Relativity and Quantization

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1 Galilean relativity

1.1 Classical mechanics

1.1.1 States

The points of a 1+1-dimensional Galilean spacetime can be represented by the points (t, x) of $\mathbb{R}^2$. In this model, the trajectory of a free particle with mass m is represented by a straight line, and the only restriction on trajectories is that they can't be parallel to the line $\{(0, \lambda)\}_{\lambda \in \mathbb{R}}$. You could say that the state space $\mathcal{S}_m$ of a free particle with mass m is the set of lines that aren't parallel to $\{(0, \lambda)\}_{\lambda \in \mathbb{R}}$.

1.1.2 Observables

For any line $L \in \mathcal{S}_m$, there are unique real numbers $w(L)$ and $p(L)$ such that

$$L = \left\{ \left(\lambda, \frac{w(L) + p(L)\lambda}{m} \right) \right\}_{\lambda \in \mathbb{R}}.$$

Since these numbers describe observable properties of the particle, let's call them observables. The observable p is the particle's mass times its velocity—in other words, it's the particle's momentum. The observable w is the particle's mass times its position at time zero. There's no standard name for this quantity, so I'll call it the particle's "weighted position."

1.1.3 Transformations

If you move ξ units to the right, every point in spacetime appears to move ξ units to the left. Your change in position can therefore be represented by the transformation $(t, x) \mapsto (t, x - \xi)$. This transformation turns the line

$$L = \left\{ \left(\lambda, \frac{w(L) + p(L)\lambda}{m} \right) \right\}_{\lambda \in \mathbb{R}}$$

into the line

$$T_\xi L = \left\{\left(\lambda, \frac{w(L)+p(L)\lambda}{m} - \xi\right)\right\}_{\lambda\in\mathbb{R}}$$
$$= \left\{\left(\lambda, \frac{[w(L)-m\xi]+p(L)\lambda}{m}\right)\right\}_{\lambda\in\mathbb{R}}.$$

Notice that

$$w(T_\xi L) = w(L) - m\xi$$
$$p(T_\xi L) = p(L).$$

If you move τ units into the future, every point in spacetime appears to move τ units into the past. Your change in time can therefore be represented by the transformation $(t,x) \mapsto (t-\tau, x)$. This transformation turns the line

$$L = \left\{\left(\lambda, \frac{w(L)+p(L)\lambda}{m}\right)\right\}_{\lambda\in\mathbb{R}}$$

into the line

$$U_\tau L = \left\{\left(\lambda-\tau, \frac{w(L)+p(L)\lambda}{m}\right)\right\}_{\lambda\in\mathbb{R}}$$
$$= \left\{\left(\lambda, \frac{w(L)+p(L)[\lambda+\tau]}{m}\right)\right\}_{\lambda\in\mathbb{R}}.$$

Notice that

$$w(U_\tau L) = w(L) + \tau p(L)$$
$$p(U_\tau L) = p(L).$$

If you boost your velocity v units to the right, a point t units ahead of you in time appears to move vt units to the left. Your change in velocity can therefore be represented by the transformation $(t,x) \mapsto (t, x-vt)$. This transformation turns the line

$$L = \left\{\left(\lambda, \frac{w(L)+p(L)\lambda}{m}\right)\right\}_{\lambda\in\mathbb{R}}$$

into the line

$$B_v L = \left\{\left(\lambda, \frac{w(L)+p(L)\lambda}{m} - v\lambda\right)\right\}_{\lambda\in\mathbb{R}}$$
$$= \left\{\left(\lambda, \frac{w(L)+[p(L)-mv]\lambda}{m}\right)\right\}_{\lambda\in\mathbb{R}}.$$

Notice that

$$w(B_v L) = w(L)$$
$$p(B_v L) = p(L) - mv.$$

1.1.4 Group action

The space translations, time translations, and velocity boosts discussed in the last section form a set of generators for the Galilei group. The maps T_ξ, U_τ, and B_v therefore describe an action of the Galilei group on the state space $\mathcal{S}_m$.

