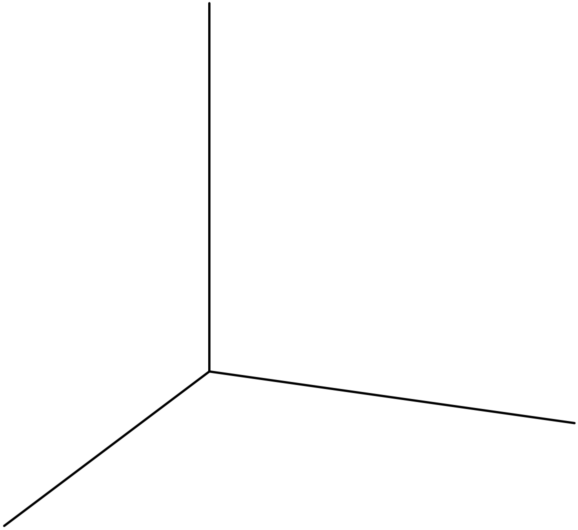
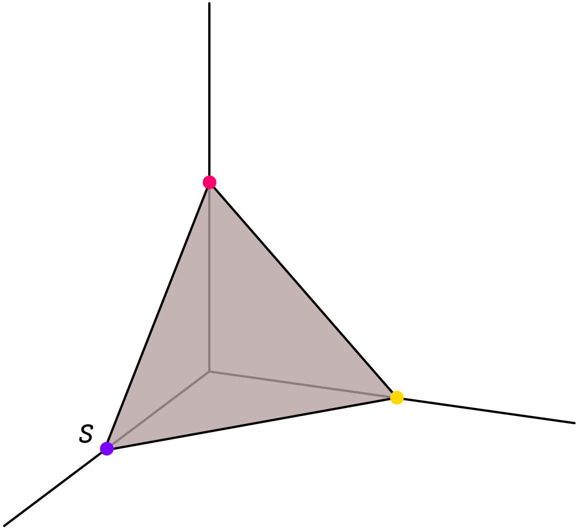
The Perron-Frobenius Theorem

Consider a non-zero linear operator B on $\mathbf{R}^n$ that sends the non-negative orthant into itself.



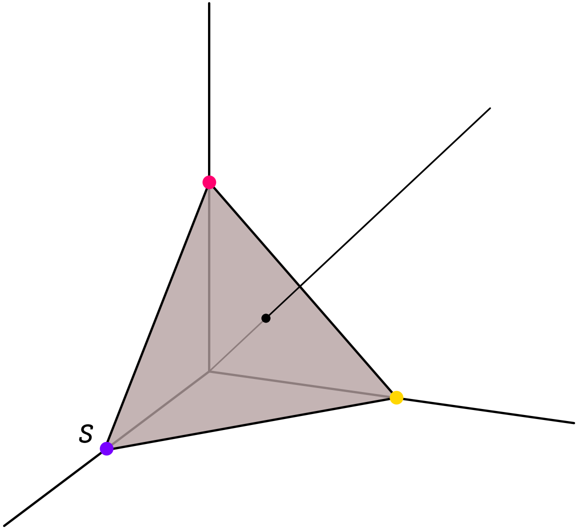
The Perron-Frobenius Theorem



Consider a non-zero linear operator B on $\mathbf{R}^n$ that sends the non-negative orthant into itself.

Let S be the simplex whose corners are the standard basis vectors.

The Perron-Frobenius Theorem

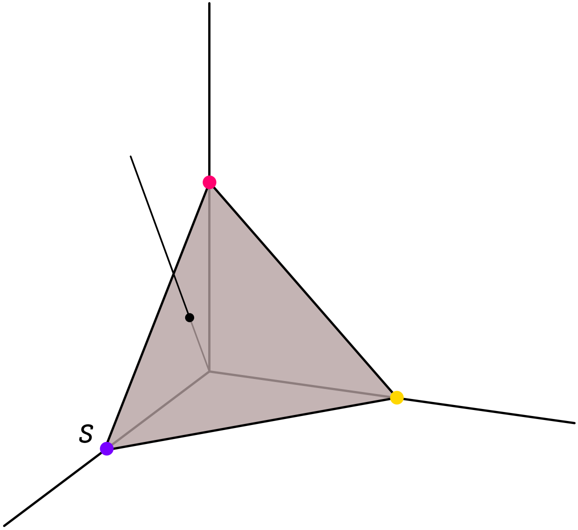


Consider a non-zero linear operator B on $\mathbf{R}^n$ that sends the non-negative orthant into itself.

Let S be the simplex whose corners are the standard basis vectors.

Every ray from the origin into the non-negative orthant hits S at exactly one point, so we can identify S with the space of such rays.

