Symmetry, isogeny, homology and filtration properties for finite group schemes over perfect fields

Ofer Gabber and Adrian Vasiu

May 27, 2022

Abstract. Let k be a perfect field of characteristic $p > 0$. Let $\mathbb{B}$ be a finite flat separable $\mathbb{Z}_p$-algebra. Let D and E be two Barsotti–Tate groups over k which are left $\mathbb{B}$-modules, i.e., we have ring homomorphisms from $\mathbb{B}$ to $\mathrm{End}(D)$ and $\mathrm{End}(E)$. We show that the non-decreasing sequence $(\dim(\mathbf{Hom}_{\mathbb{B}}(D[p^n], E[p^n])))_{n\geq 0}$ becomes constant of constant value that does not change if D and E are interchanged or replaced by Barsotti–Tate groups D' and E' which are left $\mathbb{B}$-modules isogenous to D and E (respectively). The case when D and E have the same dimension and codimension is generalized to certain relative contexts provided by Barsotti–Tate groups over k endowed with a group in the sense of [GV]. If a truncated Barsotti–Tate group G of level m over k and a finite commutative group scheme H over k annihilated by p^m are also left $\mathbb{B}$-modules, then we classify all cases when $\dim(\mathbf{Hom}_{\mathbb{B}}(G, H)) = \dim(\mathbf{Hom}_{\mathbb{B}}(H, G))$; for instance, the identity holds always if $\mathbb{B} = \mathbb{Z}_p$. We also identify general cases when a stronger form of this identity holds, which is an identity that involves the Lie algebras of $\mathbf{Hom}_{\mathbb{B}}(G, H)_{\mathrm{red}}$ and $\mathbf{Hom}_{\mathbb{B}}(H, G)_{\mathrm{red}}$ and a quotient of the Grothendieck group of the multiplicative monoid scheme over k associated to the reduced ring scheme $\mathbf{End}_{\mathbb{B}}(G)_{\mathrm{red}} \times_k \mathbf{End}_{\mathbb{B}}(H)^{\mathrm{op}}_{\mathrm{red}}$. In the process, we introduce and characterize several subclasses of 'balanced' objects H in terms of homological and filtration properties, including the subclass of all H which lift to the ring of p-typical Witt vectors with coefficients in k.

Key words: Barsotti–Tate group, category, Dieudonné module, filtration, Grothendieck group, group scheme, lattice, Lie algebra, matrix, module, monoid, quasi-algebraic group, proalgebraic group, representation, ring, truncated Barsotti–Tate group.

MSC 2020: 11F22, 11G10, 11G18, 14D07, 14F30, 14G35, 14K10, 14L05, 14L15, 17B45, 18B35, 18F30, 18G05, 18G15, 18G50, 20G15, 20M13, and 20M32.

1 Introduction

Let p be a prime, let k be a perfect field of characteristic p and let $\bar{k}$ be an algebraic closure of k. We will abbreviate a Barsotti–Tate group (i.e., a p-divisible group) by BT and a truncated Barsotti–Tate group of level $m \in \mathbb{N}^*$ by BT_m. Let G and H be two objects of the category $\mathcal{F}_k$ of finite commutative group schemes over k of p power order. Let $\mathbf{Hom}(G, H)$ be the affine group scheme over k of homomorphisms from G to H. We have $\mathrm{Hom}(G, H) = \mathbf{Hom}(G, H)(k)$ and we define

$$\gamma(G, H) := \dim(\mathbf{Hom}(G, H)).$$

Let ${}^{\mathrm{t}} : \mathcal{F}_k \to \mathcal{F}_k$ be the Cartier duality involutory antiequivalence of categories. So $\delta^{\mathrm{t}} : H^{\mathrm{t}} \to G^{\mathrm{t}}$ is the Cartier dual of a homomorphism $\delta : G \to H$. The canonical identification $\mathbf{Hom}(G, H) = \mathbf{Hom}(H^{\mathrm{t}}, G^{\mathrm{t}})$ in particular gives

$$\gamma(G, H) = \gamma(H^{\mathrm{t}}, G^{\mathrm{t}}). \tag{1}$$

Let $\boldsymbol{\alpha}_p := \boldsymbol{\alpha}_{p,k}$ be the kernel of the Frobenius endomorphism of $\mathbb{G}_{a,k}$; we have $\boldsymbol{\alpha}_p = \boldsymbol{\alpha}_p^{\mathrm{t}}$. We recall that the a-number of G is

$$a_G := \gamma(\boldsymbol{\alpha}_p, G).$$

Equivalently, a_G is the largest integer such that $\boldsymbol{\alpha}_p^{a_G}$ is isomorphic to a subgroup scheme of G, and such a subgroup scheme of G is unique and it will be denoted by $\boldsymbol{\alpha}_p(G)$. In general, $a_G \neq a_{G^{\mathrm{t}}}$ (see Example 1 of Subsection 2.1) but it is well-known that if G is a truncated BT, then we have an equality

$$a_G = a_{G^{\mathrm{t}}}$$

(for instance, see [GV], Subsect. 3.5), which, based on Equation (1), can be rewritten more symmetrically as

$$\gamma(\boldsymbol{\alpha}_p, G) = \gamma(G, \boldsymbol{\alpha}_p). \tag{2}$$

The goal of the paper is to generalize Equation (2) in order to get symmetry and isogeny properties for several subclasses of objects of $\mathcal{F}_k$. To achieve this goal, as main tools, a few filtrations and duality and homological theories, notions and properties are used or developed.

We begin with the 'sequence context' provided by two BTs D and E over k. Following [GV], Subsect. 6.1, let $n_{D,E}$ be the smallest non-negative integer for which the following statement holds: if $\mathcal{E}$ is a BT over $\bar{k}$ endowed with a short exact sequence $0 \to E_{\bar{k}} \to \mathcal{E} \to D_{\bar{k}} \to 0$ whose truncation of level $n_{D,E}$ splits, then this short exact sequence splits. From properties (i) and (ii) of [GV], Subsect. 6.1 we get that $n_{D,E}$ is also the smallest non-negative integer for which the sequence $(\gamma_{D,E}(n))_{n \geq n_{D,E}}$ is constant, where, following [GV], for $n \in \mathbb{N}$ we will use the shorter notation

$$\gamma_{D,E}(n) := \gamma(D[p^n], E[p^n]).[1]$$

Following [LNV], Def. 7.9 we denote

$$s_{D,E} := \gamma_{D,E}(n_{D,E}) \quad \text{and} \quad s_D := s_{D,D}.$$

We generalize the invariants $n_{D,E}$ and $s_{D,E}$ as follows. Let $\mathbb{B}$ be a finite flat separable $\mathbb{Z}_p$-algebra, i.e., a $\mathbb{Z}_p$-algebra isomorphic to a finite product of matrix algebras over finite unramified extensions of $\mathbb{Z}_p$.

Definition 1. *We say that a commutative group scheme Υ over k annihilated by a power of p is a left $\mathbb{B}$-module if it is endowed with a ring homomorphism $\rho_\Upsilon : \mathbb{B} \to \mathrm{End}(\Upsilon)$. If $\delta : \Upsilon_1 \to \Upsilon_2$ is a homomorphism between commutative group schemes over k annihilated by a power of p and if $\rho_{\Upsilon_1} : \mathbb{B} \to \mathrm{End}(\Upsilon_1)$ and $\rho_{\Upsilon_2} : \mathbb{B} \to \mathrm{End}(\Upsilon_2)$ are two homomorphisms defining left $\mathbb{B}$-module structures, we say that δ is a left $\mathbb{B}$-module homomorphism if for all $b \in \mathbb{B}$ we have $\rho_{\Upsilon_2}(b) \circ \delta = \delta \circ \rho_{\Upsilon_1}(b)$.*

We similarly speak about BTs over k which are left $\mathbb{B}$-modules and about left $\mathbb{B}$-module homomorphisms and isogenies between them.

Let $\mathcal{F}_k^{\mathbb{B}}$ be the category of finite commutative group schemes over k of p power order endowed with left $\mathbb{B}$-actions; we identify canonically $\mathcal{F}_k^{\mathbb{Z}_p} = \mathcal{F}_k$. If (G, ρ_G) and (H, ρ_H) are objects of $\mathcal{F}_k^{\mathbb{B}}$, then we will usually not mention explicitly the ring homomorphisms $\rho_G : \mathbb{B} \to \mathrm{End}(G)$ and $\rho_H : \mathbb{B} \to \mathrm{End}(H)$

[1] We have $\gamma_{D,E}(1) = 0 \Leftrightarrow a_D a_E = 0 \Leftrightarrow D$ or E is ordinary (i.e., is a direct sum of étale and multiplicative type BTs over k), and these imply that $n_{D,E} = 0$.

and we let $\mathbf{Hom}_{\mathbb{B}}(G,H)$ be the closed subgroup scheme of $\mathbf{Hom}(G,H)$ of left $\mathbb{B}$-module homomorphisms: for a k-algebra R, we have

$$\mathbf{Hom}_{\mathbb{B}}(G,H)(R) := \{\delta \in \mathbf{Hom}(G,H)(R) | \rho_H(b)_R \circ \delta = \delta \circ \rho_G(b)_R \ \ \forall b \in \mathbb{B}\}.$$

Let $\mathrm{Hom}_{\mathbb{B}}(G,H) := \mathrm{Hom}((G,\rho_G),(H,\rho_H)) = \mathbf{Hom}_{\mathbb{B}}(G,H)(k) \subset \mathrm{Hom}(G,H)$. Let

$$\gamma_{\mathbb{B}}(G,H) := \dim(\mathbf{Hom}_{\mathbb{B}}(G,H)).$$

If D and E are also left $\mathbb{B}$-modules, let

$$\gamma^{\mathbb{B}}_{D,E}(n) := \gamma_{\mathbb{B}}(D[p^n], E[p^n]).$$

The abstract groups $\mathrm{Ext}^1_{\mathbb{B}}(D,E)$ and $\mathrm{Ext}^1_{\mathbb{B}}(G,H)$ are defined similarly to $\mathrm{Ext}^1(D,E)$ and $\mathrm{Ext}^1(G,H)$ (respectively) but using short exact sequences of left $\mathbb{B}$-modules. So $\mathbf{Hom}(G,H) = \mathbf{Hom}_{\mathbb{Z}_p}(G,H)$, $\gamma(G,H) = \gamma_{\mathbb{Z}_p}(G,H)$, $\mathrm{Ext}^1(G,H) = \mathrm{Ext}^1_{\mathbb{Z}_p}(G,H)$ and $\mathbf{End}_{\mathbb{B}}(G)$ is a ring scheme over k.

Theorem 1. *If the* BT*s D and E over k are left $\mathbb{B}$-modules, then the following three properties hold:*

(a) *There exists a smallest non-negative integer $n^{\mathbb{B}}_{D,E}$ for which the following statement holds: if $\mathcal{E}$ is a* BT *over $\bar{k}$ which is a left $\mathbb{B}$-module and we have a short exact sequence $0 \to E_{\bar{k}} \to \mathcal{E} \to D_{\bar{k}} \to 0$ of left $\mathbb{B}$-modules whose truncation of level $n^{\mathbb{B}}_{D,E}$ splits, then this short exact sequence splits.*

(b) *The number $n^{\mathbb{B}}_{D,E}$ is the smallest non-negative integer for which the sequence $(\gamma^{\mathbb{B}}_{D,E}(n))_{n \geq n^{\mathbb{B}}_{D,E}}$ is constant.*

(c) *The dimension*

$$s^{\mathbb{B}}_{D,E} := \gamma^{\mathbb{B}}_{D,E}(n^{\mathbb{B}}_{D,E})$$

is a symmetric isogeny invariant: if $D' \dashrightarrow D$ and $E' \dashrightarrow E$ are left $\mathbb{B}$-module isogenies between BT*s over k, then we have $s^{\mathbb{B}}_{D,E} = s^{\mathbb{B}}_{E,D} = s^{\mathbb{B}}_{D',E'}$. Moreover, we have the symmetry property $n^{\mathbb{B}}_{D,E} = n^{\mathbb{B}}_{E,D}$.*

The case $D = E$, $D' = E'$ and $\mathbb{B} = \mathbb{Z}_p$ of Theorem 1 (i.e., the equality $s_D = s_{D'}$) was first proved in [V2], Thm. 1.2 (e) (see also [GV], Rm. 4.5). From the very definition of $n^{\mathbb{B}}_{D,E}$ in terms of extensions and the symmetry property $n^{\mathbb{B}}_{D,E} = n^{\mathbb{B}}_{E,D}$ we get:

Corollary 1. *We assume that $k = \bar{k}$. For $n \in \mathbb{N}$ we consider the two homomorphisms of abstract groups $\mathrm{Ext}^1_{\mathbb{B}}(D,E) \to \mathrm{Ext}^1_{\mathbb{B}}(D[p^n], E[p^n])$ and $\mathrm{Ext}^1_{\mathbb{B}}(E,D) \to \mathrm{Ext}^1_{\mathbb{B}}(E[p^n], D[p^n])$. Then one is injective if and only if the other one is injective.*

Subsection 3.1 proves Theorem 1 (a) and (b) based on [GV]. Theorem 1 (c) is proved is Subsections 3.2 to 3.4 using general properties of quasi-algebraic and affine proalgebraic groups over k recalled or proved in Subsections 2.2 to 2.5. In particular, we will use extensively the Serre duality for unipotent connected commutative quasi-algebraic groups over k and its connection to duals of finitely generated modules over the ring $W(k)$ of p-typical Witt vectors with coefficients in k as recalled in Subsection 2.5.