1.2 Quantum mechanics

1.2.1 States and transformations

Now, let's consider a free quantum particle with mass m in a 1+1-dimensional Galilean spacetime. We can safely assume that the states of the particle should be represented by the rays of a complex Hilbert space $\mathcal{H}_m$. The fact that the particle lives in a Galilean spacetime, where observers can be translated and boosted into different frames of reference, strongly suggests that the Galilei group should act on $\mathcal{H}_m$ with transformations T_ξ, U_τ, and B_v, just like in the classical case. It turns out that if T_ξ, U_τ, and B_v are to preserve transition probabilities, they must generate a unitary or antiunitary projective representation of the Galilei group on $\mathcal{H}_m$ [3, §2.2]. We should therefore be looking for a complex Hilbert space $\mathcal{H}_m$ that carries a unitary or antiunitary projective representation of the Galilei group.

The set of solutions $\Psi \colon \mathbb{R}^2 \to \mathbb{C}$ to the equation

$$\left[i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right]\Psi = 0$$

turns out to have almost all of the properties we want. The equation is linear, so the solutions form a complex vector space, and the operators

$$\begin{aligned}
[T_\xi\Psi](t,x) &= \Psi(t, x+\xi)\\
[U_\tau\Psi](t,x) &= \Psi(t+\tau, x)\\
[B_v\Psi](t,x) &= \exp\left[-i\tfrac{m}{\hbar}\left(\tfrac{1}{2}v^2t + vx\right)\right]\Psi(t, x+vt)
\end{aligned}$$

generate a projective representation of the Galilei group on the solution set [1]. The sesquilinear form

$$\langle\Phi,\Psi\rangle = \int_{-\infty}^{\infty} \overline{\Phi}(0,x)\Psi(0,x)\,dx$$

is not an inner product on the solution set, because it's not always well-defined, but it is an inner product on the set of square-integrable solutions—that is, the set of solutions Ψ for which $\langle\Psi,\Psi\rangle$ is well-defined. We now have an inner product space, which can be completed to a Hilbert space. The operators T_ξ, U_τ, and B_v are unitary with respect to $\langle\cdot,\cdot\rangle$.

So, let $\mathcal{H}_m$ be the $\langle\cdot,\cdot\rangle$-completion of the set of square-integrable solutions $\Psi \colon \mathbb{R}^2 \to \mathbb{C}$ to the equation

$$\left[i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right]\Psi = 0.$$

1.2.2 Observables

It would be nice to have some quantum observables W and P whose expectation values act like the classical observables w and p under the action of the Galilei group. Knowing that w and p are the generators of boosts and space translations, respectively, in Hamiltonian mechanics, we might make the inspired guess that

$$\begin{aligned} W\Psi &= i\hbar \, \frac{d}{dv} B_v \Psi \bigg|_{v=0} \\ P\Psi &= -i\hbar \, \frac{d}{d\xi} T_\xi \Psi \bigg|_{\xi=0}. \end{aligned}$$

Explicitly,

$$\begin{aligned} W\Psi &= [mx - tP]\Psi \\ P\Psi &= -i\hbar \frac{\partial}{\partial x} \Psi. \end{aligned}$$

To check our guess, let's see how the expectation values of W and P change under the action of the Galilei group. In general, if A is an observable and F is a unitary map,

$$\langle A \rangle_{F\Psi} = \langle A \rangle_\Psi + \langle F^{-1}[A, F] \rangle_\Psi,$$

where $\langle A \rangle_\psi$ is shorthand for $\langle \Psi, A\Psi \rangle$. It's straightforward to show that

$$\begin{aligned} T_\xi^{-1}[W, T_\xi] &= -m\xi \\ T_\xi^{-1}[P, T_\xi] &= 0 \\ \\ U_\tau^{-1}[W, U_\tau] &= \tau P \\ U_\tau^{-1}[P, U_\tau] &= 0 \\ \\ B_v^{-1}[W, B_v] &= 0 \\ B_v^{-1}[P, B_v] &= -mv. \end{aligned}$$

Hence,

$$\begin{aligned} \langle W \rangle_{T_\xi \Psi} &= \langle W \rangle_\Psi - m\xi \\ \langle P \rangle_{T_\xi \Psi} &= \langle P \rangle_\Psi \\ \\ \langle W \rangle_{U_\tau \Psi} &= \langle W \rangle_\Psi + \tau \langle P \rangle_\Psi \\ \langle P \rangle_{U_\tau \Psi} &= \langle P \rangle_\Psi \\ \\ \langle W \rangle_{B_v \Psi} &= \langle W \rangle_\Psi \\ \langle P \rangle_{B_v \Psi} &= \langle P \rangle_\Psi - mv \end{aligned}$$

These relations match the classical ones found in §1.1.3.