The Perron-Frobenius Theorem



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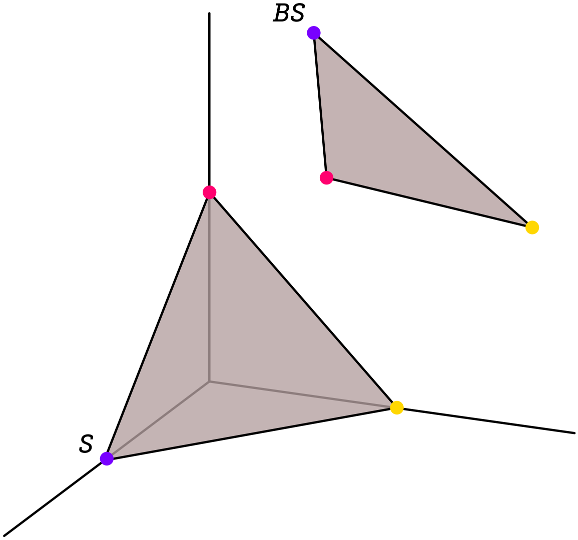
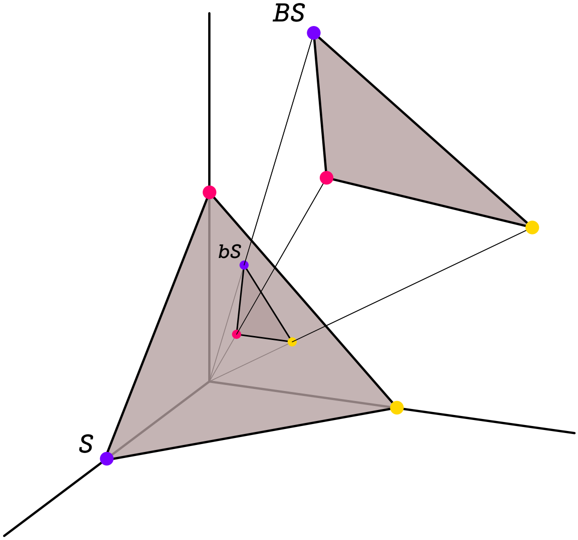
The Perron-Frobenius Theorem

The map B sends S to a triangle floating in the non-negative orthant.



The Perron-Frobenius Theorem



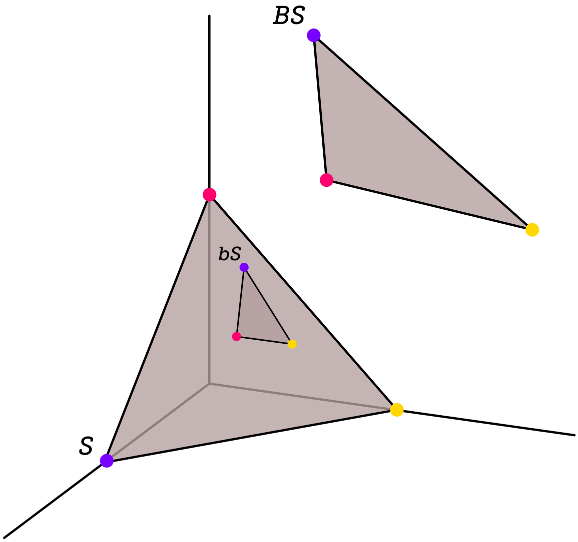
The map B sends S to a triangle floating in the non-negative orthant.

Project that triange back onto S along rays through the origin.

This gives a map b from S to itself, which describes what B does to rays in the non-negative orthant.

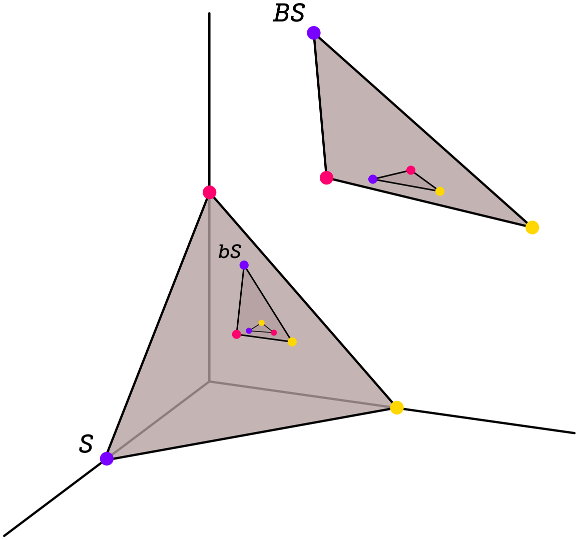
By construction, b is a projective transformation.

The Perron-Frobenius Theorem



In particular, b is continuous. Since S is a convex, compact subset of a Euclidean space, the Brouwer fixed-point theorem guarantees that b has a fixed point.

