Operator ordering for quantum curves

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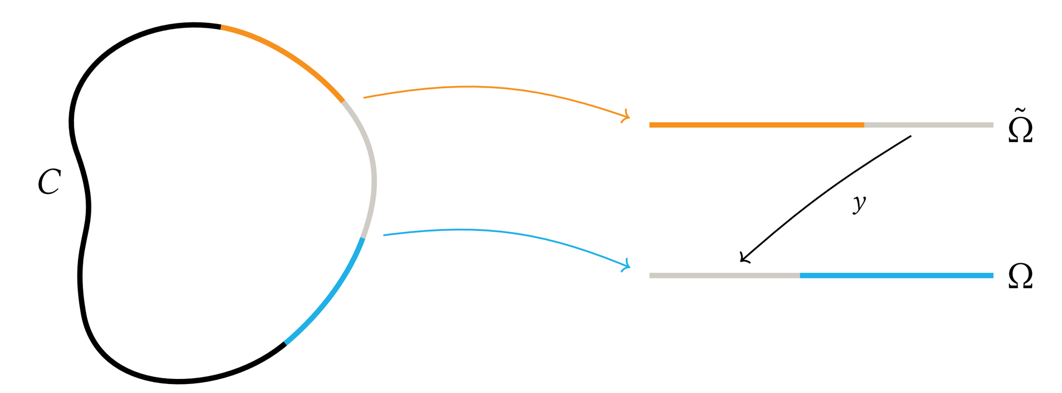
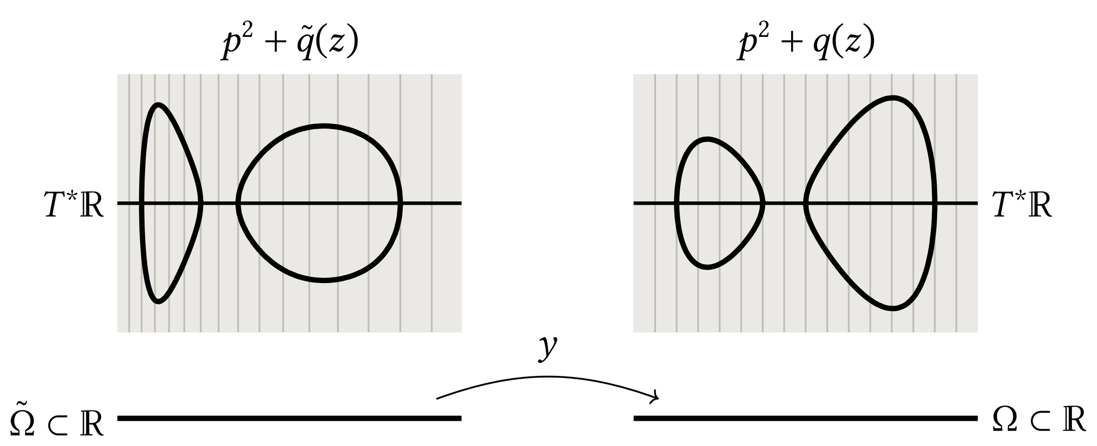
Local quantization

A particle on a curve



Let's study a particle moving on a real curve C.

Classical energy levels

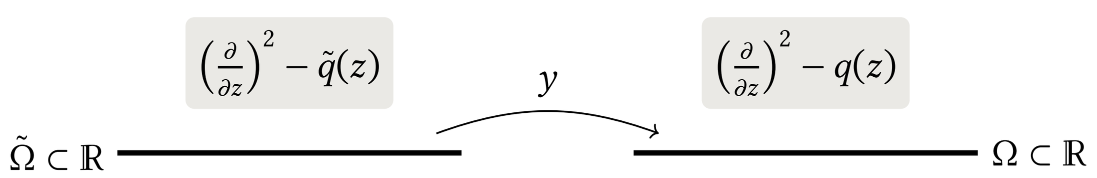


A *classical energy level* is a curve in T^*C, cut out locally by

$$\tfrac{1}{2}p^2 + V(z) = E \quad \rightsquigarrow \quad p^2 + q(z) = 0$$

For consistency between charts, we need $\tilde q = y^*q\, y_z^2$.

