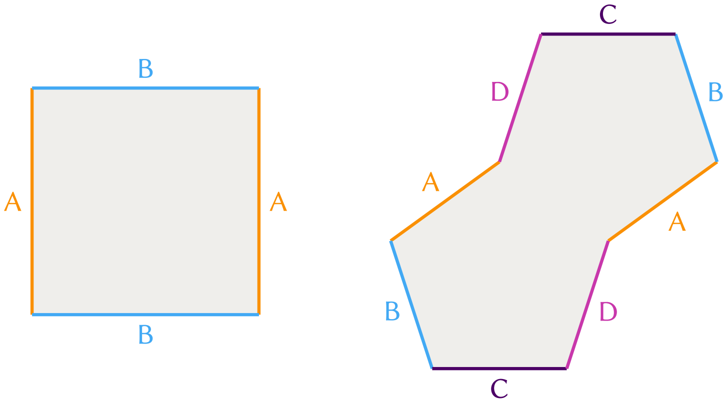
# Pleated hyperbolic surfaces in condensed matter physics

Aaron Fenyes (IHÉS)

Heidelberg Geometry Seminar

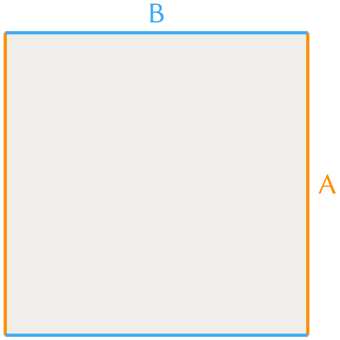
16 February, 2020

# Translation surfaces




Build a *translation surface* $\Sigma$ by gluing polygons along parallel sides.

# Toy 1D quasicrystals from flat surfaces



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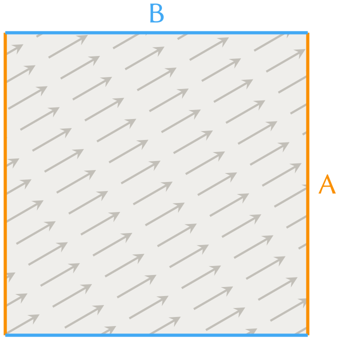


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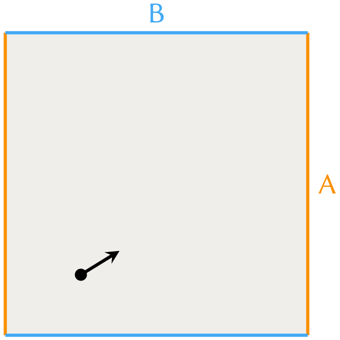


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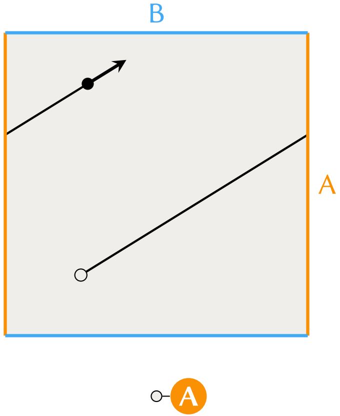


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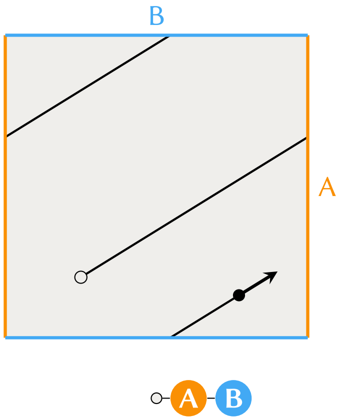


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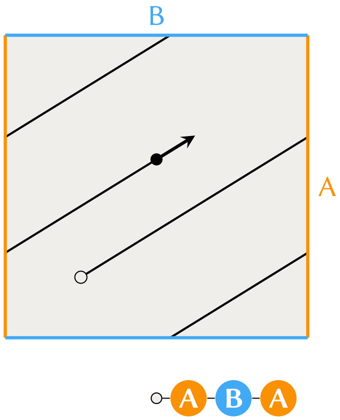


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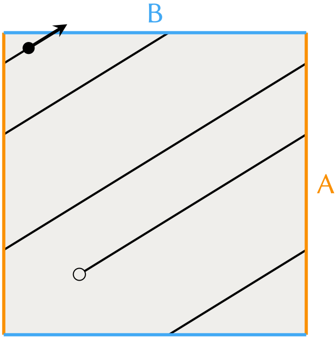



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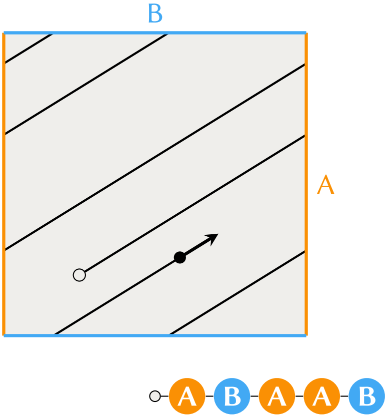


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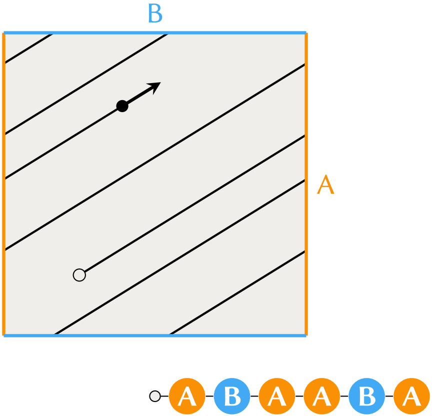


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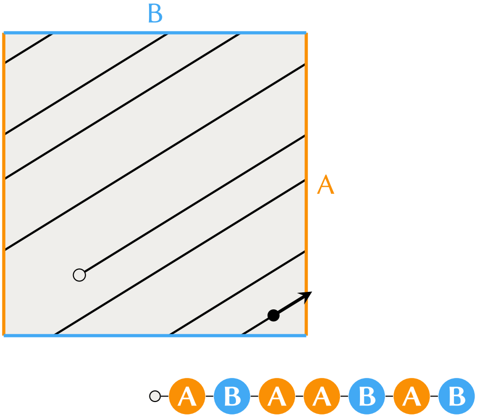


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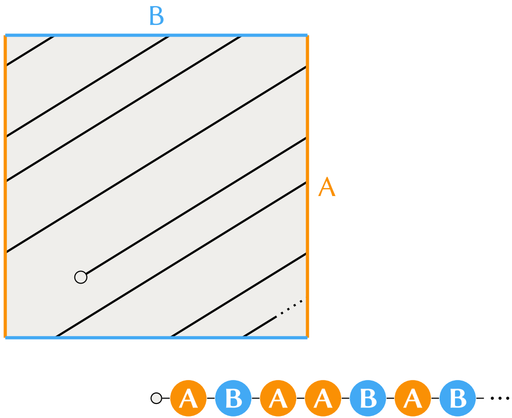


