Operator ordering for quantum curves

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1 Overview

A “quantum spectral curve” over a Riemann surface C is typically supposed to be a flat connection on $(T^*C)^{1/2} \oplus (T^*C)^{-1/2}$. It can also be seen, more or less equivalently,[1] as a *complex projective structure*, which has two equivalent definitions [1, §2].

- **Analytic.** An atlas of charts to $\mathbb{C}$ which comes with an operator of the form $-\left(\frac{\partial}{\partial z}\right)^2 + q(z)$ on each chart. The transitions between charts are required to be *Liouville transformations* relating the operators [1, §1.2].
- **Geometric.** An atlas of charts to $\mathbb{CP}^1$ whose transitions are Möbius transformations.

Why do we use the bundle $(T^*C)^{1/2} \oplus (T^*C)^{-1/2}$? Equivalently, why do we use Liouville transformations instead of some other transformation rule for differential operators?

I think there’s a special operator ordering rule for observables and coordinate transformations which leads to these choices. I’ll describe it here in an *ad hoc* way, with the hope that it can be put on a firmer footing later.

2 The cotangent lift

2.1 Classical

Let z and p be the standard position and momentum coordinates on $T^*\mathbb{C}$. A conformal map $f\colon \tilde{\Omega} \to \Omega$ between two regions in $\mathbb{C}$ lifts to a map $F\colon T^*\tilde{\Omega} \to T^*\Omega$ defined by

$$\begin{aligned} F^*z &= f \\ F^*p &= p f_z^{-1}. \end{aligned}$$

Let’s view the Liouville form $p\,dz$ as a $\mathrm{Sym}(T^*\mathbb{C})$-valued function on $T^*\mathbb{C}$. That means we can write it as a polynomial in dz whose coefficients are holomorphic functions in z and p. We extend F^* to $\mathrm{Sym}(T^*\mathbb{C})$-valued functions by

[1] The holonomy map is a covering map from the space of complex projective structures to the space of flat $\mathrm{SL}_2\,\mathbb{C}$ connections. Two complex projective structures have the same holonomy connection if their holonomies differ by $2\pi\mathbb{Z}$ rotations.

letting it act like f^* on differentials:

$$F^* dz = f_z\, dz.$$

This gives $F^*(p\, dz) = p\, dz$, confirming that F is a symplectic map.

2.2 Quantum

Let's try to define a quantum version of F by applying the canonical quantization

$$\begin{aligned} z &\rightsquigarrow Z := z \\ p &\rightsquigarrow P := -i \tfrac{\partial}{\partial z} \end{aligned}$$

to the formulas above. For operator ordering, we'll use the "momentum sandwich" rule

$$p^\alpha z^\beta \rightsquigarrow Z^{\beta/2} \circ P^\alpha \circ Z^{\beta/2},$$

giving

$$\begin{aligned} F^* Z &= f \\ F^* P &= f_z^{-1/2} \circ P \circ f_z^{-1/2}. \end{aligned}$$

To quantize $\mathrm{Sym}(T^*\mathbb{C})$-valued functions, we introduce a formal symbol dZ which orders like Z. It pulls back with no ordering ambiguity:

$$F^* dZ = f_z\, dZ.$$

As an example, let's see how the quantum Liouville form pulls back along the quantum cotangent lift of f. First, we quantize the Liouville form.

$$p\, dz \rightsquigarrow dZ^{1/2} \circ P \circ dZ^{1/2}.$$

Then we pull it back, by applying F^* to each factor in order.

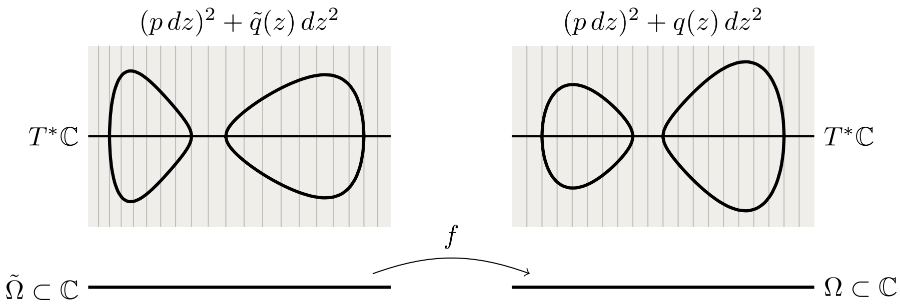
$$\begin{aligned} F^*(dZ^{1/2} \circ P \circ dZ^{1/2}) &= (f_z\, dZ)^{1/2} \circ \left[f_z^{-1/2} \circ P \circ f_z^{-1/2} \right] \circ (f_z\, dZ)^{1/2} \\ &= dZ^{1/2} \circ P \circ dZ^{1/2}. \end{aligned}$$

Our quantum cotangent lift, like the classical one, preserves the Liouville form.