Quantization



According to the *canonical quantization* rules

$$z \rightsquigarrow Z := z \qquad p \rightsquigarrow P := -i\tfrac{\partial}{\partial z}$$

a quantum energy level should be described locally by

$$\tfrac{1}{2}\left(i\tfrac{\partial}{\partial z}\right)^2 + [V(z) - E] \quad \rightsquigarrow \quad \underbrace{\left(\tfrac{\partial}{\partial z}\right)^2 - q(z)}_{\textit{Hill's operator}}$$

What condition should we impose for consistency between charts?

Liouville equivalence is a time-honored choice.

$$\left(\tfrac{\partial}{\partial z}\right)^2 - \tilde q \quad = \quad y_z^{3/2} \circ \left[\left(\tfrac{\partial}{\partial y}\right)^2 - y^*q\right] \circ y_z^{1/2}$$

The geometry of quantum energy levels

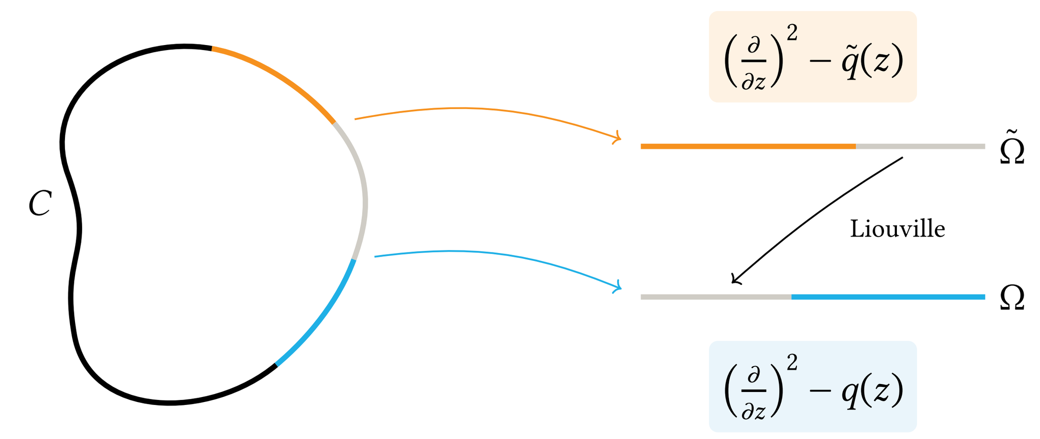
This gluing condition produces a convenient space of quantum curves with a beautiful geometric interpretation.

Operator ordering

Its unusual form can be motivated using an operator-ordering rule.

The geometry of quantum energy levels

Quantum energy levels

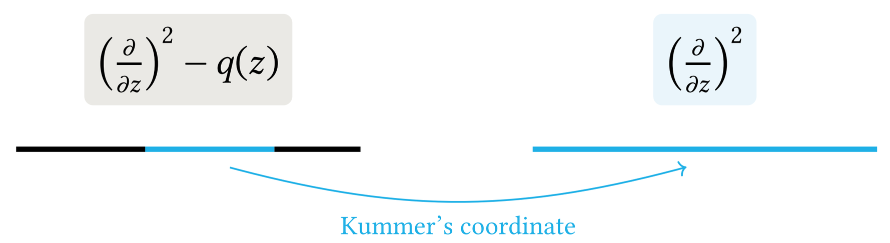


A *quantum energy level* on C consists of:

- **Data** An atlas with a Hill's operator on the image of each chart.
- **Condition** Transitions are Liouville equivalences.

It's a geometric structure!