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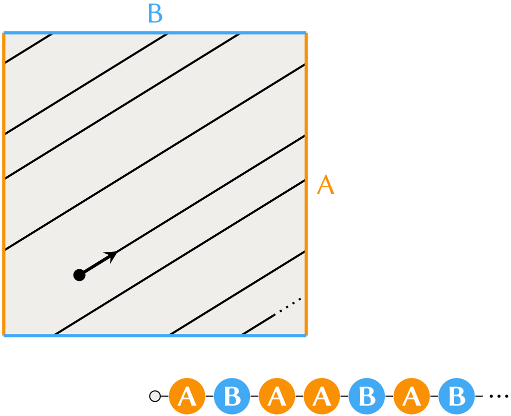


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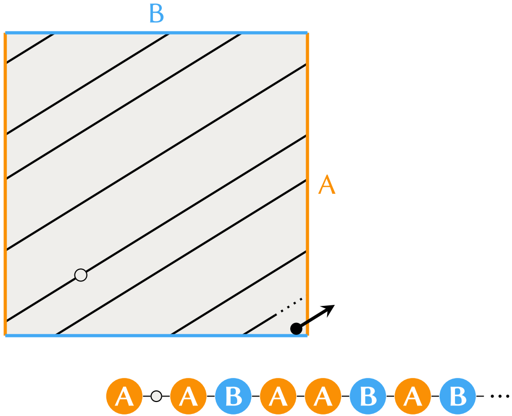


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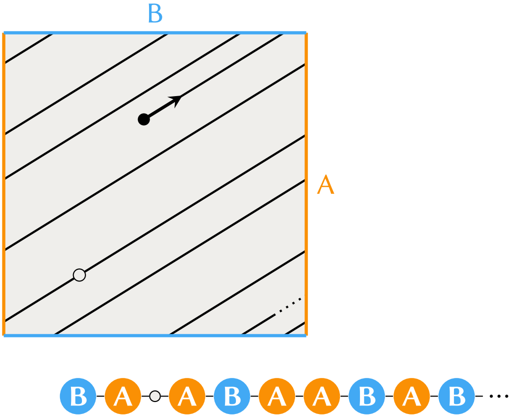


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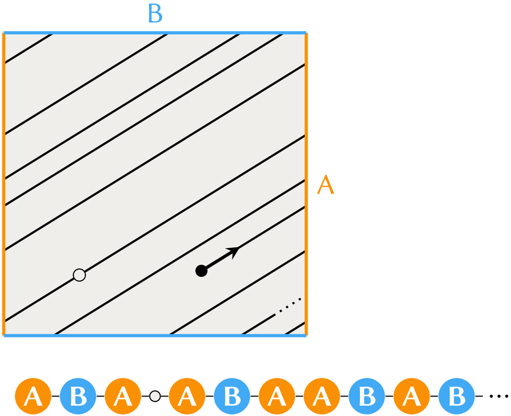


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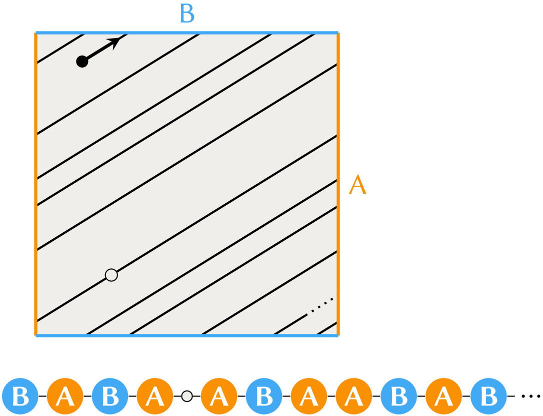


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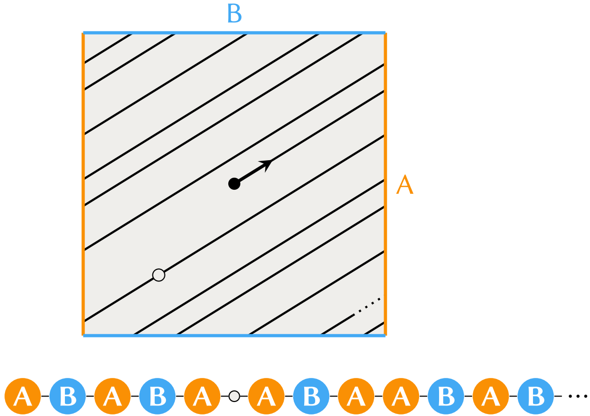


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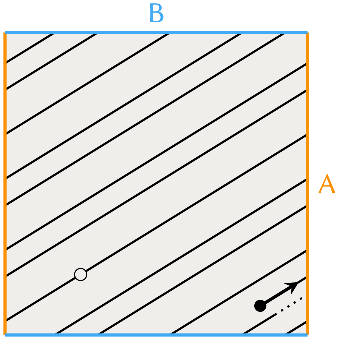



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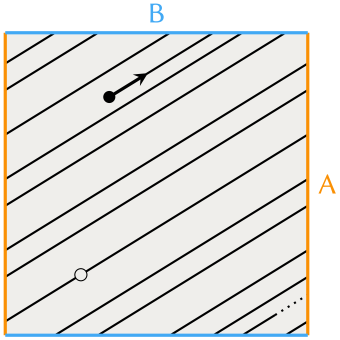


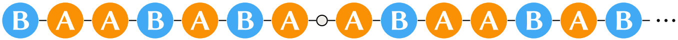


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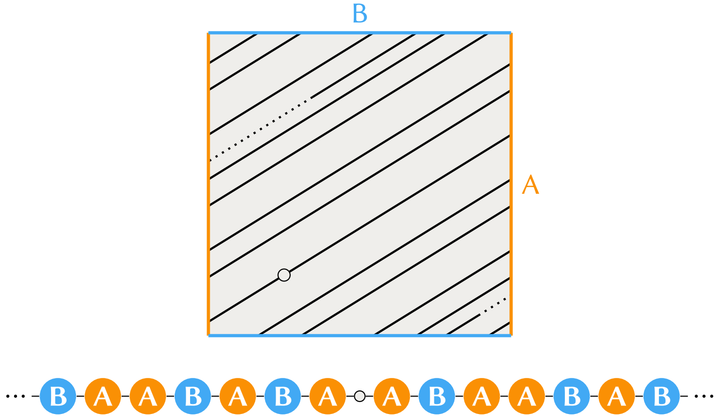


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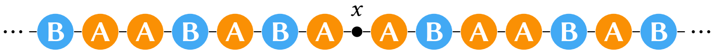
# Cutting sequences are quasiperiodic

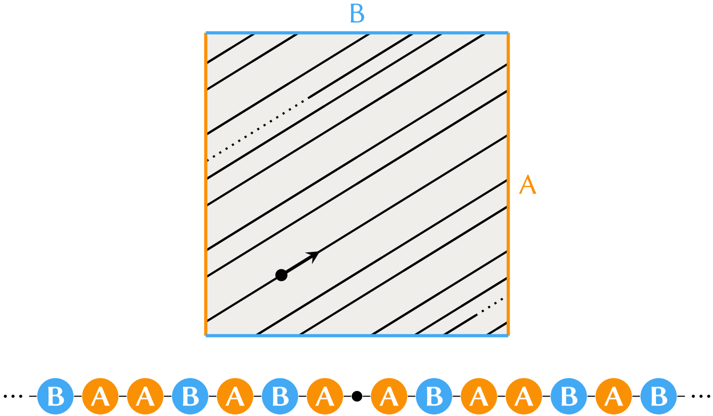



Every word you see in a cutting sequence has occurred before, and will occur again.