2 Lorentzian relativity

2.1 Classical mechanics

2.1.1 States

The points of a 1+1-dimensional Lorentzian spacetime can be represented by the points (t, x) of $\mathbb{R}^2$. In this model, the trajectory of a free particle with mass m is represented by a straight line. The line through the points (t_1, x_1) and (t_2, x_2) is a possible trajectory if and only if $(t_2 - t_1)^2 - (x_2 - x_1)^2 > 0$. The state space $\mathcal{S}_m$ of a free particle with mass m is the set of all lines that are possible trajectories according to the rule above.

2.1.2 Observables

For any line $L \in \mathcal{S}_m$, there are unique real numbers $w(L)$ and $p(L)$ such that

$$L = \left\{ \left(\lambda, \frac{w(L) + p(L)\lambda}{\sqrt{m^2 + p(L)^2}} \right) \right\}_{\lambda \in \mathbb{R}}.$$

It will be convenient to define $h(L) = \sqrt{m^2 + p(L)^2}$.

The vector $(h(L), p(L))$ is parallel to the trajectory of the particle, and has Minkowski norm m. In other words, it's the particle's spacetime momentum. The observables h and p are therefore the particle's energy and spatial momentum, respectively. The observable w is the particle's energy times its position at time zero. In analogy with the Galilean case, I'll refer to w as the particle's "weighted position."

2.1.3 Transformations

If you move ξ units to the right, every point in spacetime appears to move ξ units to the left. Your change in position can therefore be represented by the transformation $(t, x) \mapsto (t, x - \xi)$. This transformation turns the line

$$L = \left\{ \left(\lambda, \frac{w(L) + p(L)\lambda}{h(L)} \right) \right\}_{\lambda \in \mathbb{R}}$$

into the line

$$\begin{aligned} T_\xi L &= \left\{ \left(\lambda, \frac{w(L) + p(L)\lambda}{h(L)} - \xi \right) \right\}_{\lambda \in \mathbb{R}} \\ &= \left\{ \left(\lambda, \frac{[w(L) - h(L)\xi] + p(L)\lambda}{h(L)} \right) \right\}_{\lambda \in \mathbb{R}}. \end{aligned}$$

Notice that

$$\begin{aligned} w(T_\xi L) &= w(L) - h(L)\xi \\ h(T_\xi L) &= h(L) \\ p(T_\xi L) &= p(L). \end{aligned}$$

If you move τ units into the future, every point in spacetime appears to move τ units into the past. Your change in time can therefore be represented by the transformation $(t,x) \mapsto (t-\tau, x)$. This transformation turns the line

$$L = \left\{ \left(\lambda, \frac{w(L)+p(L)\lambda}{h(L)} \right) \right\}_{\lambda\in\mathbb{R}}$$

into the line

$$\begin{aligned} U_\tau L &= \left\{ \left(\lambda - \tau, \frac{w(L)+p(L)\lambda}{h(L)} \right) \right\}_{\lambda\in\mathbb{R}} \\ &= \left\{ \left(\lambda, \frac{w(L)+p(L)[\lambda+\tau]}{h(L)} \right) \right\}_{\lambda\in\mathbb{R}}. \end{aligned}$$

Notice that

$$\begin{aligned} w(U_\tau L) &= w(L) + \tau p(L) \\ h(U_\tau L) &= h(L) \\ p(U_\tau L) &= p(L). \end{aligned}$$