Simplification

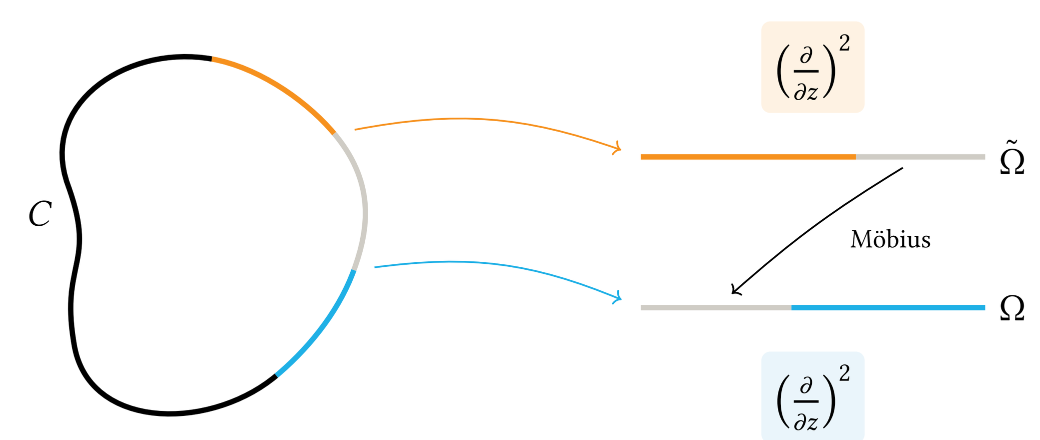


Every Hill's operator is locally Liouville-equivalent to $\left(\frac{\partial}{\partial z}\right)^2$.

Hence, we can describe a quantum energy level using only *projective charts*, which come with $\left(\frac{\partial}{\partial z}\right)^2$ on their images.

The Liouville self-equivalences of this operator are the Möbius transformations.

Real projective structures



If we stick to projective charts, we can forget about Hill's operators.

A quantum energy level on C becomes:

- **Data** An atlas of charts to $\mathbb{RP}^1$.
- **Condition** Transitions are Möbius transformations.

This is known as a *real projective structure*.

G. Segal, "The geometry of the KdV equation" (1991). doi:10.1142/S0217751X91001416.

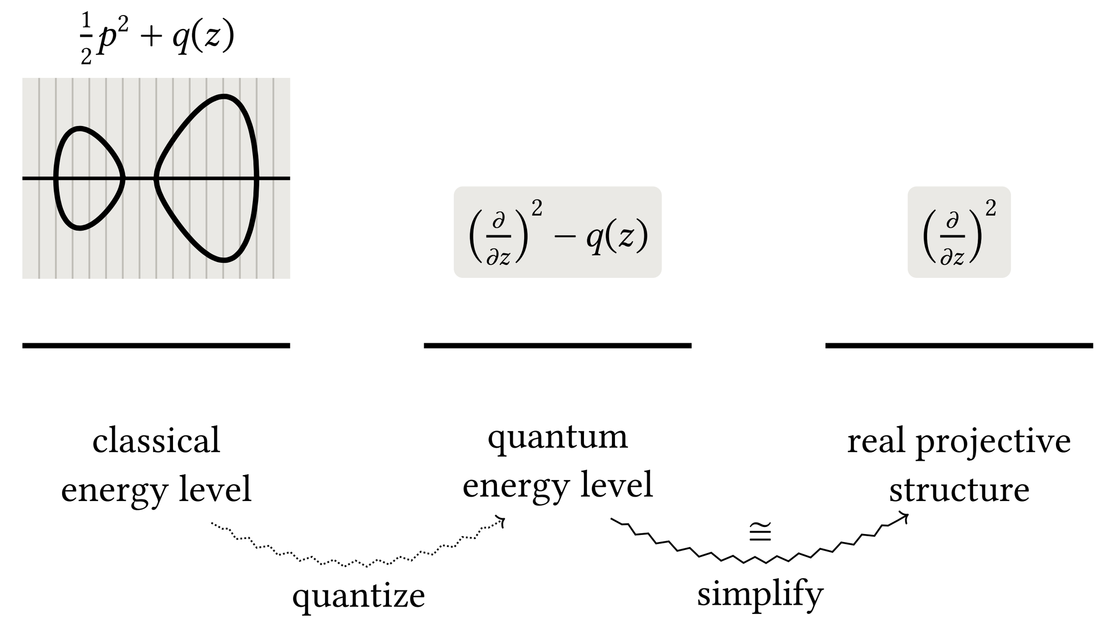
A differential point of view

We can view $\left(\frac{\partial}{\partial z}\right)^2$ globally on $\mathbb{RP}^1$ as a differential operator Δ that sends sections of the anti-tautological bundle $\mathcal{O}(1)$ to sections of $\mathcal{O}(-3)$.

A real projective structure on C globalizes $\mathcal{O}(1)$ to a line bundle $\mathcal{L}$ with an isomorphism $\mathcal{L}^{\otimes 2} \leftrightarrow TC$. It globalizes Δ to a differential operator sending sections of $\mathcal{L}$ to sections of $\mathcal{L}^{\otimes(-3)}$.