The distance to the next occurrance is bounded above and below.

This is a kind of *quasiperiodicity*.

# Family resemblance among cutting sequences

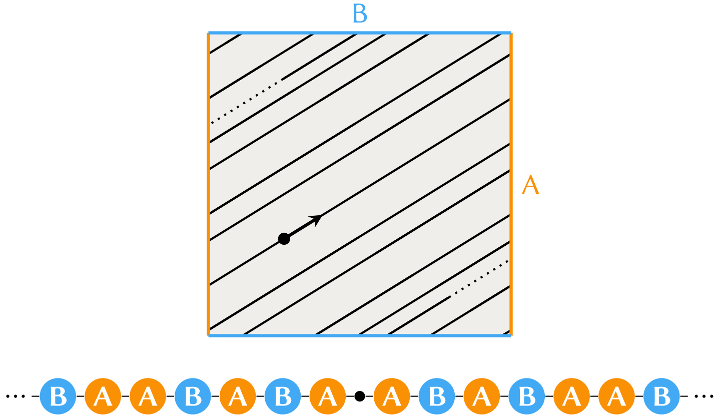



Varying $x \in \Sigma$ gives a family of cutting sequences.

They all contain the same words.

Each word's upper and lower periods are uniform over the family.

# Family resemblance among cutting sequences

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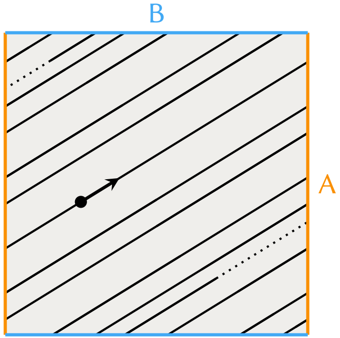


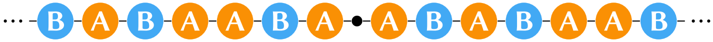


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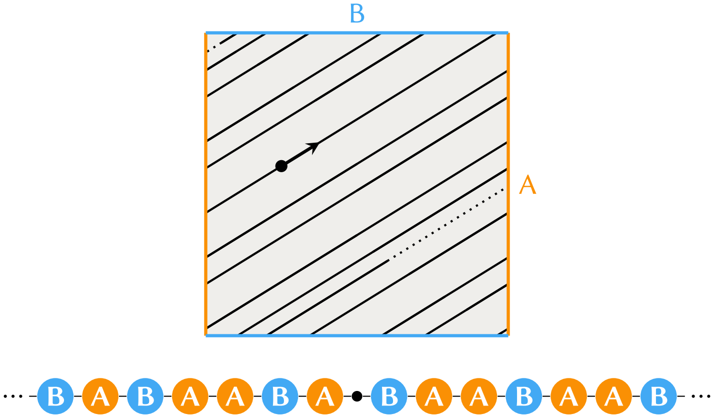


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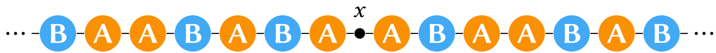
# Cutting sequences as quasicrystals




Imagine A and B are types of atoms.

Then a cutting sequence is a quasiperiodic chain of atoms.

Physicists call this a one-dimensional *quasicrystal*.

Let’s investigate its physical properties.

We’ll model the motion of an electron hopping from atom to atom.

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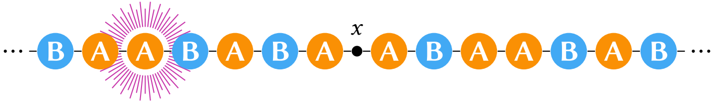


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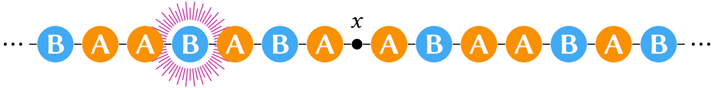


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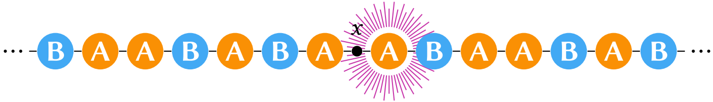


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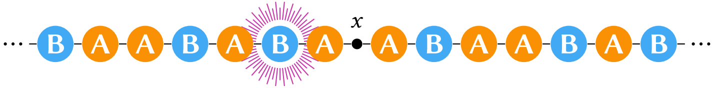


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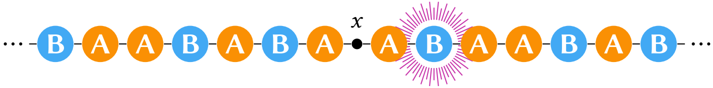


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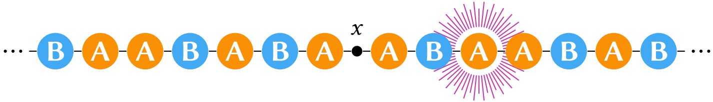


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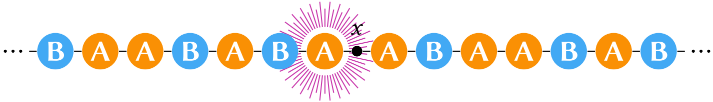


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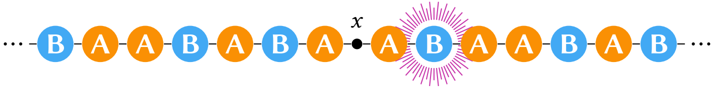


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# Quasicrystals are interesting materials

When $\Sigma$ is a flat torus, the electron is known to move strangely.

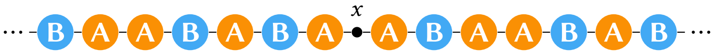
- Its allowed energies form a Cantor set of zero measure.
  Bellissard, Iochum, Scoppola, Testard (1989).
- In some cases, it displays *anomalous transport*—it doesn't move steadily, or do a random walk, or sit still.
  Damanik, Tcheremchantsev (2007); Marin (2010).

Quasicrystals from other translation surfaces might be just as weird.

Even the well-studied flat torus case has some mysteries left.

See Damanik, "Schrödinger operators with dynamically defined potentials," §§7, 8.3.

# A model for a hopping electron




In the *tight-binding model*, the electron's state is a vector $\psi \in L^2(\mathbb{Z})$.

Its motion is described by the difference operator

$$(H_x\psi)^n = -(\psi^{n+1} + \psi^{n-1}) + u_x^n \psi^n,$$

where

$$u_x^n = \begin{cases} \alpha & \text{atom } n \text{ is type A} \\ \beta & \text{atom } n \text{ is type B} \end{cases}$$

is its potential energy at site $n$.