The particular case of Theorem 1 in which D and E have the same dimension and codimension is generalized in Subsection 3.5 to relative contexts provided by quadruples $(L, \phi, \vartheta, \mathbf{G})$ and $(L, g\phi, \vartheta g^{-1}, \mathbf{G})$, where (L, ϕ, ϑ) and $(L, g\phi, \vartheta g^{-1})$ are the (contravariant) Dieudonné modules of two BTs over k, where $\mathbf{G}$ is a smooth integral closed subgroup scheme of $\mathbf{GL}_L$ subject to the two axioms of [GV], Sect. 5, and where $g \in \mathbf{G}(W(k))$ is such that the Lie algebra of $\mathbf{G}$ is invariant under the left (equivalently, right) multiplication by g; this invariance holds, for instance, if $\mathbf{G}$ is the group scheme of invertible elements of a $W(k)$-subalgebra of $\mathrm{End}_{W(k)}(L)$ which as a $W(k)$-module is a direct summand of $\mathrm{End}_{W(k)}(L)$.

The motivation for Theorem 1 and these generalizations stems out from applications to level m stratifications of special fibers of good integral models of Shimura varieties of Hodge type in unramified mixed characteristic $(0, p)$ (see [V2], Sect. 4) and [V4], Cor. 1.4). Theorem 5 can be used for applications to all unitary Shimura varieties which involve adjoint group schemes over $\mathrm{Spec}\,\mathbb{Z}_p$ that are products of Weil restrictions of $\mathbf{PGL}_n$ group schemes over unramified finite covers of $\mathrm{Spec}\,\mathbb{Z}_p$ (here $n \geq 2$).

For a ring A (with identity), let $\mathit{Mod}_{\mathit{fin.length}}(A)$ be the category of left A-modules of finite length; we will omit 'left' from its notation as it will be used only for self-dual rings A, i.e., for rings A isomorphic to their dual A^{op}. The classical (contravariant) Dieudonné theory constructs a self-dual *Cartier–Dieudonné ring* $\mathbb{E} = \mathbb{E}(k)$ (see Subsection 2.1) and an antiequivalence of categories

$$\mathbb{D} : \mathcal{F}_k \to \mathit{Mod}_{\mathit{fin.length}}(\mathbb{E}).$$

Let $\mathbb{J} = \mathbb{J}(k) := \mathbb{E} \otimes_{\mathbb{Z}_p} \mathbb{B}^{\mathrm{op}}$. For $m \in \mathbb{N}^*$, we have self-dual rings $\mathbb{B}_m := \mathbb{B}/p^m\mathbb{B}$, $\mathbb{E}_m = \mathbb{E}_m(k) := \mathbb{E}/p^m\mathbb{E}$ and $\mathbb{J}_m = \mathbb{J}_m(k) := \mathbb{J}/p^m\mathbb{J} = \mathbb{E}_m \otimes_{\mathbb{Z}/p^m\mathbb{Z}} \mathbb{B}_m^{\mathrm{op}}$. If $\mathcal{F}_{k,m}$ and $\mathcal{F}_{m,k}^{\mathbb{B}}$ are the full subcategories of $\mathcal{F}_k$ and $\mathcal{F}_k^{\mathbb{B}}$ (respectively) defined by all objects annihilated by p^m, then we obtain antiequivalences of categories

$$\mathbb{D} : \mathcal{F}_{k,m} \to \mathit{Mod}_{\mathit{fin.length}}(\mathbb{E}_m),$$

$$\mathbb{D}^{\mathbb{B}} : \mathcal{F}_k^{\mathbb{B}} \to \mathit{Mod}_{\mathit{fin.length}}(\mathbb{J}) \quad \text{and} \quad \mathbb{D}^{\mathbb{B}} : \mathcal{F}_{k,m}^{\mathbb{B}} \to \mathit{Mod}_{\mathit{fin.length}}(\mathbb{J}_m).$$

The identification $\mathcal{F}_k^{\mathbb{Z}_p} = \mathcal{F}_k$ restricts to an identification $\mathcal{F}_{k,m}^{\mathbb{Z}_p} = \mathcal{F}_{k,m}$ and we also identify canonically $\mathbb{D}^{\mathbb{Z}_p} = \mathbb{D}$.

Due to standard Morita equivalences, it is often more convenient to work with the subring $\mathbb{I}_m := \mathbb{E}_m \otimes_{\mathbb{Z}/p^m\mathbb{Z}} \mathbb{C}_m$ of $\mathbb{J}_m$, where $\mathbb{C}_m$ is the center of $\mathbb{B}_m^{\mathrm{op}}$. The Grothendieck group $K_0(\mathbb{J}_m)$ of the monoid of isomorphism classes of finitely generated projective left $\mathbb{J}_m$-modules is a free abelian group whose rank is a positive integer easily computable in terms of k and $\mathbb{C}_1$ (see Corollary 4 (a)). Moreover, the Frobenius automorphism σ of k acts naturally on $K_0(\mathbb{J}_m)$. If $G_0(\mathbb{J}_m)$ is the Grothendieck group of the abelian category $\mathit{Mod}_{\mathit{fin.length}}(\mathbb{J}_m)$, then we have a natural bilinear map

$$\langle\ ,\rangle : K_0(\mathbb{J}_m) \times G_0(\mathbb{J}_m) \to \mathbb{Z} \tag{3}$$

(see Equation (28) of Subsection 6.2). If G and H are left $\mathbb{B}$-module annihilated by p^m, we define their classes $[G] := [\mathbb{D}^{\mathbb{B}}(G)], [H] := [\mathbb{D}^{\mathbb{B}}(H)]$ in $G_0(\mathbb{J}_m)$ and we remark that both $\boldsymbol{\alpha}_p(G)$ and $(\boldsymbol{\alpha}_p(G^{\mathrm{t}}))^{\mathrm{t}}$ are semisimple left $\mathbb{B}$-modules. Moreover, if G is also a BT_m, then we show that there exists a projective resolution $0 \to P_1 \to P_0 \to \mathbb{D}^{\mathbb{B}}(G) \to 0$ with P_1 and P_0 finitely generated projective left $\mathbb{J}_m$-modules (see Proposition 4) and we define a class

$$[G] := [\mathbb{D}^{\mathbb{B}}(G)] = [P_0] - [P_1] \in K_0(\mathbb{J}_m) = K_0(\mathbb{I}_m)$$

satisfying $\mathrm{index}([G]) = 0$, where the homomorphism $\mathrm{index} : K_0(\mathbb{J}_m) \to \mathbb{Z}$ maps the isomorphism class of each indecomposable finitely generated projective left $\mathbb{J}_m$-module to 1 (see Corollary 4 (b)). The class $[G]$ is zero if and only if it is fixed by σ.

If $\mathcal{F}_{k,m,\text{ét}}^{\mathbb{B}}, \mathcal{F}_{k,m,\text{mult}}^{\mathbb{B}}$ and $\mathcal{F}_{k,m,\text{loc-loc}}^{\mathbb{B}}$ are the full subcategories of $\mathcal{F}_{k,m}^{\mathbb{B}}$ formed by objects which as finite groups over k are étale, of multiplicative type and local-local (respectively), then we have a natural equivalence of categories

$$\mathcal{F}_{k,m}^{\mathbb{B}} \to \mathcal{F}_{k,m,\text{ét}}^{\mathbb{B}} \times \mathcal{F}_{k,m,\text{mult}}^{\mathbb{B}} \times \mathcal{F}_{k,m,\text{loc-loc}}^{\mathbb{B}}$$

which induces a 'product decomposition' isomorphism

$$G_0(\mathbb{J}_m) \simeq G_0^{\text{ét}}(\mathbb{J}_m) \times G_0^{\text{mult}}(\mathbb{J}_m) \times G_0^{\text{loc-loc}}(\mathbb{J}_m).$$

Definition 2. (a) *If G is a BT_m over k and a left $\mathbb{B}$-module, we say it is projectively balanced if we have $[G] = 0 \in K_0(\mathbb{I}_m)$, equivalently, we have $[G[p]] = 0 \in K_0(\mathbb{I}_1)$.*

(b) *If G is a BT_m over k endowed with a left $\mathbb{B}$-action, we say it is fully projectively balanced if we have $[G[p]] = 0 \in K_0(\mathbb{E}_1 \otimes_{\mathbb{F}_p} \mathbb{C}_{G,\mathbb{B}})$, where $\mathbb{C}_{G,\mathbb{B}}$ is the center of the semisimplification of $\mathrm{Im}(\mathrm{End}_{\mathbb{B}}(G_{\bar{k}}) \to \mathrm{End}_{\mathbb{B}}(G[p]_{\bar{k}}))$ and $G[p]_{\bar{k}}$ is endowed with a unique up to isomorphism left $\mathbb{C}_{G,\mathbb{B}}$-action via every $\mathbb{C}_1$-algebra homomorphism $\mathbb{C}_{G,\mathbb{B}} \to \mathrm{Im}(\mathrm{End}_{\mathbb{B}}(G_{\bar{k}}) \to \mathrm{End}_{\mathbb{B}}(G[p]_{\bar{k}}))$ that lifts the identity automorphism of $\mathbb{C}_{G,\mathbb{B}}$ (see Subsection 5.3).*

(c) *An object H of $\mathcal{F}^{\mathbb{B}}_{k,m}$ is called balanced if $[H] = [H^{(p)}] \in G_0(\mathbb{J}_m)$ and is called $\boldsymbol{\alpha}_p$-balanced if the left $\mathbb{B}$-modules $\boldsymbol{\alpha}_p(H)$ and $(\boldsymbol{\alpha}_p(H^{\mathrm{t}}))^{\mathrm{t}}$ are isomorphic (equivalently, $\boldsymbol{\alpha}_p(H)$ and $(\boldsymbol{\alpha}_p(H^{\mathrm{t}}))^{\mathrm{t}}$ are isomorphic when endowed with left $\mathbb{C}_1$-actions).*

(d) *An object H of $\mathcal{F}^{\mathbb{B}}_{k,m}$ is called fully balanced if the classes of H and $H^{(p)}$ coincide in the Grothendieck group of the abelian category of finite commutative group schemes over k annihilated by p^m and endowed with a left action by the reduced multiplicative monoid scheme over k associated to the ring scheme $\mathbf{End}_{\mathbb{B}}(H)$ and is called fully $\boldsymbol{\alpha}_p$-balanced if $\boldsymbol{\alpha}_p(H)$ and $(\boldsymbol{\alpha}_p(H^{\mathrm{t}}))^{\mathrm{t}}$ are isomorphic as left modules over the mentioned monoid scheme.*

(e) *An object H of $\mathcal{F}^{\mathbb{B}}_{k,m}$ is called BT-balanced if it has an increasing separated and exhaustive filtration in $\mathcal{F}^{\mathbb{B}}_{k,m}$ whose quotients are BT_1s endowed with a left $\mathbb{B}$-action.*

(f) *An object of $\mathcal{F}_k$ is called fully BT-balanced or unramified liftable if it is the fiber over k of a finite flat commutative group scheme over $\mathrm{Spec}\, W(k)$.*

If G is a BT_m over k endowed with a left $\mathbb{B}$-action, then it is projectively balanced (resp. fully projectively balanced) if $\mathbb{C}_1$ (resp. $\mathbb{C}_{G,\mathbb{B}}$) is a product of finite fields which are linearly disjoint from a given field l over $\mathbb{F}_p$, where l can be any subfield of $\bar{k}$ such that $G[p]_{\bar{k}}$ is defined over l (see Corollary 4 (a) and Example 7); also it is $\boldsymbol{\alpha}_p$-balanced if and only if the object $G[p]$ of $\mathcal{F}_{k,1}$ is so. An object H of $\mathcal{F}^{\mathbb{B}}_{k,m}$ is fully balanced if either is defined over $\mathbb{F}_p$ (i.e., is the pull back of an object of $\mathcal{F}^{\mathbb{B}}_{\mathbb{F}_p,m}$) or (see Lemma 10 (b)) is BT-balanced or (see Lemma 10 (a)) as an object of $\mathcal{F}_{k,m}$ is fully BT-balanced.