A change in your velocity can be represented by the transformation $(t,x) \mapsto (t\cosh\sigma - x\sinh\sigma, -t\sinh\sigma + x\cosh\sigma)$. (The parameter σ is called the rapidity, and it's positive when your velocity is boosted to the right.) This transformation turns the line

$$L = \left\{ \left(\lambda, \frac{w(L)+p(L)\lambda}{h(L)} \right) \right\}_{\lambda\in\mathbb{R}}$$

into the line

$$\begin{aligned} B_\sigma L &= \left\{ \left(\lambda\cosh\sigma - \frac{w(L)+p(L)\lambda}{h(L)}\sinh\sigma, -\lambda\sinh\sigma + \frac{w(L)+p(L)\lambda}{h(L)}\cosh\sigma \right) \right\}_{\lambda\in\mathbb{R}} \\ &= \left\{ \left(-\frac{w(L)\sinh\sigma}{h(L)} + \frac{[h(L)\cosh\sigma - p(L)\sinh\sigma]\lambda}{h(L)}, \right.\right. \\ &\qquad \left.\left. \frac{w(L)\cosh\sigma}{h(L)} + \frac{[-h(L)\sinh\sigma + p(L)\cosh\sigma]\lambda}{h(L)} \right) \right\}_{\lambda\in\mathbb{R}} \\ &= \left\{ \left(-\frac{w(L)\sinh\sigma}{h(L)} + \lambda, \frac{w(L)\cosh\sigma}{h(L)} + \frac{[-h(L)\sinh\sigma + p(L)\cosh\sigma]\lambda}{h(L)\cosh\sigma - p(L)\sinh\sigma} \right) \right\}_{\lambda\in\mathbb{R}} \\ &= \left\{ \left(\lambda, \frac{w(L)\cosh\sigma}{h(L)} + \frac{[-h(L)\sinh\sigma + p(L)\cosh\sigma]\lambda}{h(L)\cosh\sigma - p(L)\sinh\sigma} \right.\right. \\ &\qquad \left.\left. + \frac{[-h(L)\sinh\sigma + p(L)\cosh\sigma]w(L)\sinh(\sigma)}{[h(L)\cosh\sigma - p(L)\sinh\sigma]h(L)} \right) \right\}_{\lambda\in\mathbb{R}} \\ &= \left\{ \left(\lambda, \frac{w(L)h(L)[\cosh^2\sigma - \sinh^2\sigma]}{[h(L)\cosh\sigma - p(L)\sinh\sigma]h(L)} + \frac{[-h(L)\sinh\sigma + p(L)\cosh\sigma]\lambda}{h(L)\cosh\sigma - p(L)\sinh\sigma} \right) \right\}_{\lambda\in\mathbb{R}} \\ &= \left\{ \left(\lambda, \frac{w(L) + [-h(L)\sinh\sigma + p(L)\cosh\sigma]\lambda}{h(L)\cosh\sigma - p(L)\sinh\sigma} \right) \right\}_{\lambda\in\mathbb{R}}. \end{aligned}$$

It's not hard to verify that

$$
\begin{aligned}
w(B_\sigma L) &= w(L) \\
h(B_\sigma L) &= h(L)\cosh\sigma - p(L)\sinh\sigma \\
p(B_\sigma L) &= -h(L)\sinh\sigma + p(L)\cosh\sigma.
\end{aligned}
$$

2.1.4 Group action

The space translations, time translations, and velocity boosts discussed in the last section form a set of generators for the Poincaré group. The maps T_ξ, U_τ, and B_σ therefore describe an action of the Poincaré group on the state space $\mathcal{S}_m$.

2.2 Quantum mechanics

2.2.1 States and transformations

Now, let's consider a free quantum particle with mass m in a 1+1-dimensional Lorentzian spacetime. By the reasoning used in §1.2.1, the state space of the particle should be a complex Hilbert space $\mathcal{H}_m$ that carries a unitary or antiunitary projective representation of the Poincaré group.

The set of solutions $\Psi\colon \mathbb{R}^2 \to \mathbb{C}$ to the equation

$$
\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \frac{m^2}{\hbar^2}\right]\Psi = 0
$$

turns out to have almost all of the properties we want. The equation is linear, so the solutions form a complex vector space, and the operators

$$
\begin{aligned}
[T_\xi\Psi](t,x) &= \Psi(t, x+\xi) \\
[U_\tau\Psi](t,x) &= \Psi(t+\tau, x) \\
[B_\sigma\Psi](t,x) &= \Psi(t\cosh\sigma + x\sinh\sigma, t\sinh\sigma + x\cosh\sigma)
\end{aligned}
$$

generate a representation of the Poincaré group on the solution set. The sesquilinear form