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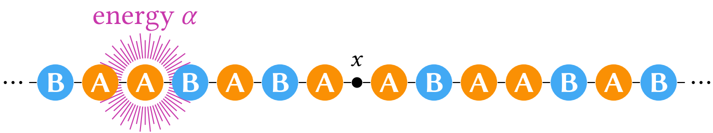


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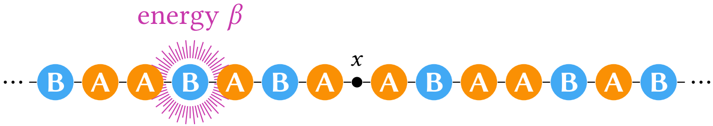


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# Flat bundles reveal the electron's energies

The spectrum of $H_x \circlearrowright L^2(\mathbb{Z})$ gives the electron's allowed energies.

Studying the $E$-eigenspace of $H_x \circlearrowright \mathbb{C}^{\mathbb{Z}}$ will lead us to a test for whether $E \in \mathbb{C}$ is in the spectrum. To build eigenvectors, solve

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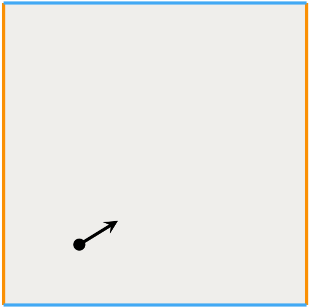
The atom type at $n$ determines the transition from $v^n$ to $v^{n+1}$.

See Viana, *Lectures on Lyapunov Exponents*, §2.1.3.

# Flat bundles reveal the electron's energies

$$B = \begin{bmatrix} \beta - E & -1 \\ 1 & \cdot \end{bmatrix}$$



$$A = \begin{bmatrix} \alpha - E & -1 \\ 1 & \cdot \end{bmatrix}$$

The transition matrices define a flat $\mathrm{SL}_2\,\mathbb{C}$ vector bundle $\mathscr{V}(E) \to \Sigma$.

Its flat sections along the $\phi$-orbit of $x$ are the $E$-eigenvectors of $H_x$.

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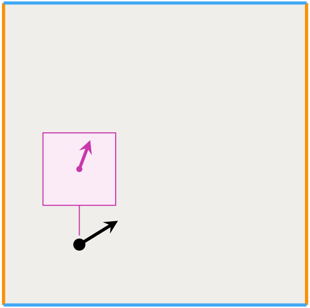

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Carrying $v^0$

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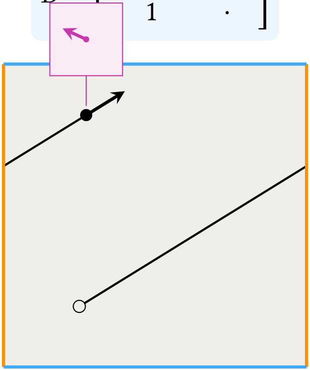

$$A = \begin{bmatrix} \alpha - E & -1 \\ 1 & \cdot \end{bmatrix}$$

$$Av^0 = v^1$$

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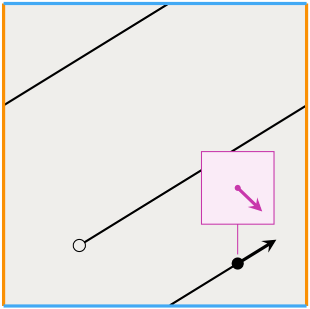

$$A = \begin{bmatrix} \alpha - E & -1 \\ 1 & \cdot \end{bmatrix}$$

$$BAv^0 = v^2$$

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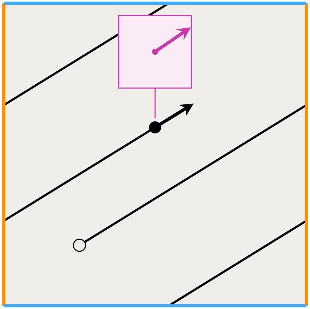

$$A = \begin{bmatrix} \alpha - E & -1 \\ 1 & \cdot \end{bmatrix}$$

$$ABAv^0 = v^3$$

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# Flat bundles reveal the electron's energies

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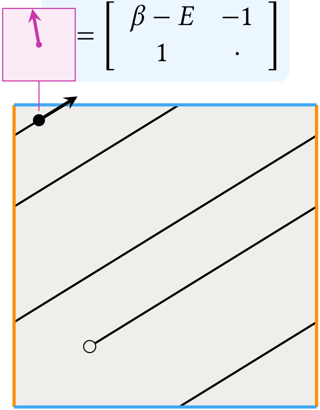

$$A = \begin{bmatrix} \alpha - E & -1 \\ 1 & \cdot \end{bmatrix}$$

$$AABAv^0 = v^4$$

The transition matrices define a flat $\mathrm{SL}_2\,\mathbb{C}$ vector bundle $\mathscr{V}(E) \to \Sigma$.

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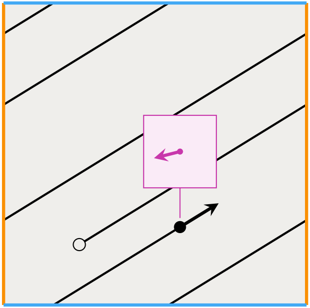

$$A = \begin{bmatrix} \alpha - E & -1 \\ 1 & \cdot \end{bmatrix}$$

$$BAABAv^0 = v^5$$

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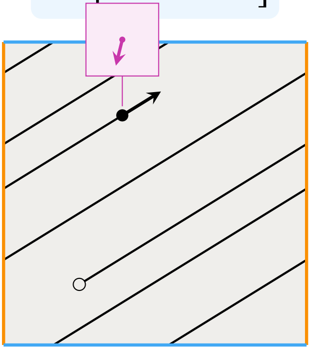

$$A = \begin{bmatrix} \alpha - E & -1 \\ 1 & \cdot \end{bmatrix}$$

$$ABAABAv^0 = v^6$$

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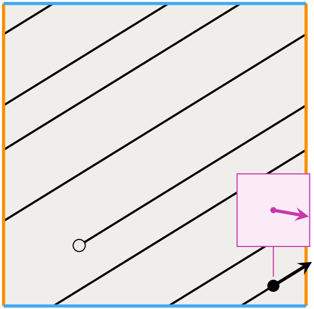

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# Flat bundles reveal the electron's energies

Theorem (special case of Johnson 1986)

If the orbit of $x \in \Sigma$ is dense, the spectrum of $H_x$ is the complement of

$$\{E \in \mathbb{C} \,:\, \mathscr{V}(E) \text{ is } \textit{uniformly hyperbolic} \text{ with respect to } \phi\}.$$

Uniform hyperbolicity is a dynamical condition. It's like being Anosov, but only along $\phi$, instead of in all directions.

# Uniform hyperbolicity: dynamical definition

Lift $\phi$ along the flat connection to a flow $\Phi$ on $\mathscr{V}(E)$.