In Section 6 we will prove the following general symmetry principle.

Theorem 2. *Let $m \in \mathbb{N}^*$. Then the following three properties hold:*

(a) *Let G and H be objects of $\mathcal{F}^{\mathbb{B}}_{k,m}$. If G is a BT_m, then we have*

$$\gamma_{\mathbb{B}}(G,H) - \gamma_{\mathbb{B}}(H,G) = -\langle [G],[H] \rangle \tag{4}$$

and thus we have $\langle [G],[H]\rangle = 0$ if and only if we have the symmetric identity

$$\gamma_{\mathbb{B}}(G,H) = \gamma_{\mathbb{B}}(H,G). \tag{5}$$

(b) *We have a canonical identification $G_0^{\text{loc-loc}}(\mathbb{J}_m) = K_0(\mathbb{J}_m)$ and the bilinear map (3) factors through the projection $G_0(\mathbb{J}_m) \to G_0^{\text{loc-loc}}(\mathbb{J}_m)$ (see (a)) inducing a symmetric bilinear map*

$$\langle\ ,\ \rangle : K_0(\mathbb{J}_m) \times K_0(\mathbb{J}_m) \to \mathbb{Z} \tag{6}$$

whose matrix representation with respect to the standard basis of $K_0(\mathbb{J}_m)$ formed by classes of indecomposable finitely generated projectively left $\mathbb{J}_m$-modules is diagonal and, in case $\mathbb{B}_1$ is a simple ring, it is a positive integer times the identity matrix of the same size.

(c) *If G is a BT_m endowed with a left $\mathbb{B}$-action, then the following three statements are equivalent:*

c1 *The left $\mathbb{B}$-module $G[p]$ is projectively balanced.*

c2 *Equation (5) holds for all objects H of $\mathcal{F}^{\mathbb{B}}_{m,k}$.*

c3 *The left $\mathbb{B}$-module $G[p]$ is $\boldsymbol{\alpha}_p$-balanced.*

(d) *For an object H of $\mathcal{F}^{\mathbb{B}}_{m,k}$, the following two statements are equivalent:*

d1 *The left $\mathbb{B}$-module H is balanced.*

d2 *Equation (5) holds for each BT_m G endowed with a left $\mathbb{B}$-action.*

The injective and projective resolutions over $\mathbb{J}_m$ required to prove Theorem 2 are presented in Sections 4 and 5 (respectively). These resolutions and other homological properties will be used in Section 6 to include a noncommutative duality over $\mathbb{E}_m$ which is analogous to the fact that for an affine connected smooth curve $\operatorname{Spec} R$ over k, the R-module $\mathrm{Frac}(R)/R$ maps isomorphically to the R-torsion submodule in the k-dual of the space of one forms on $\operatorname{Spec} R$ via the 'residue at infinity' functional.

Note that Equation (5) applied with $\mathbb{B}$ equal to $\mathbb{Z}_p$ also implies that $s^{\mathbb{B}}_{D,E} = s^{\mathbb{B}}_{E,D}$ and $n^{\mathbb{B}}_{D,E} = n^{\mathbb{B}}_{E,D}$ and, when combined with [V2], Thm. 1.2 (e), that $s_{D,E} = s_{D',E'}$ (see Example 9 of Subsection 6.6)). The following proposition (proved in Subsection 6.7 based on Example 10 of Subsection 6.6) shows that Equation (5) does not hold in general if G is a truncated BT of some level n but p^n does not annihilate H.

Proposition 1. *Let $m > n > 0$ be integers. We assume that G is a* BT_n *over k and that the finite commutative group scheme H over k is annihilated by p^m. Then the following three properties hold:*

(a) *If G and H are left $\mathbb{B}$-modules such that either $G[p]$ is projectively balanced or H is balanced, then we have optimal inequalities*

$$\frac{n}{m}\gamma_{\mathbb{B}}(H,G) \leq \gamma_{\mathbb{B}}(G,H) \leq \frac{m}{n}\gamma_{\mathbb{B}}(H,G). \tag{7}$$

(b) *The difference $\gamma(G,H) - \gamma(H,G)$ can be an arbitrary integer.*

(c) *We assume that the left $\mathbb{E}$-modules $\mathbb{D}(G)$ and $\mathbb{D}(H)$ (resp. and $\mathbb{D}(H^t)$) are cyclic and that H is not annihilated by p^{m-1}. When m varies subject to the inequalities $m > n > 0$, the difference $\gamma(G,H) - \gamma(H,G)$ can be an arbitrary negative (resp. positive) integer.*

Section 7 presents a few equivalent characterizations of fully BT-balanced objects of $\mathcal{F}_k$ in terms of filtrations (see Theorem 6), including Deligne's filtrations of objects of $\mathcal{F}_k$ defined by nilpotent endomorphisms of them which are compatible with Cartier duality and which in the abstract context of abelian categories were introduced in [De], Subsect. (1.6). Subsections 7.1 and 7.2 complement loc. cit.

For a group scheme Υ over k, let Υ_{red} be the reduced group scheme of Υ and let $\mathrm{Lie}(\Upsilon)$ be its Lie algebra, viewed below just as a k-vector space.

We assume that both G and H are left $\mathbb{B}$-modules. The left $\mathbb{B}$-modules $\boldsymbol{\alpha}_p(G)$ and $(\boldsymbol{\alpha}_p(G^{\mathrm{t}}))^{\mathrm{t}}$ are isomorphic if and only if their associated left $\mathbb{B}$-modules $\mathrm{Lie}(\boldsymbol{\alpha}_p(G))$ and $\mathrm{Lie}((\boldsymbol{\alpha}_p(G^{\mathrm{t}}))^{\mathrm{t}})$ are so. An element $b \in \mathbb{B}$ acts, say, on $\mathrm{Lie}(\boldsymbol{\alpha}_p(G))$ via the Lie differential $\mathrm{Lie}(\rho_G(b)) : \mathrm{Lie}(\boldsymbol{\alpha}_p(G)) \to \mathrm{Lie}(\boldsymbol{\alpha}_p(G))$ of the endomorphism $\rho_G(b) : \boldsymbol{\alpha}_p(G) \to \boldsymbol{\alpha}_p(G)$ induced by $\rho_G(b) : G \to G$ and denoted in the same way.

Let $\mathbf{M}$ be the multiplicative monoid scheme over k associated to the reduced ring scheme

$$\mathbf{End}_{\mathbb{B}}(G)_{\mathrm{red}} \times_k \mathbf{End}_{\mathbb{B}}(H)^{\mathrm{op}}_{\mathrm{red}} = \mathbf{End}_{\mathbb{B}}(G)_{\mathrm{red}} \times_k \mathbf{End}_{\mathbb{B}^{\mathrm{op}}}(H^{\mathrm{t}})_{\mathrm{red}}.$$

By a left $\mathbf{M}$-module Z (or a representation Z of $\mathbf{M}$) we mean a k-vector space Z endowed with a homomorphism ϱ_Z from $\mathbf{M}$ to the multiplicative monoid scheme over k associated to the ring scheme $\mathbf{End}(Z) = \mathbf{End}(Z)_{\mathrm{red}}$. For a $\mathbf{M}$-module Z we get a $\mathbf{M}^{(p)}$-module $Z^{(p)}$ and we let $Z^{(\sigma)}$ be the $\mathbf{M}$-module which is the k-vector space $Z^{(p)}$ but endowed with the homomorphism $\varrho_{Z^{(\sigma)}} := (\varrho_Z)^{(p)} \circ \mathrm{Frob}_{\mathbf{M}}$, where $\mathrm{Frob}_{\mathbf{M}} : \mathbf{M} \to \mathbf{M}^{(p)}$ is the relative Frobenius.

Let $K_0(\mathbf{M})$ be the Grothendieck group of the abelian category of finite dimensional left $\mathbf{M}$-modules Z; let $[Z] \in K_0(\mathbf{M})$ be the element corresponding to Z. Let $I_0(\mathbf{M})$ be the subgroup of $K_0(\mathbf{M})$ generated by all elements of the form $[Z^{(\sigma)}] - [Z]$ with Z an arbitrary finite dimensional left $\mathbf{M}$-module.

Let

$$\mathbf{L}_1 := \mathrm{Lie}(\mathbf{Hom}_{\mathbb{B}}(G,H)_{\mathrm{red}}) \quad \text{and} \quad \mathbf{L}_2 := \mathrm{Lie}(\mathbf{Hom}_{\mathbb{B}}(H,G)_{\mathrm{red}}).$$

Let $\mathbf{L}_1^\vee := \mathrm{Hom}_k(\mathbf{L}_1, k)$ be the dual k-vector space.

Both $\mathbf{L}_1^\vee$ and $\mathbf{L}_2$ are naturally left $\mathbf{M}$-modules. For instance, working just with $\mathbf{L}_2$, if $(\eta_1, \eta_2) \in \mathbf{M}(k) = \mathbf{End}_{\mathbb{B}}(G)(k) \times \mathbf{End}_{\mathbb{B}}(H)^{\mathrm{op}}(k)$, then we have an endomorphism $c_{\eta_1,\eta_2} : \mathbf{Hom}_{\mathbb{B}}(H,G)_{\mathrm{red}} \to \mathbf{Hom}_{\mathbb{B}}(H,G)_{\mathrm{red}}$ which maps each $\delta \in \mathbf{Hom}_{\mathbb{B}}(H,G)_{\mathrm{red}}(k)$ to $\eta_1 \circ \delta \circ \eta_2 \in \mathbf{Hom}_{\mathbb{B}}(H,G)_{\mathrm{red}}(k)$ and (η_1, η_2) acts on $\mathbf{L}_2$ via the Lie differential $\mathrm{Lie}(c_{\eta_1,\eta_2}) : \mathbf{L}_2 \to \mathbf{L}_2$ of c_{η_1,η_2}. We have the following refinement of Theorem 2 which is proved in Section 9.

Theorem 3. *Let $m \in \mathbb{N}^*$. Let G and H be objects of $\mathcal{F}^{\mathbb{B}}_{k,m}$. We assume that G is a BT_m over k. We also assume that either G is fully projectively balanced or H is fully balanced. Then the image of $[\mathbf{L}_1^\vee] - [\mathbf{L}_2]$ in $K_0(\mathbf{M})/I_0(\mathbf{M})$ is 0.*

The need to mod out by $I_0(\mathbf{M})$ is seen already in the simplest local-local case when $k = \bar{k}$, $m = 1$, $G = H = \mathcal{S}[p]$ with $\mathcal{S}$ a supersingular elliptic curve over k, and $\mathbb{B} = W(\mathbb{F}_{p^2})$: the left actions of $\mathbf{M}$ on the 1 dimensional k-vector spaces $\mathbf{L}_1^\vee$ and $\mathbf{L}_2$ factor through an epimorphism $\mathbf{M} \to \mathbb{F}_{p^2}^2 = \mathbb{B}_1^2$ of monoids, and as $\mathbb{F}_{p^2}^2$-modules we have $\mathbf{L}_1^\vee = \mathbf{L}_2^{(\sigma)} = (\mathbf{L}_1^\vee)^{(\sigma^2)} := [(\mathbf{L}_1^\vee)^{(\sigma)}]^{(\sigma)}$.