$$
\langle\Phi,\Psi\rangle = i\int_{-\infty}^{\infty}\left[\overline{\Phi}(0,x)\frac{\partial\Psi}{\partial x}(0,x) - \Psi(0,x)\frac{\partial\overline{\Phi}}{\partial x}(0,x)\right]dx
$$

is not an inner product on the solution set, because it's not always well-defined—and when it is defined, it's not positive definite. We can solve both problems at once by letting $\mathcal{H}_m$ be the $\langle\cdot,\cdot\rangle$-completion of the set of functions

$$
\Psi(t,x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(p)\exp\left[\frac{i}{\hbar}\left(-\sqrt{m^2+p^2}\,t + px\right)\right]dp
$$

for which $\langle\Psi,\Psi\rangle$ is well-defined [2, §14.2].

2.2.2 Observables

It would be nice to have some quantum observables W, H and P whose expectation values act like the classical observables w, h and p under the action of the Poincaré group. Knowing that w, h and p are the generators of boosts, time translations, and space translations in Hamiltonian mechanics, we might make the inspired guess that

$$\begin{aligned} W\Psi &= i\hbar \left.\frac{d}{d\sigma} B_\sigma \Psi\right|_{\sigma=0} \\ H\Psi &= i\hbar \left.\frac{d}{d\tau} U_\tau \Psi\right|_{\tau=0} \\ P\Psi &= -i\hbar \left.\frac{d}{d\xi} T_\xi \Psi\right|_{\xi=0}. \end{aligned}$$

Explicitly,

$$\begin{aligned} W\Psi &= i\hbar \left[x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} \right] \Psi \\ H\Psi &= i\hbar \frac{\partial}{\partial t}\Psi \\ P\Psi &= -i\hbar \frac{\partial}{\partial x}\Psi. \end{aligned}$$

To check our guess, let's see how the expectation values of W, H and P change under the action of the Poincaré group. It's straightforward to show that

$$\begin{aligned} T_\xi^{-1}[W, T_\xi] &= -\xi H \\ T_\xi^{-1}[H, T_\xi] &= 0 \\ T_\xi^{-1}[P, T_\xi] &= 0 \end{aligned}$$

$$\begin{aligned} U_\tau^{-1}[W, U_\tau] &= \tau P \\ U_\tau^{-1}[P, U_\tau] &= 0 \\ U_\tau^{-1}[P, U_\tau] &= 0 \end{aligned}$$

$$\begin{aligned} B_\sigma^{-1}[W, B_\sigma] &= 0 \\ B_\sigma^{-1}[H, B_\sigma] &= (\cosh\sigma)H - (\sinh\sigma)P - H \\ B_\sigma^{-1}[P, B_\sigma] &= -(\sinh\sigma)H + (\cosh\sigma)P - P. \end{aligned}$$

Using the formula from §1.2.2, it follows that

$$\begin{aligned}
\langle W\rangle_{T_\xi\Psi} &= \langle W\rangle_\Psi - \langle H\rangle_\Psi \xi \\
\langle H\rangle_{T_\xi\Psi} &= \langle H\rangle_\Psi \\
\langle P\rangle_{T_\xi\Psi} &= \langle P\rangle_\Psi \\
\\
\langle W\rangle_{U_\tau\Psi} &= \langle W\rangle_\Psi + \tau\langle P\rangle_\Psi \\
\langle H\rangle_{U_\tau\Psi} &= \langle H\rangle_\Psi \\
\langle P\rangle_{U_\tau\Psi} &= \langle P\rangle_\Psi \\
\\
\langle W\rangle_{B_\sigma\Psi} &= \langle W\rangle_\Psi \\
\langle H\rangle_{B_\sigma\Psi} &= \langle H\rangle_\Psi \cosh\sigma - \langle P\rangle_\Psi \sinh\sigma \\
\langle P\rangle_{B_\sigma\Psi} &= -\langle H\rangle_\Psi \sinh\sigma + \langle P\rangle_\Psi \cosh\sigma.
\end{aligned}$$

These relations match the classical ones found in §2.1.3.

3 Representations on $L^2(\mathbb{R})$

In the last few pages, I argued that the state space of a free quantum particle with mass m in a 1+1-dimensional Galilean or Lorentzian spacetime should be a complex Hilbert space that carries a unitary or antiunitary projective representation of the spacetime symmetry group. I then showed you two differential equations whose solution sets, when suitably restricted, completed, and equipped with inner products, become Hilbert spaces that carry unitary projective representations of the Galilei and Lorentz groups, respectively. In short, I showed you Hilbert spaces that can be used to describe free quantum particles in 1+1-dimensional Galilean and Lorentzian spacetimes.