We say $\mathscr{V}(E)$ is *uniformly hyperbolic* with respect to $\phi$ if it splits into line sub-bundles $\mathscr{V}^{\pm}(E)$, preserved by $\Phi$, with

$$\|\Phi_x^{\pm t} v\| \lesssim e^{-Kt}\|v\|$$

over all $x \in \Sigma$, $v \in \mathscr{V}^{\pm}(E)_x$, and $t \in [0, \infty)$.

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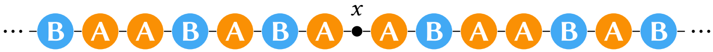


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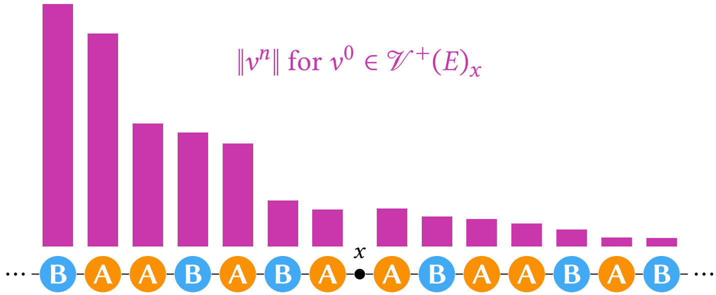


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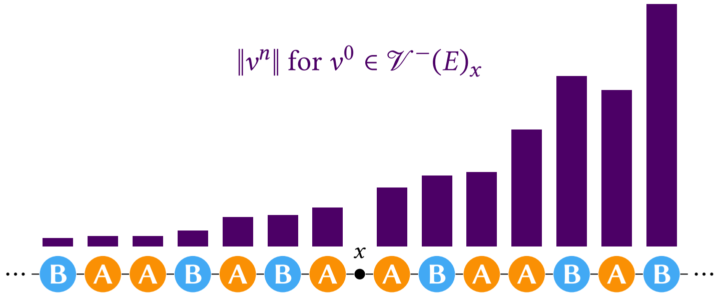


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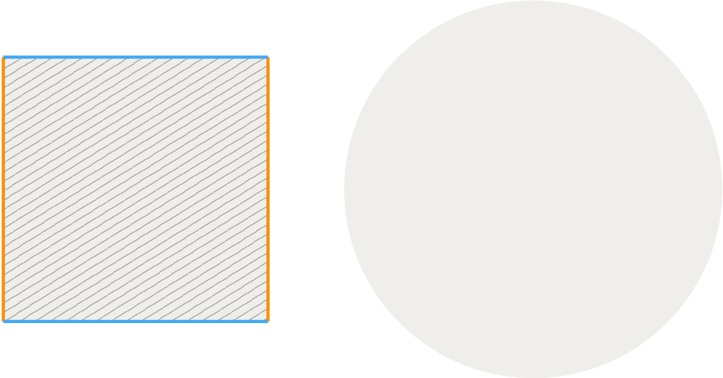

Our construction gives $\mathscr{V}(E)_x \xrightarrow{\cong} \mathbb{C}^2$ over the fundamental polygon.

That makes $\mathscr{V}^+(E)_x$ and $\mathscr{V}^-(E)_x$ points in $\mathrm{P}\mathbb{C}^2 \cong \partial\mathbb{H}^3$, giving

$$\text{orbit segments in polygon} \xrightarrow[\text{equivariant}]{} \text{geodesics in } \mathbb{H}^3$$

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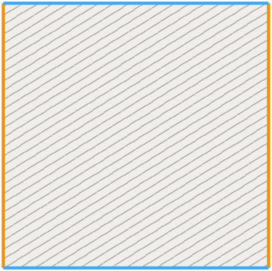

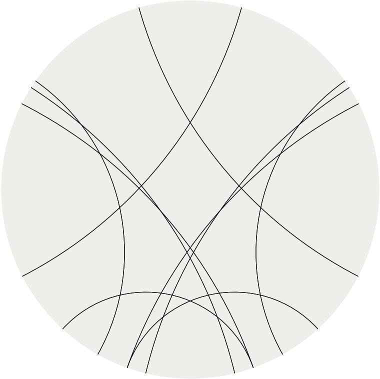

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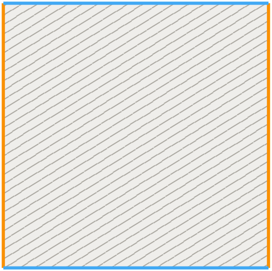

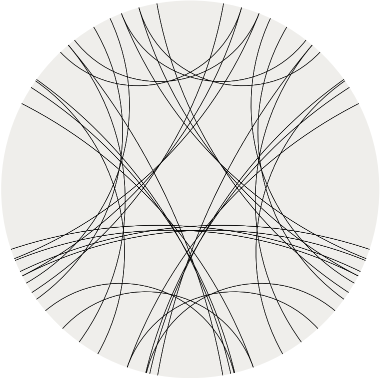

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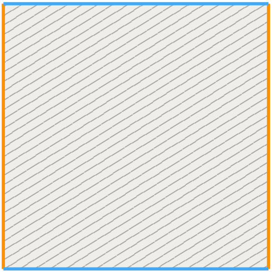

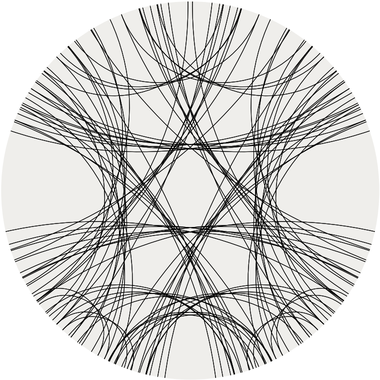

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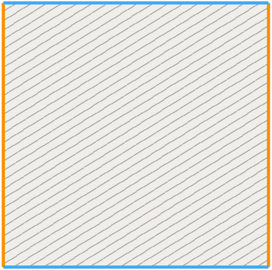

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They should form the pleating locus of a *pleated hyperbolic surface* with holonomy bundle $\mathscr{V}(E)$.