The kernel of the dimension homomorphism $\dim : K_0(\mathbf{M}) \to \mathbb{Z}$ contains $I_0(\mathbf{M})$ and thus it induces a dimension homomorphism

$$\dim : K_0(\mathbf{M})/I_0(\mathbf{M}) \to \mathbb{Z}$$

denoted in the same way. Thus Theorem 3 implies Theorem 2 (a) when the hypotheses of Theorem 3 are satisfied. But we emphasize that the proof of Theorem 3 we present does rely on Theorem 2 applied to left $\mathbb{B}_0 \otimes_{\mathbb{Z}_p} W(l_0)$-modules G_0 and H_0, where $\mathbb{B}_0$ runs through the simple factors of $\mathbb{B}$, where G_0 and H_0 run through certain direct summands of G and H (respectively) which are left $\mathbb{B}_0$-modules, and where l_0 runs through a finite number of finite fields of characteristic p (and this explains the 'fully' assumptions in Theorem 3). It also relies on the following general result which is proved in Section 8 and which we think it is of interest in its own.

Theorem 4. *We assume that k is a finite field. Let Y be a reduced scheme of finite type over k and let $f, h \in \mathcal{O}(Y_{\bar{k}})$ be such that for every finite field extension $k \to k'$ there exists an element $a \in \mathrm{Gal}(k'/\mathbb{F}_p)$ such that we have $f = a \circ h$ on $Y(k')$, i.e., as elements of* $\mathrm{Maps}(Y(k'), \bar{k})$*. Then there exists $\tau \in W(\bar{k}/\mathbb{F}_p)$, i.e., an integral power of the Frobenius generator* $\mathrm{Frob}_p \in \mathrm{Gal}(\mathbb{F}_p) := \mathrm{Gal}(\bar{k}/\mathbb{F}_p)$*, such that we have $f = \tau \circ h \in \mathrm{Maps}(Y(\bar{k}), \bar{k})$.*

2 Group scheme theoretical preliminaries

We recall that $W(k)$ is the ring of p-typical Witt vectors with coefficients in k. Let $B(k) := \mathrm{Frac}(W(k))$ be the field of fractions of $W(k)$. We denote also by σ the Frobenius automorphism of $W(k)$ or $B(k)$. Let

$$\mathbb{L} = \mathbb{L}(k) := B(k)_\sigma[F, F^{-1}]$$

be the skew (twisted) Laurent polynomial ring in which for all $\alpha \in B(k)$ we have $F\alpha F^{-1} = \sigma(\alpha)$, i.e., it is the group of formal sums

$$\{\sum_{i\in\mathbb{Z}} \alpha_i F^i | \alpha_i \in B(k), \alpha_i = 0 \text{ for all but a finite number of } i \in \mathbb{Z}\}$$

with multiplication $(\sum_{i\in\mathbb{Z}} \alpha_i F^i)(\sum_{i\in\mathbb{Z}} \beta_i F^i) = \sum_{i\in\mathbb{Z}}(\sum_{j\in\mathbb{Z}} \alpha_j \sigma^j(\beta_{i-j}))F^i$. Let $V := pF^{-1} \in \mathbb{L}$ and let

$$\mathbb{E} = \mathbb{E}(k) := W(k)[F, V]$$

as a subring of $\mathbb{L}$. We have an involutory antiautomorphism $\iota : \mathbb{E} \to \mathbb{E}$ that interchanges F and V and that fixes $W(k)$.

For $m \in \mathbb{N}^*$, we recall that $\mathbb{E}_m = \mathbb{E}_m(k) = \mathbb{E}/p^m\mathbb{E}$ and let

$$W_m(k) := W(k)/p^m W(k).$$

The reduction $\iota_m = \iota_m(k) : \mathbb{E}_m \to \mathbb{E}_m$ of ι modulo p^m is an involutory antiautomorphism of $\mathbb{E}_m$ which allows us to transform right $\mathbb{E}_m$-modules into left $\mathbb{E}_m$-modules.

Let $M := \mathbb{D}(G)$: it is a left $\mathbb{E}$-module M which as a $W(k)$-module is torsion and finitely generated. If G is annihilated by p^m, then M is a left $\mathbb{E}_m$-module. If G is a BT_m, then M is a free $W_m(k)$-module of finite rank.

2.1 On the a-number

We have

$$a_G = \dim_k(M/(FM + VM)),$$
$$a_{G^{\mathrm{t}}} = \dim_k(\mathrm{Ker}(F : M \to M) \cap \mathrm{Ker}(V : M \to M)).$$

If G is local-local (i.e., both G and G^t are connected), then a_G is the minimal number of generators of the left $\mathbb{E}$-module M.

We recall that $\boldsymbol{\alpha}_p(G)$ was defined in Section 1. Let $\boldsymbol{\beta}_p(G)$ be the unique closed subgroup scheme of G such that $G/\boldsymbol{\beta}_p(G)$ is isomorphic to $\boldsymbol{\alpha}_p^{a_{G^{\mathrm{t}}}}$, i.e., we have $(G/\boldsymbol{\beta}_p(G))^{\mathrm{t}} = \boldsymbol{\alpha}_p(G^{\mathrm{t}})$. We obtain two increasing filtrations $(\boldsymbol{\alpha}_p(G)_j)_{j\in\mathbb{Z}}$ and $(\boldsymbol{\beta}_p(G)_j)_{j\in\mathbb{Z}}$ of G defined by the initial rules

$$\boldsymbol{\alpha}_p(G)_0 := 0,\ \ \boldsymbol{\alpha}_p(G)_1 := \boldsymbol{\alpha}_p(G),\ \ \boldsymbol{\beta}_p(G)_{-1} := \boldsymbol{\beta}_p(G),\ \ \boldsymbol{\beta}_p(G)_0 := G,$$

and the inductive rules

$$\boldsymbol{\alpha}_p(G)_{j+1}/\boldsymbol{\alpha}_p(G)_j := \boldsymbol{\alpha}_p(G/\boldsymbol{\alpha}_p(G)_j) \ \text{ and } \ \boldsymbol{\beta}_p(G)_{-j-1} := \boldsymbol{\beta}_p(\boldsymbol{\beta}_p(G)_{-j}) \ \ \forall j \in \mathbb{N}.$$

The filtration $(\boldsymbol{\alpha}_p(G)_j)_{j\in\mathbb{Z}}$ (resp. $(\boldsymbol{\beta}_p(G)_j)_{j\in\mathbb{Z}}$) of G is exhaustive (resp. separated) if and only if G is local-local.

Example 1. *Let G be such that M is a k-vector space of dimension 3 which has a basis $\{v_1, v_2, v_3\}$ with the properties that $Fv_1 = Fv_2 = Vv_1 = Vv_2 = 0$, $Fv_3 = v_1$, and $Vv_3 = v_2$. Then $a_G = 1$ while $a_{G^{\mathrm{t}}} = 2$. Thus we have $\boldsymbol{\alpha}_p(G) = \boldsymbol{\beta}_p(G) \simeq \boldsymbol{\alpha}_p$, $\boldsymbol{\alpha}_p(G^{\mathrm{t}}) = \boldsymbol{\beta}_p(G^{\mathrm{t}}) \simeq \boldsymbol{\alpha}_p^2$ and short exact sequences $0 \to \boldsymbol{\alpha}_p \to G \to \boldsymbol{\alpha}_p^2 \to 0$ and $0 \to \boldsymbol{\alpha}_p^2 \to G^{\mathrm{t}} \to \boldsymbol{\alpha}_p \to 0$. Thus $\boldsymbol{\alpha}_p(G)_2 = G$ and $\boldsymbol{\beta}_p(G)_{-2} = 0$. Moreover, we have $\gamma(G, G^{\mathrm{t}}) = 4$ and $\gamma(G^t, G) = 3$.*

2.2 Brief review of quasi-algebraic groups over k

Following [Se1], Subsect. 1.4 we recall several ways to introduce a quasi-algebraic group Q over k. The simplest way is to define Q to be a group object of the category of perfect varieties over k, i.e., of the full subcategory of the category of schemes over k whose objects admit finite open covers by spectra of perfections of finitely generated k-algebras (equivalently, of reduced finitely generated k-algebras) and thus are quasi-compact and topologically noetherian, hence quasi-separated. Each perfect variety over k is the perfection of a (reduced) scheme over k of finite type, see [Se1], Prop. 9 and Cor. after it.[2] The arguments of loc. cit. and [G1], Ch. 1, Sect. 6, Prop.

[2]Loc. cit. works only with separated schemes and (perfect) varieties but following the proof – partly left to the reader – there exists no usage of the "separated" assumption.

(6.3.3) show that every affine perfect variety over k is the perfection of an affine reduced scheme of finite type over k, and every affine quasi-algebraic group over k is the perfection of an affine smooth algebraic group over k. Thus Q can be identified with a covariant functor from the category of commutative perfect k-algebras to the category of groups which is representable by a perfect variety over k. Each quasi-algebraic group over k is the perfection of a group scheme over k of finite type (see [Se1], Prop. 10; the proof of loc. cit. applies in the non-commutative case over k as well).

We also recall that an affine proalgebraic group over k is an affine perfect group scheme over k and can be viewed as a filtered projective limit (as a functor on the category of perfect k-algebras) of affine quasi-algebraic groups over k or as a pro-object of the category of affine quasi-algebraic groups over k (see [Se1], Def. 1 of Subsect. 2.1 and [SY], Subsect. 3.3). Each affine quasi-algebraic group over k is an affine proalgebraic group over k.

Let $\mathcal{Q}$ be the abelian category of commutative quasi-algebraic groups over k (see [Se1], Prop. 5 applied over $\bar{k}$).

Fact 1. *Let $f : Q_1 \to Q_2$ be a morphism of the category $\mathcal{Q}$. Then f is an isomorphism if and only if the abstract homomorphism $f(\bar{k}) : Q_1(\bar{k}) \to Q_2(\bar{k})$ is an isomorphism.*

Proof: This follows from the fact that each one of these two statements is equivalent to the third statement that both $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ are trivial.$\square$

2.3 The χ_p function on prounipotent groups

For this and the next subsection we will assume that k is algebraically closed. Let $\mathcal{P}$ be the abelian category of commutative affine proalgebraic groups over k (see [Se1], Prop. 7; see [DG], Ch. III, Sect. 3, Thm. 5.6 and Thm. of Subsect. 7.2 for why cokernels are affine). Note that the full subcategory of $\mathcal{Q}$ formed by affine objects is a full subcategory of $\mathcal{P}$. We have a thick (Serre) subcategory of $\mathcal{P}$ whose objects are finite dimensional affine proalgebraic groups (those which are projective limits of commutative affine quasi-algebraic groups of bounded dimension) and on it the dimension function dim (defined in the obvious way) is additive. Similar to Fact 1, if K is an algebraically closed field which contains k, then the K-valued functor from $\mathcal{P}$ to the category of abelian groups is a faithful exact functor.

For unipotent group schemes over k we refer to [SGA3-2], Exp. XVII. By a prounipotent group over k we mean a projective limit of affine perfect unipotent group scheme over k.

We now specialize to a commutative prounipotent group U over k. For each $n \in \mathbb{N}$ we have a natural multiplication by p epimorphism

$$(p^n U)/(p^{n+1}U) \to (p^{n+1}U)/(p^{n+2}U).$$

Thus, if U/pU is finite dimensional, then we have a decreasing sequence $(\dim((p^n U)/(p^{n+1}U)))_{n\in\mathbb{N}}$ of non-negative integers and hence there exists a smallest $n_U \in \mathbb{N}$ with the property that $(\dim((p^n U)/(p^{n+1}U)))_{n\geq n_U}$ is a constant sequence. Thus the connected component of the identity element of $\mathrm{Ker}(p : p^{n_U}U \to p^{n_U}U)$ is contained in all $p^n U$ with $n \geq n_U$ and hence it is trivial. Thus we have $\dim(\mathrm{Ker}(p : p^{n_U}U \to p^{n_U}U)) = 0$. By a decreasing induction on $i \in \{0, \ldots, n_U\}$ we get that $\mathrm{Ker}(p : p^i U \to p^i U)$ is finite dimensional.