The Perron-Frobenius Theorem



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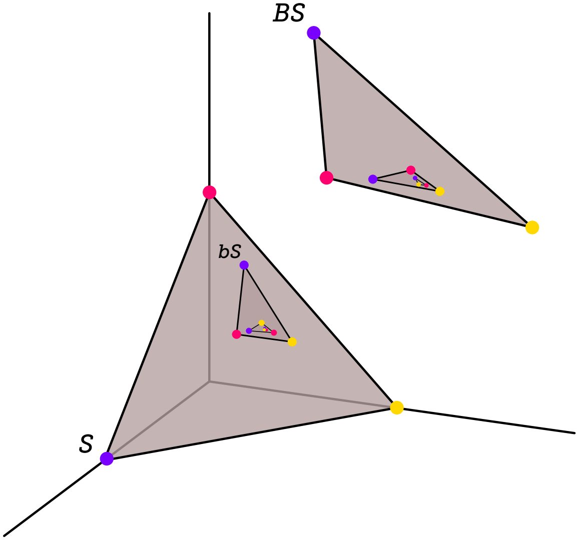

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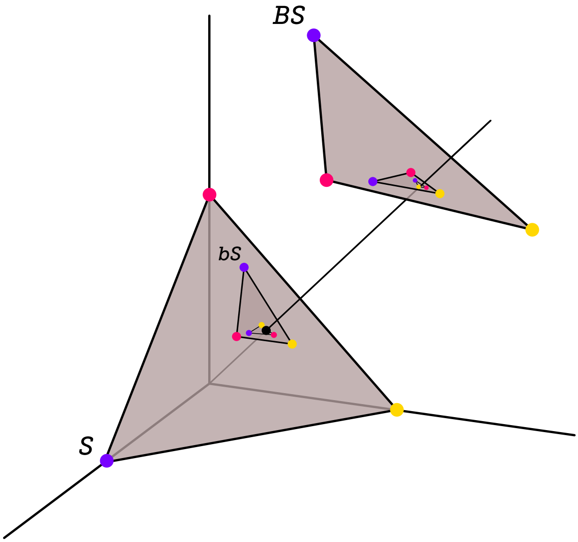

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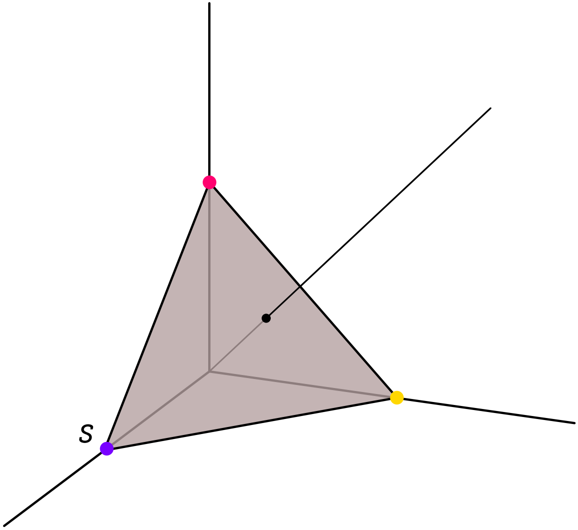
Every fixed point of b corresponds to an eigenray of B.

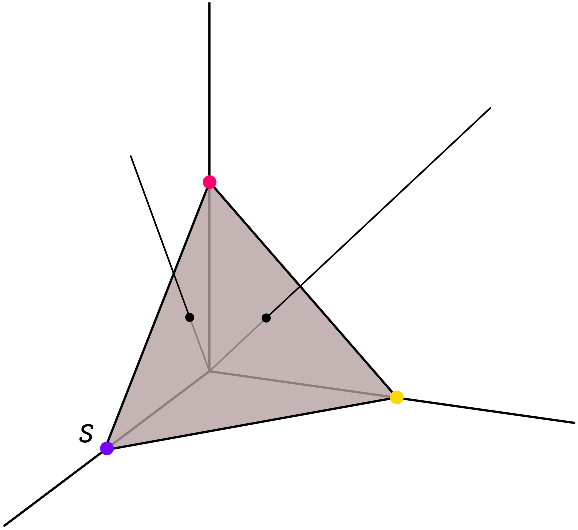
The Perron-Frobenius Theorem



In general, b can have many fixed points.

If some power of B maps the non-negative orthant into the positive orthant, however, then b has just one fixed point.