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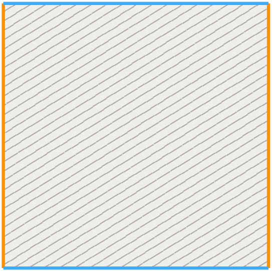



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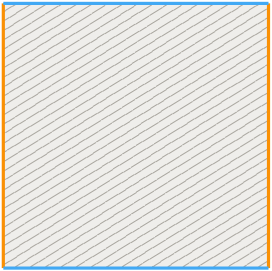

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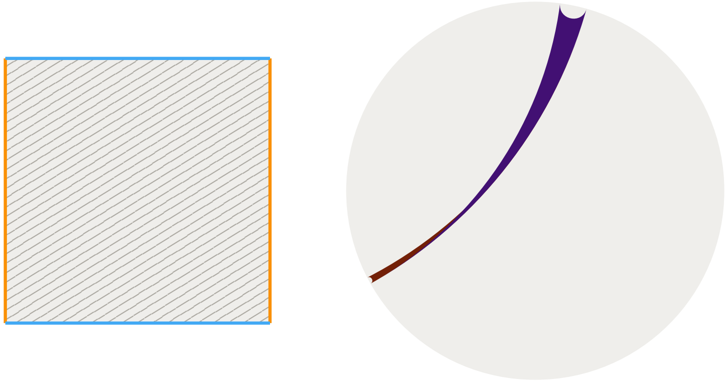

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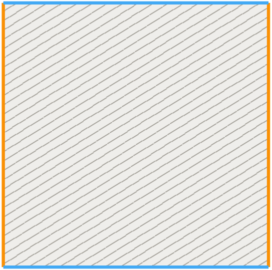

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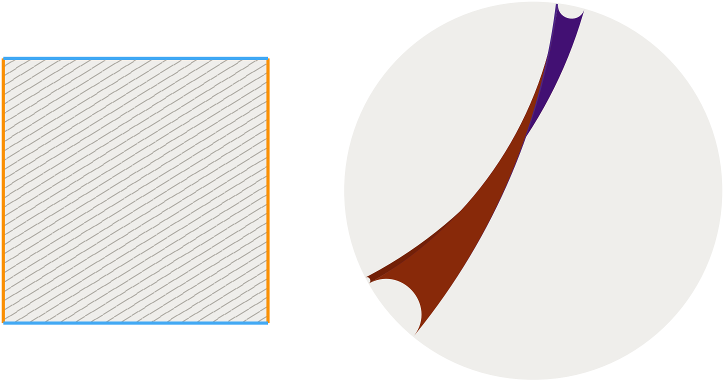

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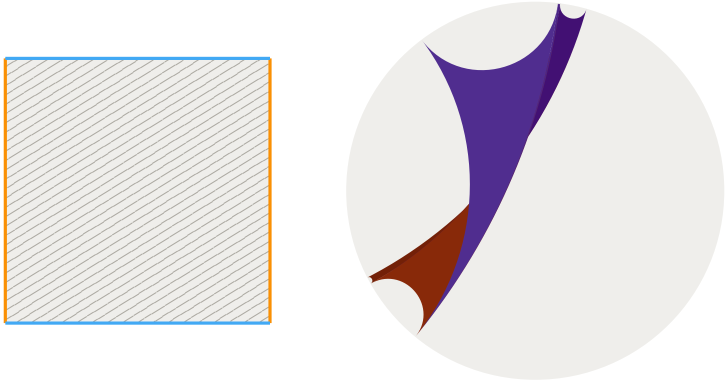

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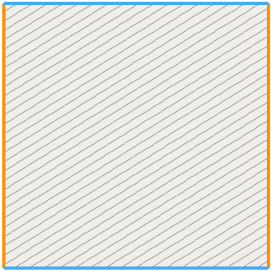

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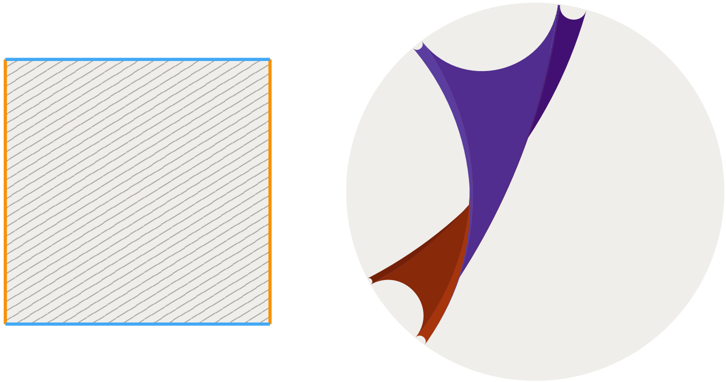

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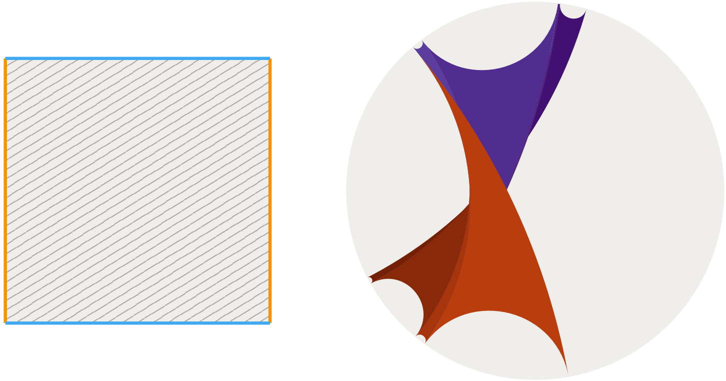

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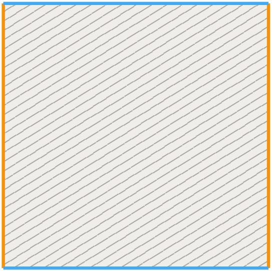

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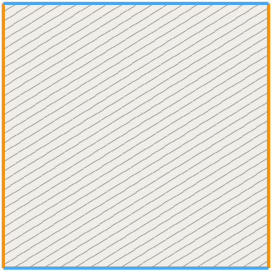

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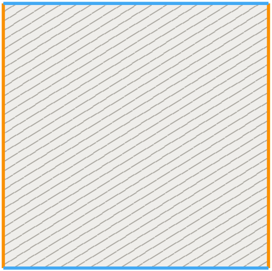

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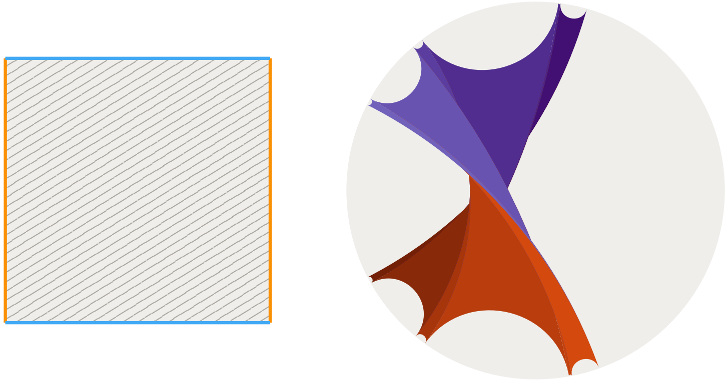

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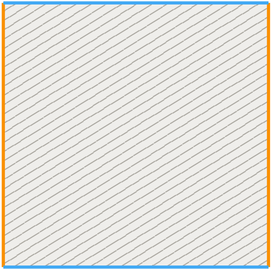

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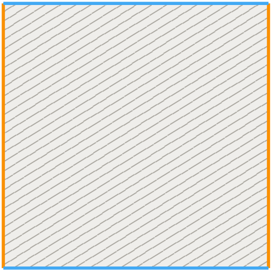

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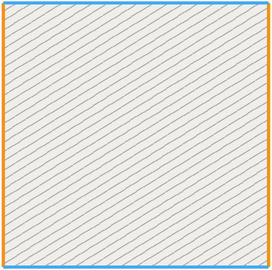