We can recover the real projective structure from $\mathcal{L}$, Δ, and $\mathcal{L}^{\otimes 2} \leftrightarrow TC$.

Summary



A quantum energy level is a real projective structure.

Interpretation as quantum curves

Background

Quantum curves were first defined for algebraic curves in $T^*\mathbb{C}$.

P. Norbury, "Quantum curves and topological recursion" (2015). arXiv:1502.04394.

Dumitrescu and Mulase have now generalized them to Hitchin spectral curves in T^*C, where C is an algebraic curve with a Higgs bundle.

O. Dumitrescu and M. Mulase, "Lectures on the topological recursion for Higgs bundles…," §1.6 (2016). arXiv:1509.09007.

—, "Interplay between Opers, Quantum Curves, WKB Analysis, and…" (2021). doi:10.3842/SIGMA.2021.036.

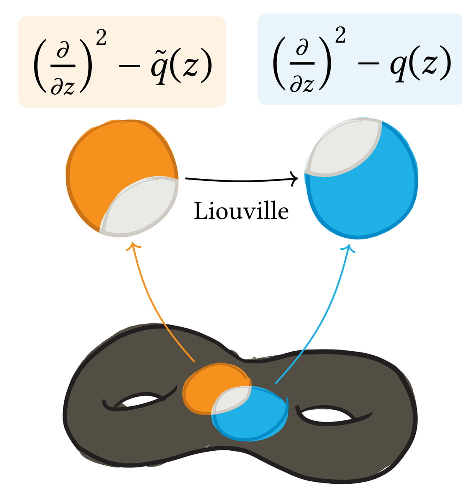
Our quantum energy levels are quantum curves in that sense.

Complex projective structures

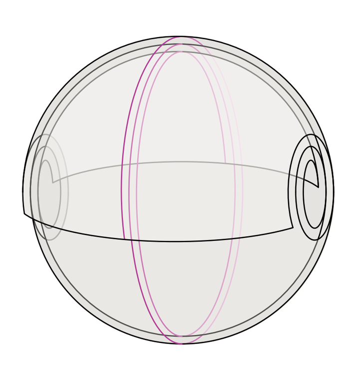
Make C a complex curve.

All our definitions carry over. A quantum energy level is now a *complex projective structure.*

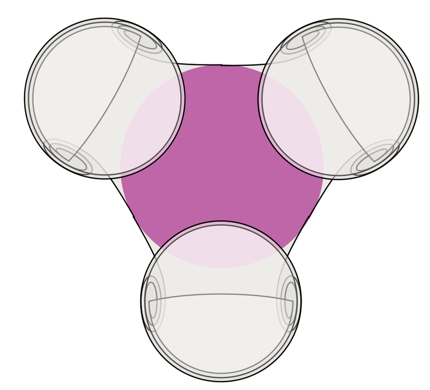
Its $\mathcal{L}$, Δ, and $\mathcal{L}^{\otimes 2} \leftrightarrow TC$ form an $\mathrm{SL}_2(\mathbb{C})$ *oper*, which Dumitrescu and Mulase read as a quantum curve.



Examples



free particle: $p^2 + \frac{1}{4}$



falling particle: $p^2 + z$

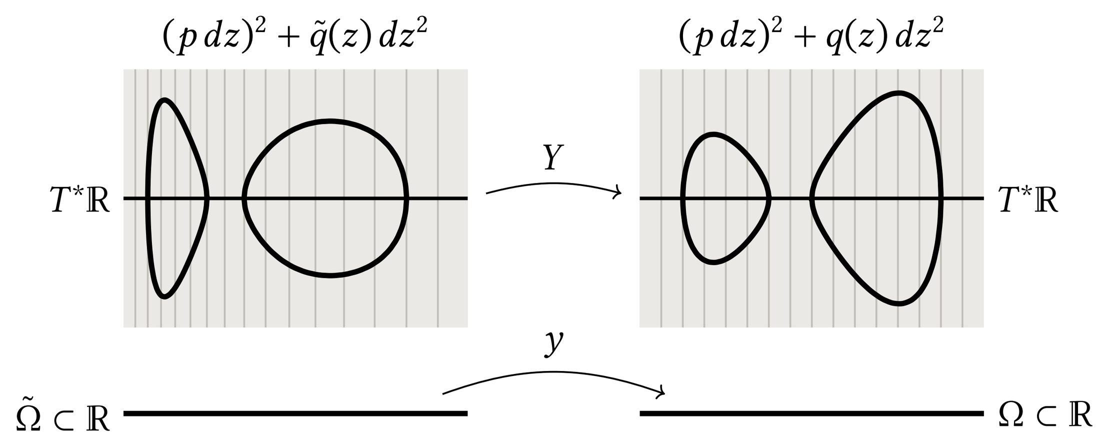
In the complex setting, even the most basic systems have interesting energy level geometry. This plays a role in the exact WKB method.