3 Gluing spectral curves

3.1 Classical

Consider local spectral curves over Ω and $\tilde{\Omega}$, cut out by the ideals of $\mathrm{Sym}(T^*\mathbb{C})$-valued functions generated by $(p\, dz)^2 + q(z)\, dz^2$ and $(p\, dz)^2 + \tilde{q}(z)\, dz^2$ respectively.



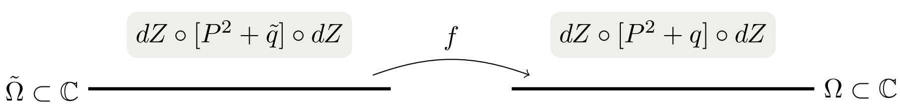
These curves glue properly along f if and only if $(p\,dz)^2 + \tilde{q}\,dz^2$ generates the same ideal as

$$F^*\left[(p\,dz)^2 + q\,dz^2\right] = (p\,dz)^2 + f^*(q\,dz^2)$$

Hence, the spectral curves glue if and only if f pulls $q\,dz^2$ back to $\tilde{q}\,dz^2$.

3.2 Quantum

Our quantization rules turn the ideals representing the classical spectral curves over Ω and $\tilde{\Omega}$ into the ideals of differential operators whose generators are shown below.



Imitating the gluing condition for classical spectral curves, we'll consider our quantum spectral curves to glue properly along f if and only if

$$dZ \circ [P^2 + \tilde{q}] \circ dZ$$

generates the same ideal as

$$\begin{aligned}
F^*\left(dZ \circ [P^2 + q] \circ dZ\right) &= (f_z\,dZ) \circ \left[\left(f_z^{-1/2} \circ P \circ f_z^{-1/2}\right)^2 + f^*q\right] \circ (f_z\,dZ) \\
&= (f_z^{1/2}\,dZ) \circ P \circ f_z^{-1} \circ P \circ (f_z^{1/2}\,dZ) + f^*q\; f_z^2\,dZ^2 \\
&= (f_z^{3/2}\,dZ) \circ \left[f_z^{-1} \circ P\right] \circ \left[f_z^{-1} \circ P\right] \circ (f_z^{1/2}\,dZ) + f^*q\; f_z^2\,dZ^2 \\
&= (f_z^{3/2}\,dZ) \circ \left[(f_z^{-1} \circ P)^2 + f^*q\right] \circ (f_z^{1/2}\,dZ).
\end{aligned}$$

Writing out P, we see that the quantum spectral curves glue if and only if

$$dZ \circ \left[\left(-i\frac{\partial}{\partial z}\right)^2 + \tilde{q}\right] \circ dZ$$

generates the same ideal as

$$(f_z^{3/2}\, dZ) \circ \left[\left(-i \frac{\partial}{\partial f} \right)^2 + q \right] \circ (f_z^{1/2}\, dZ).$$

This condition is familiar from the study of second-order differential operators. It's the condition that f is a Liouville transformation relating $dZ \circ [P^2 + \tilde{q}] \circ dZ$ to $dZ \circ [P^2 + q] \circ dZ$ [1, §1.2].

4 Questions

- Is there a more conceptual way to understand the formal symbol dZ? Maybe as something like a Kähler differential?
- Is there a coordinate-free description of the momentum sandwich ordering? Something like the Toeplitz operator description of the anti-Wick ordering?
- Are there other operator ordering rules that put the quantum spectral curve on other bundles? Can flat connections on those other bundles be described analytically with something like Liouville transformations, or geometrically as something like complex projective structures?

References

[1] A. Fenyes, "The complex geometry of the free particle, and its perturbations," `arXiv:2008.03836 [math.GT]`.