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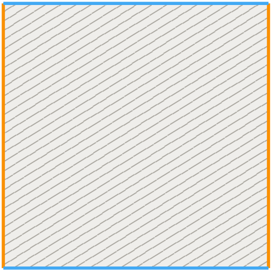

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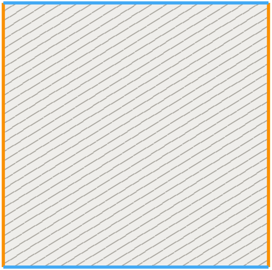

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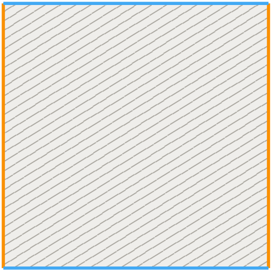

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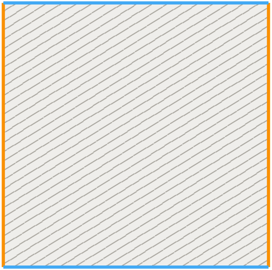

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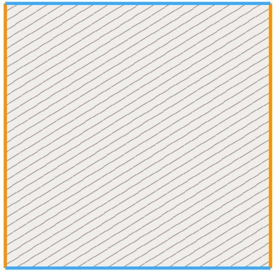

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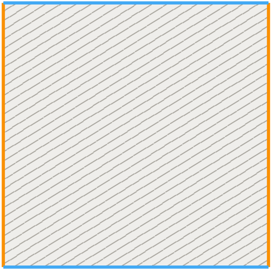

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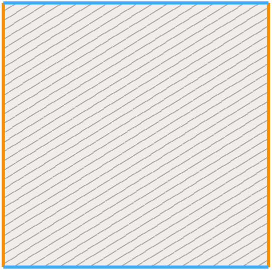

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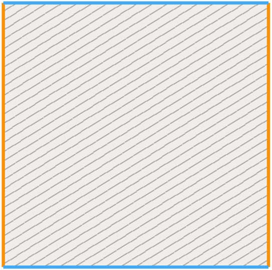

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They should form the pleating locus of a *pleated hyperbolic surface* with holonomy bundle $\mathscr{V}(E)$.

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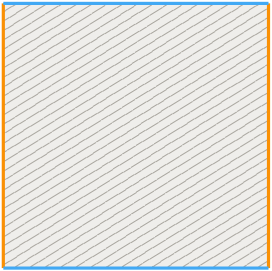

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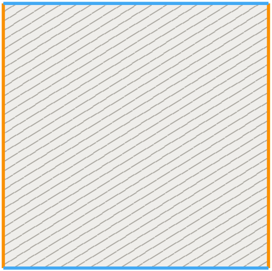

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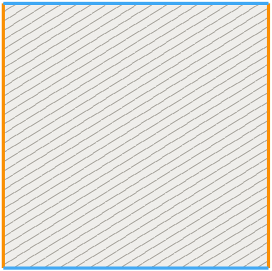

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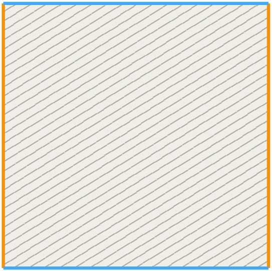

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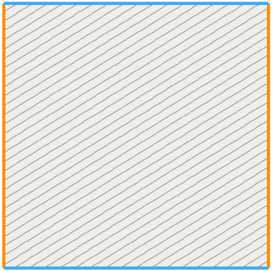


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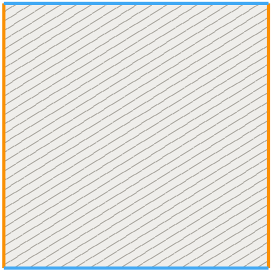

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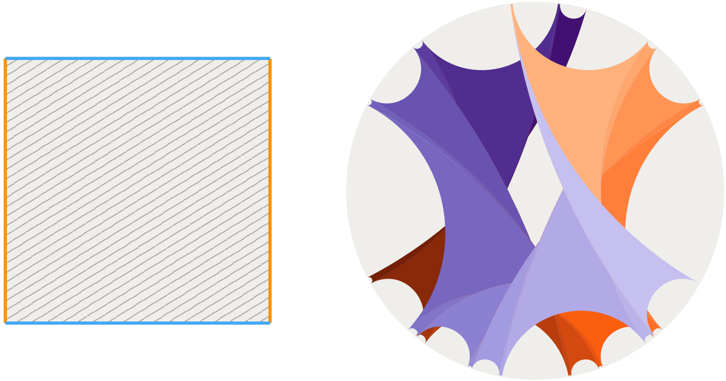

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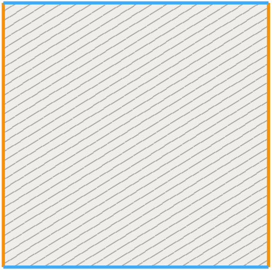

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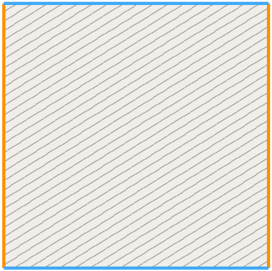

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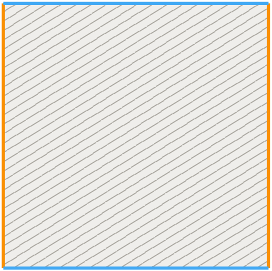

$E = -0.95$

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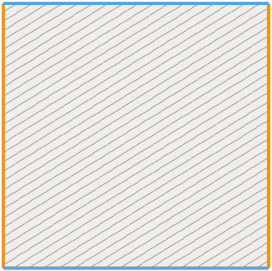

$E = +0.50$

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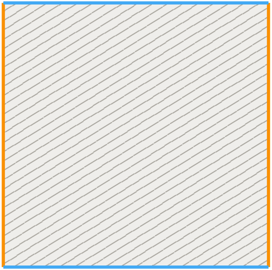

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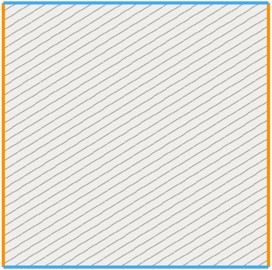

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# Geometry encodes the electron's motion

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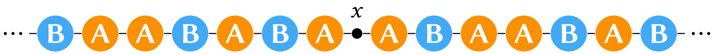


Our $E$-dependent pleated surface encodes $(H_x - E)^{-1}$ for all $x \in \Sigma$.

Choose $E$-eigenvectors $\psi_\pm \in \mathbb{C}^{\mathbb{Z}}$ with $\psi_+^0 = \psi_-^0$ and $v_\pm^0 \in \mathscr{V}^{\pm}(E)_x$.

Splice and rescale to get unique $\psi_\star \in L^2(\mathbb{Z})$ with $(H_x - E)\psi_\star = \delta_0$.