The full subcategory of $\mathcal{P}$ whose objects are those commutative prounipotent groups U for which U/pU is finite dimensional is thick and on it we have an integral valued additive function (see snake lemma) defined by the rule

$$\chi_p(U) := \dim(U/pU) - \dim(\mathrm{Ker}(p : U \to U)).$$

Fact 2. *Let U be a commutative prounipotent group over k such that U/pU is finite dimensional. Let $n_U \in \mathbb{N}$ be as above. Then the following two properties hold:*

(a) *We have $\chi_p(U) = \dim(p^{n_U}U/p^{n_U+1}U) \geq 0$.*

(b) *The following three statements are equivalent:*

b1 *We have $\chi_p(U) = 0$.*

b2 *The commutative prounipotent group $p^{n_U}U$ is zero dimensional.*

b3 *The commutative prounipotent group U is finite dimensional.*

Proof: If $pU = 0$, then $\chi_p(U) = \dim(U) - \dim(U) = 0$. Based on the additivity of χ_p, a simple induction on $i \in \mathbb{N}^*$ shows that if $p^i U = 0$, then $\chi_p(U) = 0$ and U is finite dimensional.

In general, from the additive equality $\chi_p(U) = \chi_p(U/p^{n_U}U) + \chi_p(p^{n_U}U)$ and from the facts that $\dim(\mathrm{Ker}(p : p^{n_U}U \to p^{n_U}U)) = 0$ (see above) and $\chi_p(U/p^{n_U}U) = 0$ (see previous paragraph), we get that

$$\chi_p(U) = \chi_p(p^{n_U}U) = \dim(p^{n_U}U/p^{n_U+1}U) \geq 0,$$

i.e., part (a) holds.

It is clear that (b2) $\Rightarrow$ (b3). If (b3) holds, then $(\dim((p^nU)/(p^{n+1}U)))_{n\geq n_U}$ is a constant sequence of constant value 0 and from part (a) we get that $\chi_p(U) = 0$, i.e., (b1) holds. If (b1) holds, then the endomorphism $p^{n_U}U \xrightarrow{p} p^{n_U}U$ has zero dimensional cokernel (see part (a)) and it is easy to see that this implies that the connected component of the identity element of $p^{n_U}U$ is trivial, hence (b2) holds. Thus part (b) holds. $\square$

2.4 Prounipotent groups associated to $W(k)$-modules

Each finitely generated $W(k)$-module O defines a commutative prounipotent group $\underline{O}$ over k which to a k-algebra homomorphism $\delta : R_1 \to R_2$ between perfect k-algebras associates the additive homomorphism

$$W(\delta) \otimes 1_O : W(R_1) \otimes_{W(k)} O \to W(R_2) \otimes_{W(k)} O.$$

The quotient $\underline{O}/p\underline{O}$ is finite dimensional and we have

$$\chi_p(\underline{O}) = \dim_{B(k)}(O \otimes_{W(k)} B(k)).$$

If $\chi_p(\underline{O}) = 0$ (i.e., if O has finite length), then $\underline{O}$ is a commutative quasi-algebraic group over k and we have

$$\dim(\underline{O}) = \mathrm{length}_{W(k)}(O).$$

If X is a finite dimensional $B(k)$-vector space, then we view it as an inductive limit $\underline{X}$ of commutative affine proalgebraic groups $\underline{L}$ given by lattices L of X (i.e., by free $W(k)$-submodules L of X of rank equal to $\dim_{B(k)}(X)$).

Definition 3. *We say that a subgroup U of $\underline{X}$ is admissible if and only if there exist lattices L_1 and L_2 of X such that U is an affine proalgebraic subgroup of $\underline{L_1}$ that contains $\underline{L_2}$, i.e., we have $\underline{L_2} \subset U \subset \underline{L_1}$.*

If U_1 and U_2 are admissible subgroups of $\underline{X}$, we can define the index

$$\chi(U_1, U_2) := \dim(U_1/U_3) - \dim(U_2/U_3) \in \mathbb{Z}$$

for each admissible subgroup U_3 of $\underline{X}$ contained in both U_1 and U_2; this generalizes the definition of $\chi(L_1, L_2)$ for lattices L_1 and L_2 of X which was introduced in [Se2], Ch. II, Subsect. 1.2 and which equals $\chi(\underline{L_1}, \underline{L_2})$. For all integers i we have

$$\chi(p^i U_1, p^i U_2) = \chi(U_1, U_2) = -\chi(U_2, U_1). \tag{8}$$

For four admissible subgroups U_1, U_2, U_3, and U_4 of $\underline{X}$ we also have the following interchanging identity

$$\chi(\underline{U_1}, \underline{U_2}) - \chi(\underline{U_3}, \underline{U_4}) = \chi(\underline{U_1}, \underline{U_3}) - \chi(\underline{U_2}, \underline{U_4}). \tag{9}$$

Example 2. *Let U be an admissible subgroup of $\underline{X}$, let L be a lattice of X, and let $n \in \mathbb{N}$. As $0 = \chi(U, \underline{L}) - \chi(p^n U, p^n \underline{L}) = \chi(U, p^n U) - \chi(\underline{L}, p^n \underline{L})$ (see Equations (8) and (9)) we get that*

$$\chi(U, p^n U) = \chi(\underline{L}, p^n \underline{L}) = \dim(\underline{L}/p^n \underline{L}) = \mathrm{length}_{W(k)}(L/p^n L) = n \dim_{B(k)}(X).$$

Definition 4. *Let $T : X \to X'$ be a homomorphism between the underlying abelian groups of two finite dimensional $B(k)$-vector spaces. We say that T is proalgebraic if and only if it comes from a proalgebraic homomorphism between lattices $T_0 : \underline{L} \to \underline{L'}$, i.e., we have*

$$T = X \to X' = T_0(k) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] : L \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \to L' \otimes_{\mathbb{Z}} \mathbb{Z}[1/p].$$

If $T : X \to X'$ is proalgebraic and if L_1 and L_1' are lattices of X and X' (respectively) such that $T(L_1) \subset L_1'$, then T comes from a proalgebraic homomorphism $T_1 : \underline{L_1} \to \underline{L_1'}$. To check this, by multiplying by some power of p we can assume that $L_1 \subset L$ and $L_1' \subset L'$; the composite of the restriction $T_{01} : \underline{L_1} \to \underline{L'}$ of T_0 with the projection $\underline{L'} \to \underline{L'}/\underline{L_1'}$ is 0 on k-valued points and hence it is 0, and thus we can take T_1 to be the factorization of T_{01}.

If T is proalgebraic, we get an inductive homomorphism

$$\underline{T} : \underline{X} \to \underline{X'} = T_0 \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] : \underline{L} \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \to \underline{L'} \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$$

which will be also called proalgebraic.

If T is proalgebraic, we consider for each $n \in \mathbb{N}^*$ the kernel of

$$(T_0 \bmod p^n) : \underline{L}/p^n \underline{L} \to \underline{L'}/p^n \underline{L'}.$$

The images in $\underline{L}/p\underline{L}$ of such kernels form a decreasing sequence of quasi-algebraic subgroups, and thus they become constant for $n \geq n_1$ for some $n_1 \in \mathbb{N}^*$, with constant value equal to the image in $\underline{L}/p\underline{L}$ of $\mathrm{Ker}(T_0)$ (we recall that filtered inverse limits are exact in our abelian category $\mathcal{P}$ of commutative affine proalgebraic groups, see [Se1], Subsects. 2.3 and 2.5).

The proof of the following basic elementary fact is left to the reader:

Fact 3. *We assume that $T : X \to X'$ is proalgebraic. Let $T_0 : \underline{L} \to \underline{L'}$ and $n_1 \in \mathbb{N}^*$ be as above. Then the following two properties hold:*

(a) *The proalgebraic group $p^{n_1-1}\mathrm{Coker}(T_0)$ is torsion free.*

(b) *The following three statements are equivalent:*

(b1) *The homomorphism T is surjective.*

(b2) *The cokernel $\mathrm{Coker}(T_0)$ is killed by a power of p.*

(b3) *The cokernel $\mathrm{Coker}(T_0)$ is finite dimensional.*

Definition 5. *Let $T : X \to X'$ be a proalgebraic homomorphism between the underlying groups of two finite dimensional $B(k)$-vector spaces. We say that $T : X \to X'$ (or $\underline{T} : \underline{X} \to \underline{X'}$) is admissible if and only if it comes from a proalgebraic homomorphism between lattices $T_0 : \underline{L} \to \underline{L'}$ whose kernel $Ker(T_0)$ and cokernel $\mathrm{Coker}(T_0)$ are finite dimensional (this is independent of the choice of lattices L and L').*

As $\mathrm{Ker}(T_0) \xrightarrow{p} \mathrm{Ker}(T_0)$ is a monomorphism, $\mathrm{Ker}(T_0)$ is finite dimensional if and only if $\dim(\mathrm{Ker}(T_0)) = 0$ and if and only if $\mathrm{Ker}(T_0)$ is a free finitely generated $\mathbb{Z}_p$-module. If T is admissible, then T is surjective (see Fact 3 (b)).

The additivity of the function χ_p gives that

$$\chi_p(\mathrm{Ker}(T_0)) - \chi_p(\mathrm{Coker}(T_0)) = \chi_p(\underline{L}) - \chi_p(\underline{L'}) = \dim_{B(k)}(X) - \dim_{B(k)}(X').$$

Thus, if the $B(k)$-vector spaces X and X' have the same dimension, then $\chi_p(\mathrm{Ker}(T_0)) = \chi_p(\mathrm{Coker}(T_0))$ and from Fact 2 (b) we get that the finite dimensionality of $\mathrm{Ker}(T_0)$ is equivalent to the finite dimensionality of $\mathrm{Coker}(T_0)$.

Example 3. *Let i and j be distinct integers. Let $\delta_i : X \to X'$ be a σ^i-linear map and let $\delta_j : X \to X'$ be a σ^j-linear map; thus $T := \delta_i + \delta_j : X \to X'$ is proalgebraic and for each lattice L of X there exists a lattice L' of X' such that T comes from a proalgebraic homomorphism $T_0 : \underline{L} \to \underline{L'}$ (for instance, we can take L' to contain $\delta_i(L) + \delta_j(L)$). We assume that either δ_i or δ_j is invertible (thus X and X' have the same dimension). The Dieudonné–Manin classification of σ^ℓ-F-isocrystals over k with $\ell \in \{i-j, j-i\} \subset \mathbb{Z}\setminus\{0\}$ implies that $\mathrm{Ker}(T)$ is a finite dimensional $\mathbb{Q}_p$-vector space. Thus the intersection $L \cap \mathrm{Ker}(T)$ is a free finitely generated $\mathbb{Z}_p$-module, i.e., $\dim(\mathrm{Ker}(T_0)) = 0$ and therefore $\mathrm{Coker}(T_0)$ is finite dimensional (i.e., $\underline{T}(\underline{L})$ is an admissible subgroup of X'). Thus $\underline{T}$ is admissible.*

Lemma 1. *Let $\underline{T} : \underline{X} \to \underline{X'}$ be admissible. Let U_1 and U_2 (resp. U'_1 and U'_2) be admissible subgroups of $\underline{X}$ (resp. of $\underline{X'}$). Then the following three properties hold:*

(a) *We have $\chi(U'_2, \underline{T}(U_2)) - \chi(U'_1, \underline{T}(U_1)) = \chi(U'_2, U'_1) - \chi(U_2, U_1)$ and thus we have $\chi(U'_2, U'_1) = \chi(U_2, U_1)$ if and only if $\chi(U'_2, \underline{T}(U_2)) = \chi(U'_1, \underline{T}(U_1))$.*

(b) *We have $\dim_{B(k)}(X) = \dim_{B(k)}(X')$.*

(c) *If $\underline{T}(U) \subset U'$ and if $n_2 \in \mathbb{N}$ is such that p^{n_2} annihilates $U'/\underline{T}(U)$, then for all integers $n \geq n_2$ the kernel of the induced map $U/p^nU \to U'/p^nU'$ has dimension equal to $\dim(U'/\underline{T}(U))$.*

Proof: As $\underline{T}$ is admissible, $\mathrm{Ker}(T_0)$ is a free finitely generated $\mathbb{Z}_p$-module and this implies that we have

$$\chi(\underline{T}(U_2), \underline{T}(U_1)) = \chi(U_2, U_1). \tag{10}$$

From Equation (9) applied with $(U_1, U_2, U_3, U_4) = (U'_2, \underline{T}(U_2), U'_1, \underline{T}(U_1))$ and Equation (10) we get that part (a) holds. By taking $U_1 = pU_2$ in Equation (10), from Example 2 applied with $n = 1$ we get that part (b) holds. For part (c), we note that we have $p^nU' \subset \underline{T}(U)$ as well as two short exact sequences

$$0 \to \mathrm{Ker}(T : U/p^nU \to U'/p^nU') \to U/p^nU \to \underline{T}(U)/p^nU' \to 0$$

and $0 \to \underline{T}(U)/p^nU' \to U'/p^nU' \to U'/\underline{T}(U) \to 0$. This implies that the dimension of $\mathrm{Ker}(T : U/p^nU \to U'/p^nU')$ is equal to the expression $\dim(U'/T(U)) + \dim(U/p^nU) - \dim(U'/p^nU')$. Based on Example 2 and part (b), the two dimensions $\dim(U/p^nU)$ and $\dim(U'/p^nU')$ are equal and therefore this expression is equal to $\dim(U'/T(U))$. Thus part (c) also holds. $\square$

2.5 Serre duality

For a group scheme Υ over k, let Υ^0 be the connected component of its identity element.