A. F., "The complex geometry of the free particle, and its perturbations" (2020). arXiv:2008.03836.

—, "The geometry of quantum energy levels" (2020 talk). Slides available online.

Operator ordering

A more invariant view



Each transition $y : \tilde\Omega \to \Omega$ lifts to a map $Y : T^*\tilde\Omega \to T^*\Omega$ defined by

$$Y^*z = y \qquad Y^*p = p y_z^{-1}$$

The cotangent lift preserves the Liouville form $p\,dz$. This suggests rewriting each energy level equation in the more invariant form

$$(p\,dz)^2 + q(z)\,dz^2 = 0.$$

The consistency condition becomes $\tilde q\,dz^2 = y^*(q\,dz^2)$.

A classical energy level is a quadratic differential.

The quantum cotangent lift

Let's apply canonical quantization to the definition of Y. For operator ordering, we'll use the "momentum sandwich" rule

$$p^\alpha z^\beta \rightsquigarrow Z^{\beta/2} \circ P^\alpha \circ Z^{\beta/2}$$

This gives us the quantum cotangent lift

$$Y^*Z = y \qquad Y^*P = y_z^{-1/2} \circ P \circ y_z^{-1/2}$$

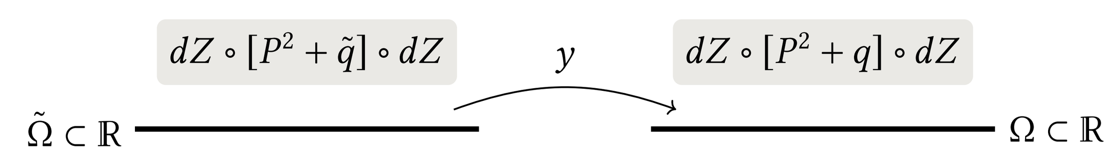
To quantize quadratic differentials, we introduce a formal symbol dZ which orders like Z. It pulls back with no ordering ambiguity:

$$Y^*dZ = y_z\,dZ$$

To generalize, we define

$$\begin{aligned} &Y^*[f_1(Z, dZ) \circ g_1(P) \circ \ldots \circ f_n(Z, dZ) \circ g_n(P)] \\ &\quad = f_1(Y^*Z,\, Y^*dZ) \circ g_1(Y^*P) \circ \ldots \circ f_n(Y^*Z,\, Y^*dZ) \circ g_n(Y^*P) \end{aligned}$$

The emergence of Liouville equivalence



Now we can quantize the local quadratic differentials describing a classical energy level. Their consistency condition quantizes to

$$\begin{aligned} dZ \circ [P^2 + \tilde q] \circ dZ &= Y^*\big(dZ \circ [P^2 + \tilde q] \circ dZ\big) \\ &= (y_z\,dZ) \circ \left[\left(y_z^{-1/2} \circ P \circ y_z^{-1/2}\right)^2 + y^*q\right] \circ (y_z\,dZ) \\ &\;\;\vdots \\ &= (y_z^{3/2}\,dZ) \circ [(y_z^{-1} \circ P)^2 + y^*q] \circ (y_z^{1/2}\,dZ) \\ dZ \circ \left[\left(-i\tfrac{\partial}{\partial z}\right)^2 + \tilde q\right] \circ dZ &= (y_z^{3/2}\,dZ) \circ \left[\left(-i\tfrac{\partial}{\partial y}\right)^2 + y^*q\right] \circ (y_z^{1/2}\,dZ) \end{aligned}$$

The quantum consistency condition is Liouville equivalence!