1)$-chain F.