Let $\mathcal{U}$ be the full subcategory of $\mathcal{Q}$ whose objects are unipotent connected commutative quasi-algebraic groups over k. For Serre duality of the category $\mathcal{U}$ we refer to [Se1], [Be], Sect. 1 and [BD], Sect. 3. We recall that Serre duality is an additive involutory antiequivalence $* : \mathcal{U} \to \mathcal{U}$ which preserves dimensions and short exact sequences (for instance, see [Be], Prop. 1.2.1).

We also recall that for a finitely generated $W_m(k)$-module O and its dual

$$O^\vee := \mathrm{Hom}_{W_m(k)}(O, W_m(k)),$$

the Serre dual of $\underline{O}$ is $\underline{O^\vee}$ in a functorial way with respect to all σ^i-linear maps with $i \in \mathbb{Z}$ (see [Se1], Subsect. 8.4, Prop. 4 and Lem. 2).

Lemma 2. *For a morphism $f : U_1 \to U_2$ of the category $\mathcal{U}$ and its Serre dual $f^* : U_2^* \to U_1^*$ the following three properties hold:*

(a) *The* $\mathrm{Ker}(f)^0$ *and* $\mathrm{Coker}(f^*)$ *are canonically Serre dual and so we have*

$$\dim(\mathrm{Ker}(f)) = \dim(\mathrm{Coker}(f^*)) = \dim(\mathrm{Ker}(f^*)) + \dim(U_1) - \dim(U_2). \quad (11)$$

(b) *The homomorphism f is an isogeny if and only if f^* is an isogeny.*[3]

Proof: Using the isogeny $U_1/\mathrm{Ker}(f)^0 \to \mathrm{Im}(f)$ and the short exact sequences

$$0 \to \mathrm{Im}(f) \to U_2 \to \mathrm{Coker}(f) \to 0$$

and

$$0 \to \mathrm{Ker}(f)^0 \to U_1 \to U_1/\mathrm{Ker}(f)^0 \to 0,$$

we get that $\dim(\mathrm{Coker}(f)) = \dim(\mathrm{Ker}(f)) - \dim(U_1) + \dim(U_2)$. As Serre duality preserves short exact sequences and dimensions, $\mathrm{Coker}(f)^*$ is a subgroup of $\mathrm{Ker}(f^*)$ and $\dim(\mathrm{Ker}(f^*)) \geq \dim(\mathrm{Coker}(f)^*) = \dim(\mathrm{Coker}(f))$. Thus $\dim(\mathrm{Ker}(f^*)) \geq \dim(\mathrm{Ker}(f)) - \dim(U_1) + \dim(U_2)$. As Serre duality is involutory, we have $f = (f^*)^*$ and thus by replacing f with f^* in the last inequality we get (based on dimension preservation) the 'reversed' inequality $\dim(\mathrm{Ker}(f)) \geq \dim(\mathrm{Ker}(f^*)) - \dim(U_2) + \dim(U_1)$. Thus Equation (11) holds and we have $\dim(\mathrm{Ker}(f^*)) = \dim(\mathrm{Coker}(f)^*)$. By reasons of dimensions, the canonical monomorphism $\mathrm{Coker}(f)^* \to \mathrm{Ker}(f^*)^0$ is an isomorphism. From this, by replacing (f, f^*) with $(f^*, (f^*)^*) = (f^*, f)$, we get that part (a) holds.

Part (b) follows from part (a) and the fact that f (or f^*) is an isogeny if and only if $\dim(U_1) = \dim(U_2)$ and $\mathrm{Ker}(f)$ (or $\mathrm{Ker}(f^*)$) has dimension 0. $\square$

For examples and future usage, let Γ be a finite abstract group with the identity element 1 and let $|\Gamma|$ be its order.

Example 4. *We assume that Γ acts on an object U of $\mathcal{U}$. Let*

$$f : U \to \mathrm{Maps}(\Gamma, U)$$

be the homomorphism that maps $u \in U(\bar{k})$ to the function

$$f_u : \Gamma \to U(\bar{k}), \quad f_u(a) = (a-1)(u) = au - u \in U(\bar{k}).$$

[3]See the proof of [Be], Prop. 1.2.1 for another argument for this.

Let $U^\Gamma := \mathrm{Ker}(f)$*: we have* $U^\Gamma(\bar{k}) = \{u \in U(\bar{k}) | au = u\ \forall a \in \Gamma\}$*. Similarly, we identify* $\mathrm{Coker}(f^*) = (U^*)_\Gamma := U^*/\sum_{a\in\Gamma} \mathrm{Im}(a-1)$*. From Lemma 2 (a) we get that* $(U^\Gamma)^0$ *and* $(U^*)_\Gamma$ *are canonically Serre dual.*

If $p \nmid |\Gamma|$*, then* $e_\Gamma := \frac{1}{|\Gamma|}(\sum_{a\in\Gamma} a) \in \mathbb{Z}_{(p)}[\Gamma]$ *is an idempotent such that* $e_\Gamma U = U^\Gamma$ *and* $\mathrm{Ker}(e_\Gamma : U \to U) = \sum_{a\in\Gamma} \mathrm{Im}(a-1)$*, hence* $U^\Gamma = (U^\Gamma)^0$ *is connected and naturally identified with* U_Γ*. From this and its analogue for* U^* *we get (see prior paragraph) that* U^Γ *and* $(U^*)^\Gamma$ *are canonically Serre dual.*

Example 5. *Let* Γ *act on two objects* U_1 *and* U_2 *of* $\mathcal{U}$*. Let* $f : U_1 \to U_2$ *be a* Γ*-invariant homomorphism. As* $\mathrm{Ker}(f)^0$ *and* $\mathrm{Coker}(f^*)$ *are canonically Serre dual (see Lemma 2 (a)), the identification* $(\mathrm{Ker}(f)^0)^* = \mathrm{Coker}(f^*)$ *is* Γ*-invariant. Thus, if* $p \nmid |\Gamma|$*, then from Example 4 applied with* $U = \mathrm{Ker}(f)^0$ *we get that* $(\mathrm{Ker}(f)^0)^\Gamma$ *and* $\mathrm{Coker}(f^*)^\Gamma$ *are canonically Serre dual.*

3 Proof of Theorems 1

If $n \in \mathbb{N}^*$ and A is a ring, let $\mathrm{M}_n(A)$ be the ring of $n \times n$ matrices with entries in A; we identify canonically $\mathrm{M}_1(A) = A$. If $\mathfrak{i}$ is a left ideal of A, let $\mathrm{J}_1(\mathfrak{i}) := \mathfrak{i}$ and for $n \geq 2$ let

$$\mathrm{J}_n(\mathfrak{i}) := \{(\alpha_{ij})_{1\leq i,j\leq n} \in M_n(A) | \alpha_{i1} \in \mathfrak{i}, \alpha_{i2} = \cdots = \alpha_{in} = 0\ \forall i \in \{1, \ldots, n\}\}.$$

Let $m \in \mathbb{N}^*$. We have $W_m(\mathbb{F}_p) = \mathbb{Z}/p^m\mathbb{Z}$. We recall that $\mathbb{B}$ is a finite flat separable $\mathbb{Z}_p$-algebra and $\mathbb{B}_m = \mathbb{B}/p^m\mathbb{B}$. We say that $\mathbb{B}$ is connected if $\mathbb{B}_1$ is a simple ring, i.e., it is isomorphic to a matrix ring over a finite field (see [Bo2], Sect. 18, Thm. 2). If $\mathbb{B}$ is connected, let $s, q \in \mathbb{N}^*$ be such that $\mathbb{B}$ is isomorphic to the matrix $\mathbb{Z}_p$-algebra $\mathrm{M}_s(W(\mathbb{F}_{p^q}))$; we will often use an identification $\mathbb{B} = \mathrm{M}_s(W(\mathbb{F}_{p^q}))$. The $\mathbb{Z}_p$-algebras $\mathbb{B}$ and $\mathbb{B}^{\mathrm{op}}$ are isomorphic but non-canonically if $\mathbb{B}$ is not commutative. We recall that

$$\mathbb{J} = \mathbb{J}(k) = \mathbb{E} \otimes_{\mathbb{Z}_p} \mathbb{B}^{\mathrm{op}} \text{ and } \mathbb{J}_m = \mathbb{J}_m(k) = \mathbb{J}/p^m\mathbb{J} = \mathbb{E}_m \otimes_{W_m(\mathbb{F}_p)} \mathbb{B}_m^{\mathrm{op}}.$$

A left module over $\mathbb{E}_m$ or $\mathbb{J}_m$ has finite length if and only if has finite length over $W_m(k)$. The classical Dieudonné theory also gives that:

• The category of BTs over k endowed with a left $\mathbb{B}$-action is antiequivalent to the category of left $\mathbb{J}$-modules which as $W(k)$-modules are free of finite rank.

To prove Theorem 1 we can assume that k is algebraically closed and that $\mathbb{B}$ is connected. So in this section we assume we have $k = \bar{k}$.

3.1 Proof of Theorem 1 (a) and (b)

Let L and J be the (contravariant) Dieudonné modules of D and E (respectively). We recall that the dual E^{t} of E has (contravariant) Dieudonné module $J^\vee := \mathrm{Hom}_{W(k)}(J, W(k))$ with F and V acting on $\delta \in J^\vee$ via the rules: $F\delta(x) = \sigma(\delta(Vx))$ and $V\delta(x) = \sigma^{-1}(\delta(Fx))$ for all $x \in J$.

In order to match our notation with the one of [GV] (modulo the replacement of M by $L \oplus J$), let $\phi : L \oplus J \to L \oplus J$ and $\vartheta : L \oplus J \to L \oplus J$ be the σ-linear map and the σ^{-1}-linear map (respectively) such that for each $x \in L \oplus J$ we have $\phi(x) = Fx$ and $\vartheta(x) = Vx$. We denote also by $\phi : \mathrm{End}_{B(k)}(L[1/p] \oplus J[1/p]) \to \mathrm{End}_{B(k)}(L[1/p] \oplus J[1/p])$ the σ-linear automorphism induced naturally by ϕ: it maps $y \in \mathrm{End}_{B(k)}(L[1/p] \oplus J[1/p])$ to $\phi \circ y \circ \phi^{-1} \in \mathrm{End}_{B(k)}(L[1/p] \oplus J[1/p])$. Therefore $\phi(y) = p^{-1}FyV$.

Let $\mathbf{G}$ be a smooth closed subgroup scheme of $\mathbf{GL}_{L\oplus J}$ with connected generic fiber $\mathbf{G}_{B(k)}$. Thus the scheme $\mathbf{G}$ is integral. Let $\mathfrak{g} := \mathrm{Lie}(\mathbf{G})$ be the Lie algebra over $W(k)$ of $\mathbf{G}$. In this section we will assume that the following two axioms introduced in [GV], Sect. 6 hold for the triple $(L \oplus J, \phi, \mathbf{G})$:

(AX1) The Lie subalgebra $\mathfrak{g}[1/p]$ of $\mathrm{End}_{W(k)}(L \oplus J)[1/p]$ (i.e., of the direct sum $\mathrm{End}_{B(k)}(L[1/p]) \oplus \mathrm{Hom}_{B(k)}(L[1/p], J[1/p]) \oplus \mathrm{End}_{B(k)}(J[1/p]) \oplus \mathrm{Hom}_{B(k)}(J[1/p], J[1/p]))$ is stable under ϕ, i.e., we have $\phi(\mathfrak{g}[1/p]) = \mathfrak{g}[1/p]$.

(AX2) There exist a direct sum decomposition $L \oplus J = P^1 \oplus P^0$ such that the following two properties hold:

(a) The kernel of the reduction of ϕ modulo p is $\overline{P^1} := P^1/pP^1$.