The Perron-Frobenius Theorem



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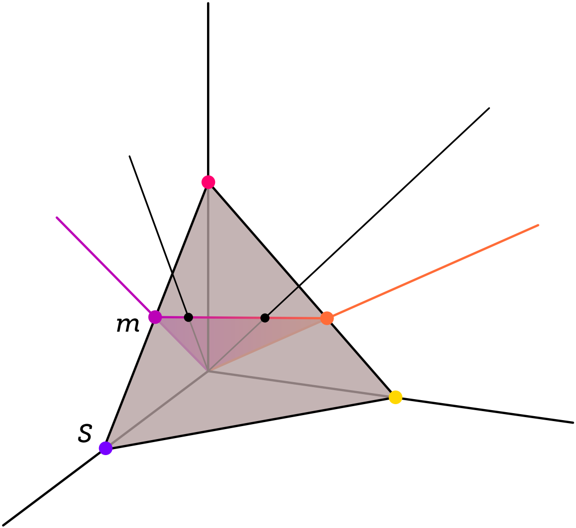


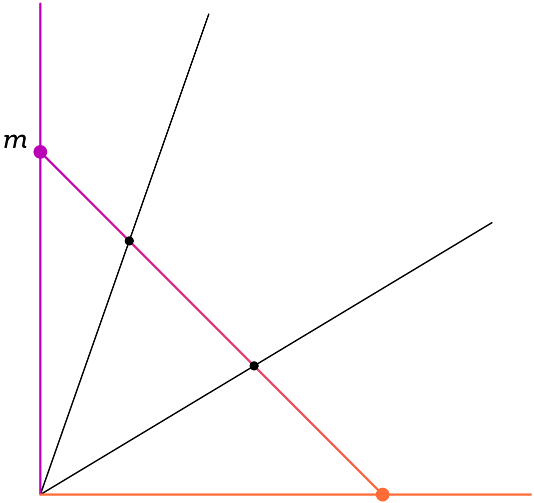
In general, b can have many fixed points.

If some power of B maps the non-negative orthant into the positive orthant, however, then b has just one fixed point.

To see why, suppose b has another fixed point.

Consider the plane M spanned by the corresponding eigenrays, which intersects S in a line segment m.

The Perron-Frobenius Theorem

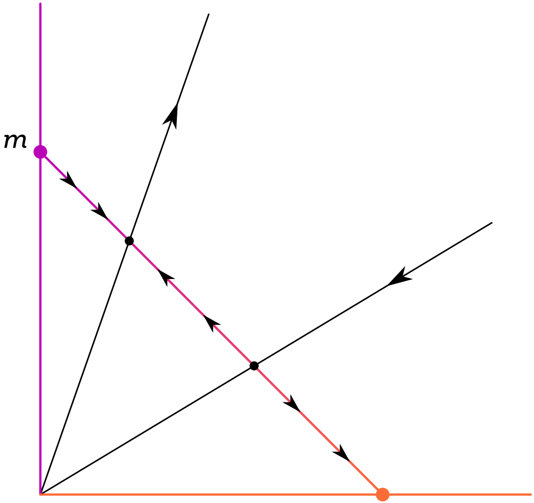


The map B restricts to a linear operator on M. Correspondingly, b restricts to a projective transformation of m.

If some power of B sends the non-negative orthant into the positive orthant, $b|_m$ cannot be the identity.

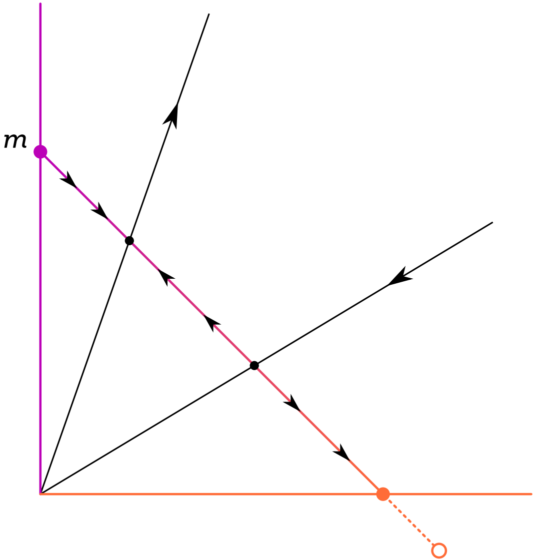
Thus, the two fixed points we started with are the only fixed points of $b|_m$. They correspond to two distinct eigenvalues of $B|_M$.

The Perron-Frobenius Theorem



The fixed point of $b|_m$ with the larger eigenvalue is attracting, and the one with the smaller eigenvalue is repelling.

The Perron-Frobenius Theorem



The fixed point of $b|_m$ with the larger eigenvalue is attracting, and the one with the smaller eigenvalue is repelling.

When $b|_m$ acts, the endpoint of m next to the repelling fixed point will get repelled clear out of the non-negative orthant. But that's impossible!

Thus, imagining that b could have more than one fixed point leads only to absurdity.