The Hilbert spaces I showed you are complicated to describe, and their inner products are difficult to work with. Can we find more user-friendly Hilbert spaces to use for our calculations?

Yes, we can! Remarkably, both of the Hilbert spaces I showed you are isomorphic to the Hilbert space $L^2(\mathbb{R})$—the set of square-integrable functions on $\mathbb{R}$, with the inner product

$$\langle \phi, \psi\rangle = \int_{-\infty}^{\infty} \overline{\phi}(p)\psi(p)\, dp.$$

The representations of the Galilei and Poincaré groups that I showed you are therefore unitarily equivalent to representations on $L^2(\mathbb{R})$. These representations are summarized below.

3.1 Galilean relativity

Any element of the Hilbert space described in §1.2.1 can be written uniquely in the form

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(p) \exp\left[\frac{i}{\hbar}\left(-\frac{p^2}{2m}\,t + px\right)\right] dp,$$

where $\psi \in L^2(\mathbb{R})$, and you can easily verify that the map

$$\psi \mapsto \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(p) \exp\left[\frac{i}{\hbar}\left(-\frac{p^2}{2m}\,t + px\right)\right] dp.$$

is unitary. Under this isomorphism, the operators T_ξ, U_τ, and B_v from §1.2.1 are equivalent to the operators

$$\begin{aligned}
[\hat{T}_\xi \psi](p) &= \exp\left[\tfrac{i}{\hbar} p\xi\right] \psi(p) \\
[\hat{U}_\tau \psi](p) &= \exp\left[-\tfrac{i}{\hbar} \tfrac{p^2}{2m}\,\tau\right] \psi(p) \\
[\hat{B}_v \psi](p) &= \psi(p + mv)
\end{aligned}$$

on $L^2(\mathbb{R})$, and the operators W and P from §1.2.2 are equivalent to the operators

$$\begin{aligned}
\hat{W}\psi &= i\hbar m \frac{\partial}{\partial p}\psi \\
\hat{P}\psi &= p\psi
\end{aligned}$$

on $L^2(\mathbb{R})$.

3.2 Lorentzian relativity

Any element of the Hilbert space described in §2.2.1 can be written uniquely in the form

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(p) \exp\left[\frac{i}{\hbar}\left(-\sqrt{m^2 + p^2}\,t + px\right)\right] dp,$$

where $\psi \in L^2(\mathbb{R})$, and you can easily verify that the map

$$\psi \mapsto \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(p) \exp\left[\frac{i}{\hbar}\left(-\sqrt{m^2 + p^2}\,t + px\right)\right] dp$$

is unitary. Under this isomorphism, the operators T_ξ, U_τ, and B_σ from §2.2.1 are equivalent to the operators

$$\begin{aligned}
[\hat{T}_\xi \psi](p) &= \exp\left[\tfrac{i}{\hbar} p\xi\right] \psi(p) \\
[\hat{U}_\tau \psi](p) &= \exp\left[-\tfrac{i}{\hbar} \sqrt{m^2 + p^2}\,\tau\right] \psi(p) \\
[\hat{B}_\sigma \psi](p) &= \left(\frac{p}{\sqrt{m^2 + p^2}} \sinh\sigma + \cosh\sigma\right) \psi\left(\sqrt{m^2 + p^2}\sinh\sigma + p\cosh\sigma\right)
\end{aligned}$$

on $L^2(\mathbb{R})$, and the operators W, H and P from §2.2.2 are equivalent to the operators

$$\begin{aligned}
\hat{W}\psi &= i\hbar\left[\sqrt{m^2+p^2}\,\frac{\partial}{\partial p} + \frac{p}{\sqrt{m^2+p^2}}\right]\psi \\
\hat{H}\psi &= \sqrt{m^2+p^2}\,\psi \\
\hat{P}\psi &= p\psi
\end{aligned}$$

on $L^2(\mathbb{R})$.

References

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[3] Steven Weinberg. *The Quantum Theory of Fields*, volume I. Cambridge University Press, 1995.