(b) The cocharacter $\mu : \mathbb{G}_{m,W(k)} \to \mathbf{GL}_{L\oplus J}$ which acts trivially on P^0 and via the inverse of the identical character of $\mathbb{G}_{m,W(k)}$ on P^1, normalizes $\mathbf{G}$.

The triple $(L \oplus J, \phi, \mathbf{G})$ is called an *F-crystal with a group* over k, see [V1], Def. 1.1 (a) and Subsect. 2.1; we always view it as just a member of the family $\{(L \oplus J, g\phi, \mathbf{G}) | g \in \mathbf{G}(W(k))\}$ of F-crystals with a group over k.

Definition 6. *For $g_1, g_2 \in \mathbf{G}(W(k))$ we say that the two F-crystals with a group $(L\oplus J, g_1\phi, \mathbf{G})$ and $(L\oplus J, g_2\phi, \mathbf{G})$ are $\mathbf{G}$-isogenous (resp. $\mathbf{G}$-isomorphic) if there exists an element $h \in \mathbf{G}(B(k))$ (resp. $h \in \mathbf{G}(W(k))$) such that $hg_1\phi = g_2\phi h$. We also call such an h as a $\mathbf{G}$-isogeny (resp. as an inner isomorphism or as a $\mathbf{G}$-isomorphism).*

We recall from [GV], Subsect. 5.1 that $n^{\mathbf{G}}_{D\oplus E}$ is the smallest non-negative integer that has the following property: for each element $g \in \mathbf{G}(W(k))$ congruent to $1_{L\oplus J}$ modulo $p^{n^{\mathbf{G}}_{D\oplus E}}$, there exists an inner isomorphism between

$(L\oplus J,\phi,\mathbf{G})$ and $(L\oplus J,g\phi,\mathbf{G})$. The existence of $n^{\mathbf{G}}_{D\oplus E}$ is implied by [V1], Main Thm. A. If $\mathbf{G}=\mathbf{GL}_{L\oplus J}$, then $n^{\mathbf{G}}_{D\oplus E}$ is the i-number (isomorphism number) $n_{D\oplus E}$ of $D\oplus E$ as used in [V1] and [GV] (see [GV], Subsect. 5.1)[4].

If E is a left $\mathbb{B}$-module, then E^{t} is a left $\mathbb{B}^{\mathrm{op}}$-module, J is a left $\mathbb{J}$-module and $J^\vee$ is a left $\mathbb{E}\otimes_{\mathbb{Z}_p}\mathbb{B}$-module. Until Subsection 3.5 we will assume that D and E are left $\mathbb{B}$-modules; hence L and J are left $\mathbb{J}$-modules. If D or E is trivial (i.e., L or J is 0), then Theorems 1 (a) and (b) hold: we can take $n^{\mathbb{B}}_{D,E}=s^{\mathbb{B}}_{D,E}=0$. So to prove Theorems 1 (a) and (b) we can assume that L and J are non-zero. Due to this, as $\mathbb{B}$ and $\mathbb{B}^{\mathrm{op}}$ are connected, the ring homomorphisms $\mathbb{B}^{\mathrm{op}}\to\mathrm{End}_{\mathbb{E}}(L)\subset\mathrm{End}_{W(k)}(L)$ and $\mathbb{B}^{\mathrm{op}}\to\mathrm{End}_{\mathbb{E}}(J)\subset\mathrm{End}_{W(k)}(J)$ are injections to be viewed as inclusions. As we have a product decomposition $W(k)\otimes_{\mathbb{Z}_p}\mathbb{B}^{\mathrm{op}}=\prod_{i=1}^q\mathbb{W}_{s,i}$ whose factors $\mathbb{W}_{s,i}$ are isomorphic to $\mathrm{M}_s(W(k))$ and are permuted transitively by $\sigma\otimes 1_{\mathbb{B}^{\mathrm{op}}}$, the $W(k)$-algebra homomorphisms $W(k)\otimes_{\mathbb{Z}_p}\mathbb{B}^{\mathrm{op}}\to\mathrm{End}_{W(k)}(L)$ and $W(k)\otimes_{\mathbb{Z}_p}\mathbb{B}^{\mathrm{op}}\to\mathrm{End}_{W(k)}(J)$ are injective. Due to this transitivity and the standard Morita equivalence between the categories of $W(k)$-modules and $\mathrm{M}_s(W(k))$-modules, there exist $s_L,s_J\in\mathbb{N}^*$ such that we have direct sum decompositions

$$L=\oplus_{i=1}^q L_i^{s_L}\quad\text{and}\quad J=\oplus_{i=1}^q J_i^{s_J}\tag{12}$$

into projective left $W(k)\otimes_{\mathbb{Z}_p}\mathbb{B}^{\mathrm{op}}$-modules with the property that each $\mathbb{W}_{s,i}$ acts as zero on L_j and J_j if $j\neq i$ and L_i and J_i are 'simple' left $\mathbb{W}_{s,i}$-modules in the sense that L_i/pL_i and J_i/pJ_i are simple left $\mathbb{W}_{s,i}/p\mathbb{W}_{s,i}$-modules; identifying $\mathbb{W}_{s,i}=\mathrm{M}_s(W(k))$, L_i and J_i become isomorphic to $\mathrm{J}_s(W(k))$. Based on decompositions (12), it is easy to see that the following two properties hold:

(i) The centralizers $\mathbf{C}_L$ and $\mathbf{C}_J$ of $\mathbb{B}^{\mathrm{op}}$ in $\mathbf{GL}_L$ and $\mathbf{GL}_J$ (respectively) are reductive group scheme over $\mathrm{Spec}\,W(k)$ isomorphic to $\mathbf{GL}^q_{s_L,W(k)}$ and $\mathbf{GL}^q_{s_J,W(k)}$ (respectively).

(ii) The centralizer $\mathfrak{b}=\mathfrak{b}_{J,L}:=\mathrm{Hom}_{W(k)\otimes_{\mathbb{Z}_p}\mathbb{B}^{\mathrm{op}}}(J,L)$ of $\mathbb{B}^{\mathrm{op}}$ in $\mathrm{Hom}_{W(k)}(J,L)$ is a direct summand of $\mathrm{Hom}_{W(k)}(J,L)$ and we have $\phi(\mathfrak{b}[1/p])=\mathfrak{b}[1/p]$.

Let $\mathbf{B}$ be the smooth closed subgroup scheme of $\mathbf{GL}_{L\oplus J}$ whose Lie algebra is $\mathfrak{b}$ and such that for each commutative k-algebra R, we have

$$\mathbf{B}(R)=1_{W(R)\otimes_{W(k)}(L\oplus J)}+W(R)\otimes_{W(k)}\mathfrak{b}.$$

[4] If $D\oplus E$ is either étale or of multiplicative type (i.e., its formal deformation space is $\mathrm{Spf}\,k$), then here, as in [GV], we have $n_{D\oplus E}=0$, and this differs from the other convention, for instance followed by [LNV], under which $n_{D\oplus E}$ would be 1 for such a $D\oplus E$.

We check that the triple $(L \oplus J, \phi, \mathbf{B})$ is an F-crystal with a group over k. Property (ii) implies that Axiom (AX1) holds for the triple $(L \oplus J, \phi, \mathbf{B})$. As each element $b \in \mathbb{B}^{\mathrm{op}}$ is fixed by ϕ, $\overline{P^1} = [\overline{P^1} \cap (L/pL)] \oplus [\overline{P^1} \cap (J/pJ)]$ is a left $k \otimes_{\mathbb{F}_p} \mathbb{B}_1^{\mathrm{op}}$-module and hence it has a direct supplement $\overline{P_0^0}$ in $(L/pL) \oplus (J/pJ)$ which is a left $k \otimes_{\mathbb{F}_p} \mathbb{B}_1^{\mathrm{op}}$-module satisfying $\overline{P_0^0} = (\overline{P_0^0} \cap L/pL) \oplus (\overline{P_0^0} \cap J/pJ)$. Thus we have a cocharacter $\mathbb{G}_{m,k} \to \mathbf{C}_{L,k} \times_k \mathbf{C}_{J,k}$ which acts trivially on $\overline{P^0}$ and via the inverse of the identical character of $\mathbb{G}_{m,k}$ on $\overline{P^1}$ and from [SGA3-2], Exp. XV, Thms. 3.6 and 7.1 we get that it lifts to a cocharacter $\mathbb{G}_{m,W(k)} \to \mathbf{C}_L \times_{\mathrm{Spec}\, W(k)} \mathbf{C}_J$. As $\mathrm{Im}(\mathbb{G}_{m,W(k)} \to \mathbf{C}_L \times_{\mathrm{Spec}\, W(k)} \mathbf{C}_J)$ normalizes $\mathbf{B}$ and $\mathfrak{b}$, Axiom (AX2) holds for the triple $(L \oplus J, \phi, \mathbf{B})$.

Moreover, we have a direct sum decomposition into $W(k)$-modules

$$\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1 \tag{13}$$

such that for $i \in \{-1, 0, 1\}$, $\mathbb{G}_{m,W(k)}$ acts on $\mathfrak{b}_i$ via the $-i$-th power of the identical character of $\mathbb{G}_{m,W(k)}$. We have

$$\phi(p\mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus p^{-1}\mathfrak{b}_1) = \mathfrak{b}. \tag{14}$$

We denote the rank of the free $W(k)$-module $\mathfrak{b}_1$ by $d_{\mathfrak{b}}$. For $\triangledown \in \{D, E\}$, if $d_\triangledown$ and $c_\triangledown$ are the dimension and the codimension of $\triangledown$ (respectively), then we have formulas

$$s_L = \frac{c_D + d_D}{sq} \quad \text{and} \quad s_J = \frac{c_E + d_E}{sq} \quad \text{and} \quad d_{\mathfrak{b}} = \frac{d_D c_E}{s^2 q^2}. \tag{15}$$

For $m \in \mathbb{N}^*$ let ϕ_m, ϑ_m be the reductions modulo p^m of ϕ and ϑ (respectively). Let $\partial^{(\sigma)}$ be the pullback (or the tensorization) of some $W(k)$-linear map or $W(k)$-module ∂ via σ. Thus $L^{(\sigma)} := W(k) \otimes_{\sigma, W(k)} L$, etc.

We consider the group scheme

$$\mathbf{Hom}(D[p^m], E[p^m])_{\mathrm{crys}}$$

over k of homomorphisms from $(J/p^m J, \phi_m, \vartheta_m)$ to $(L/p^m L, \phi_m, \vartheta_m)$. Thus, if R is a commutative k-algebra and if σ_R is the Frobenius endomorphism of the ring $W_m(R)$ of p-typical Witt vectors of length m with coefficients in R, then $\mathbf{Hom}(D[p^m], E[p^m])_{\mathrm{crys}}(R)$ is the group of those $W_m(R)$-linear maps $\natural \in W_m(R) \otimes_{W_m(k)} \mathrm{Hom}_{W_m(k)}(J/p^m J, L/p^m L)$ that satisfy the two identities

$$(1_{W_m(R)} \otimes \phi_m) \circ \natural^{(\sigma)} = \natural \circ (1_{W_m(R)} \otimes \phi_m),$$

$$\natural^{(\sigma)} \circ (1_{W_m(R)} \otimes \vartheta_m) = (1_{W_m(R)} \otimes \vartheta_m) \otimes \natural;$$

here ϕ_m and ϑ_m are viewed as $W_m(k)$-linear maps $(L/p^mL \oplus J/p^mJ)^{(\sigma)} \to L/p^mL \oplus J/p^mJ$ and $L/p^mL \oplus J/p^mJ \to (L/p^mL \oplus J/p^mJ)^{(\sigma)}$ (respectively).