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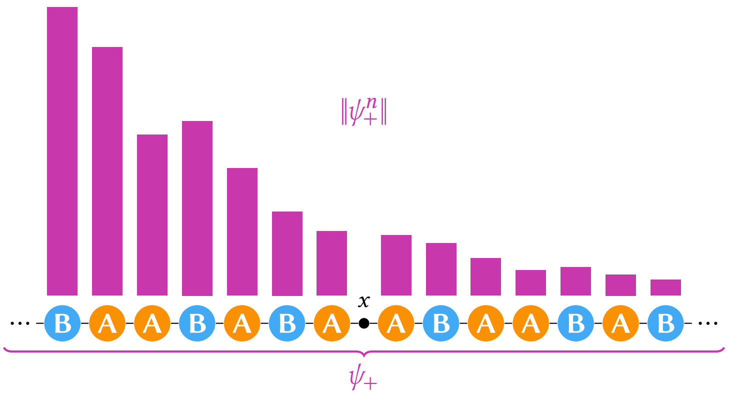


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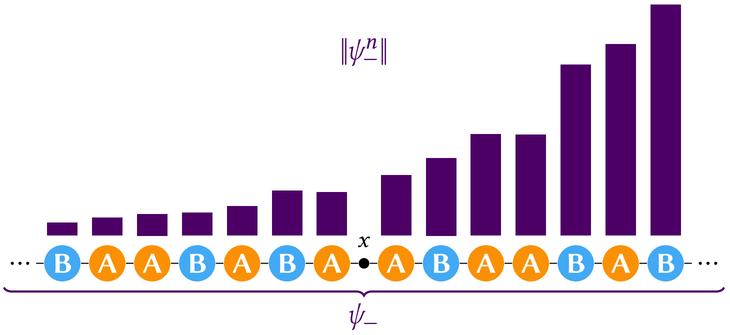


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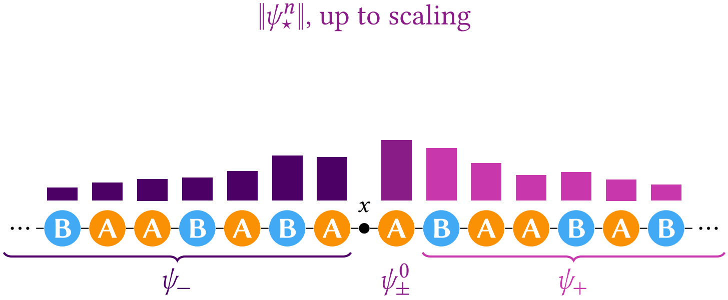


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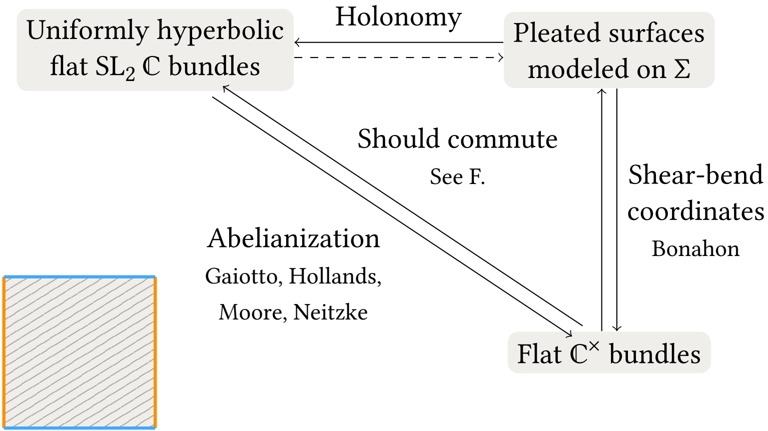
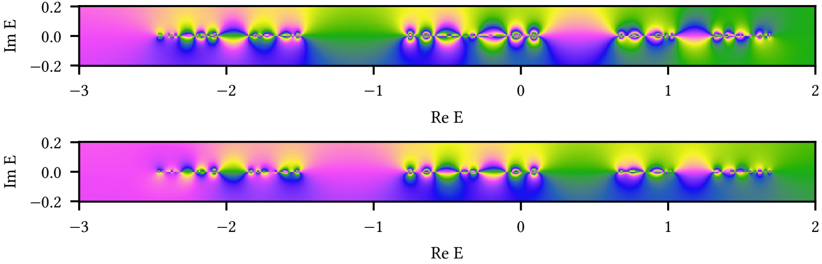
# Shear-bend coordinates describe the geometry




Translation surface $\Sigma$ with chosen direction

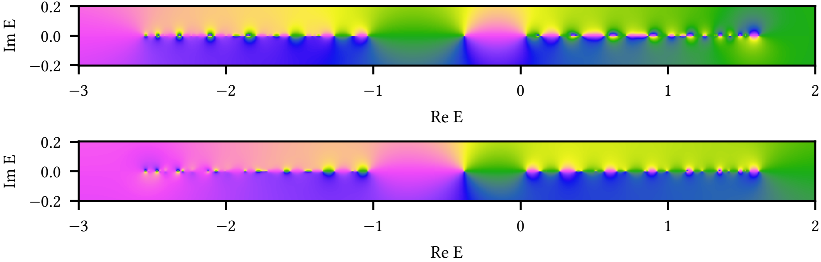
# Shear-bend coordinates encode the electron's motion

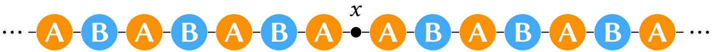



The shear-bend coordinates depend holomorphically on $E$ away from the spectrum.

Their singularities show us some part of the spectrum.

# Shear-bend coordinates depend nicely on flow direction




Varying $\phi$ changes the quasiperiodicity of our material.

For pleated surfaces with finitely many pleats, the shear-bend coordinates only change when $\phi$ connects vertices of $\Sigma$.

The same should be true in our infinitely pleated case.

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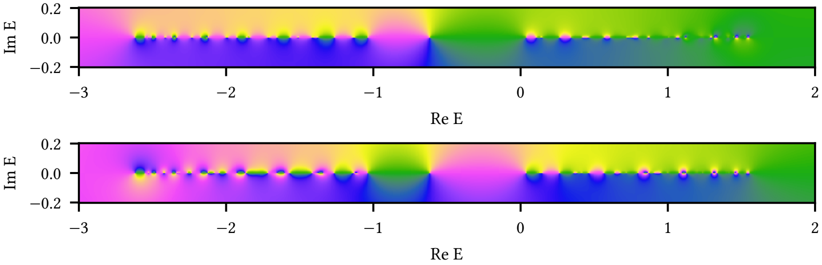


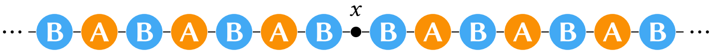


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# Questions

How do we rigorously construct a pleated surface from a uniformly hyperbolic $\mathrm{SL}_2\,\mathbb{C}$ local system?

If we change the fundamental polygon of $\Sigma$, can we interpret the new transition matrices physically?

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Quasicrystals are interesting materials


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