We consider the closed subgroup scheme

$$\mathbf{Hom}(D[p^m], E[p^m])^{\mathfrak{b}}_{\mathrm{crys}}$$

of $\mathbf{Hom}(D[p^m], E[p^m])_{\mathrm{crys}}$ such that for each commutative k-algebra R inside $W_m(R) \otimes_{W_m(k)} \mathrm{Hom}_{W_m(k)}(J/p^mJ, L/p^mL)$ we have

$$\mathbf{Hom}(D[p^m], E[p^m])^{\mathfrak{b}}_{\mathrm{crys}}(R) = \mathbf{Hom}(D[p^m], E[p^m])^{\mathfrak{b}}_{\mathrm{crys}}(R) \cap [W_m(R) \otimes_{W_m(k)} (\mathfrak{b}/p^m\mathfrak{b})].$$

If we have a short exact sequence $0 \to E \to \mathcal{E} \to D \to 0$ as in Theorem 1 (a), then the left $\mathbb{J}$-module $\mathrm{proj.lim.}_{n\to\infty}\mathbb{D}^{\mathbb{B}}(\mathcal{E}[p^n])$ is isomorphic to $(L \oplus J, g\phi, \vartheta g^{-1})$ for some arbitrary element $g \in \mathbf{B}(W(k))$, the triple $(L \oplus J, g\phi, \vartheta g^{-1}, \mathbf{B})$ being uniquely determined up to inner isomorphisms. Moreover, the short exact sequence splits modulo p^n if and only if we can assume that $g \in \mathrm{Ker}(\mathbf{B}(W(k)) \to \mathbf{B}(W_m(k)))$. The last two sentences imply that $n^{\mathbb{B}}_{D,E} = n^{\mathbf{B}}_{D\oplus E}$ (to be compared with [V1], Lem. 3.2.2 and [GV], Subsect. 6.1) and that Theorem 1 (a) holds.

For $m \in \mathbb{N}^*$ let $\mathbf{Aut}(D[p^m] \oplus E[p^m])^{\mathbf{B}}_{\mathrm{crys}}$ be the group scheme over k defined in [GV], Def. 2 (a). We have a canonical identification

$$\mathbf{Hom}(D[p^m], E[p^m])^{\mathfrak{b}}_{\mathrm{crys}} = \mathbf{Aut}(D[p^m] \oplus E[p^m])^{\mathbf{B}}_{\mathrm{crys}} \tag{16}$$

of affine group schemes over k, which for a commutative k-algebra R maps each $\natural \in \mathbf{Hom}(D[p^m], E[p^m])^{\mathfrak{g}}_{\mathrm{crys}}(R)$ to $1_{W(R)\otimes_{W(k)}(L\oplus J)} + \natural$, where $\natural$ is identified with the $W_m(R)$-linear endomorphism of $W_m(R) \otimes_{W_m(k)} (L \oplus J)$ that annihilates $W_m(R) \otimes_{W_m(k)} L$ and that maps $W_m(R) \otimes_{W_m(k)} J$ to $W_m(R) \otimes_{W_m(k)} L$ in the same way as $\natural$ does.

As $\mathfrak{b} \subset \mathrm{End}_{W(k)}(L \oplus J)$ is stable under products (we have $\mathfrak{b}^2 = 0$), the condition (i) of [GV], Thm. 6 holds for the triple $(L \oplus J, \phi, \mathbf{B})$. Thus the fact that Theorem 1 (b) holds follows from [GV], Prop. 2 (c) and Thm. 6 and from Equation (16). We only need to point out that the footnote of Subsection 1 applies in this context as well: as both $D[p]/\boldsymbol{\beta}_p(D[p])$ and $\boldsymbol{\alpha}_p(E[p])$ are left $\mathbb{B}$-modules, $\mathbf{Hom}_{\mathbb{B}}(D[p]/\boldsymbol{\beta}_p(D[p]), \boldsymbol{\alpha}_p(E[p]))$ is a closed subscheme of $\mathbf{Hom}_{\mathbb{B}}(D[p], E[p])$ and therefore the following implications hold

$$\gamma^{\mathbb{B}}_{D,E}(1) = 0 \Longleftrightarrow a_D a_E = 0 \Longleftrightarrow D \,\mathrm{or}\, E \,\mathrm{is\ ordinary} \Longrightarrow n^{\mathbf{B}}_{D\oplus E} = 0.$$

3.2 Proalgebraic constructions

If O is a finitely generated left $\mathbb{J}$-module and M is a left $\mathbb{J}$-module of finite length, then $\mathrm{Hom}_{\mathbb{J}}(O, M)$ has a natural structure of a commutative quasi-algebraic group $\underline{\mathrm{Hom}_{\mathbb{J}}(O, M)}$ over k. This can be checked easily by choosing a finite presentation of O and an expression of M as a direct sum of cyclic $W(k)$-modules and by checking directly the independence of the resulting commutative quasi-algebraic group structure $\underline{\mathrm{Hom}_{\mathbb{J}}(O, M)}$ of $\mathrm{Hom}_{\mathbb{J}}(O, M)$ from the choices made. Similarly, if O and O' are finitely generated left $\mathbb{J}$-module and O' is also finitely generated as a $W(k)$-module, then $\mathrm{Hom}_{\mathbb{J}}(O, O')$ has a natural structure of a commutative affine pro-algebraic group $\underline{\mathrm{Hom}_{\mathbb{J}}(O, O')}$ over k.

We consider the finite dimensional $B(k)$-vector space

$$X := \mathrm{Hom}_{B(k)}(J[1/p], L[1/p]) = \mathrm{Hom}_{W(k)}(J, L)[1/p] = \mathrm{Hom}_{W(k)}(J, L)^{\flat}[1/p].$$

Let $\mathrm{Hom}_{W(k)}(J, L)^{\flat}$ be the sublattice of $\mathrm{Hom}_{W(k)}(J, L)$ formed by $W(k)$-linear maps that send VJ to VL. We note that, inside X, we have an identity

$$\mathrm{Hom}_{W(k)}(J, L)^{\flat} = \mathrm{Hom}_{W(k)}(J, L) \cap \mathrm{Hom}_{W(k)}(VJ, VL)$$

and $\mathrm{Hom}_{W(k)}(J, L)^{\flat}$ is the largest sublattice of $\mathrm{Hom}_{W(k)}(J, L)$ for which we have a homomorphism of prounipotent groups

$$\underline{\Psi_{J,L}} : \underline{\mathrm{Hom}_{W(k)}(J, L)^{\flat}} \to \underline{\mathrm{Hom}_{W(k)}(J, L)}$$

defined by the abstract (proalgebraic) homomorphism

$$\Psi_{J,L} : \mathrm{Hom}_{W(k)}(J, L)^{\flat} \to \mathrm{Hom}_{W(k)}(J, L)$$

that maps $x \in \mathrm{Hom}_{W(k)}(J, L)^{\flat}$ to $x - p^{-1}FxV = x - \phi(x)$. It induces an admissible proalgebraic endomorphism $\underline{\Psi} : \underline{X} \to \underline{X}$ in the sense of Section 2 (see Example 3).

It is well-known that the kernel of $\underline{\Psi_{J,L}}$ is the affine proalgebraic group $\underline{\mathrm{Hom}_{\mathbb{E}}(J, L)} = \underline{\mathrm{Hom}(D, E)}$ of Dieudonné module homomorphisms. We check that the kernel of the reduction of $\underline{\Psi_{J,L}}$ modulo p^n is the quasi-algebraic group $\underline{\mathrm{Hom}(D[p^n], E[p^n])} = \mathbf{Hom}(D[p^n], E[p^n])^{\mathrm{perf}}$ of the abstract group

$$\mathrm{Hom}_{\mathbb{E}_n}(J/p^nJ, L/p^nL) = \mathrm{Hom}(D[p^n], E[p^n]) = \mathbf{Hom}(D[p^n], E[p^n])(k)$$

of homomorphisms between Dieudonné modules modulo p^n. The crystalline Dieudonné theory provides a natural evaluation morphism

$$\mathbb{D}_{D[p^n],E[p^n]} : \underline{\mathrm{Hom}(D[p^n], E[p^n])} \to \mathrm{Ker}(\underline{\Psi_{J,L}} \bmod p^n)$$

of the abelian category $\mathcal{Q}$. From [LNV], Lem. 8.7 we get that the abstract homomorphism $\mathbb{D}_{D[p^n],E[p^n]}(k)$ is an isomorphism and therefore from Fact 1 we conclude that $\mathbb{D}_{D[p^n],E[p^n]}$ itself is an isomorphism.

We have a $\mathbb{B}$ variant as follows. As $\mathfrak{b} = \mathrm{Hom}_{W(k)\otimes_{\mathbb{Z}_p}\mathbb{B}^{\mathrm{op}}}(J, L)$ is a direct summand of $\mathrm{Hom}_{W(k)}(J, L)$ and due to Equations (13) and (14) we have

$$\mathfrak{b}^\flat := \mathfrak{b} \cap \mathrm{Hom}_{W(k)}(J, L)^\flat = p\mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1$$

and $\mathfrak{b}^\flat$ is the largest $W(k)$-submodule of $\mathfrak{b}$ with the property that $\Psi_{J,L}$ restricts to an abstract (proalgebraic) homomorphism

$$\Psi^{\mathbb{B}}_{J,L} : \mathfrak{b}^\flat \to \mathfrak{b}$$

defining a homomorphism of prounipotent groups

$$\underline{\Psi^{\mathbb{B}}_{J,L}} : \underline{\mathfrak{b}^\flat} \to \underline{\mathfrak{b}} \tag{17}$$

whose kernel is the affine proalgebraic groups $\underline{\mathrm{Hom}_{\mathbb{J}}(J, L)} = \underline{\mathrm{Hom}_{\mathbb{B}}(D, E)}$ and which induces an admissible proalgebraic endomorphism $\underline{\Psi_{\mathbb{B}}} : \underline{X_{\mathbb{B}}} \to \underline{X_{\mathbb{B}}}$ in the sense of Section 2 (see Example 3), where

$$X_{\mathbb{B}} := \mathfrak{b}[1/p].$$

Moreover, $\mathbb{D}_{D[p^n],E[p^n]}$ restricts to an isomorphism

$$\mathbb{D}^{\mathbb{B}}_{D[p^n],E[p^n]} : \underline{\mathrm{Hom}_{\mathbb{B}}(D[p^n], E[p^n])} \to \mathrm{Ker}(\underline{\Psi^{\mathbb{B}}_{J,L}} \bmod p^n)$$

as $\mathbb{D}^{\mathbb{B}}_{D[p^n],E[p^n]}(k)$ is an isomorphism, which follows by taking centralizers of $\mathbb{B}^{\mathrm{op}}$ in the source and the target of $\mathbb{D}_{D[p^n],E[p^n]}(k)$.

Let D' and E' be BTs over k isogenous to D and E (respectively). Let L' and J' be the (contravariant) Dieudonné modules of D' and E' (respectively). We have natural identifications $L[1/p] = L'[1/p]$, $J[1/p] = J'[1/p]$ and $X = \mathrm{Hom}_{B(k)}(J')[1/p], L'[1/p]) = \mathrm{Hom}_{W(k)}(J', L')[1/p] = \mathrm{Hom}_{W(k)}(J', L')^\flat[1/p]$ and $\underline{\Psi} : \underline{X} \to \underline{X}$ is also induced by a homomorphism of prounipotent groups

$$\underline{\Psi_{J',L'}} : \underline{\mathrm{Hom}_{W(k)}(J', L')^\flat} \to \underline{\mathrm{Hom}_{W(k)}(J', L')}$$

defined by a homomorphism $\Psi_{J',L'} : \mathrm{Hom}_{W(k)}(J', L')^\flat \to \mathrm{Hom}_{W(k)}(J', L')$ which satisfies the same rule as $\Psi_{J,L}$: it maps $x \in \mathrm{Hom}_{W(k)}(J', L')^\flat$ to $x - p^{-1}FxV = x - \phi(x)$. If $\mathbb{B}^{\mathrm{op}}$ inside $\mathrm{End}_{W(k)}(L)[1/p]$ and $\mathrm{End}_{W(k)}(J)[1/p]$ leave invariant L' and J' (respectively), equivalently, if D' and E' are compatibly left $\mathbb{B}$-modules, then we similarly define

$$\mathfrak{b}' := \mathrm{Hom}_{W(k)\otimes_{\mathbb{Z}_p}\mathbb{B}^{\mathrm{op}}}(J', L') \subset \mathrm{Hom}_{W(k)}(J', L')$$

and $\underline{\Psi_{J',L'}}$ restricts to a homomorphism of prounipotent groups

$$\underline{\Psi^{\mathbb{B}}_{J',L'}} : \underline{(\mathfrak{b}')^\flat} \to \underline{\mathfrak{b}'}.$$

We have $\phi(\mathrm{Hom}_{W(k)}(J, L)) = \mathrm{Hom}_{W(k)}(FJ, FL)$. We consider the lattice

$$\mathrm{Hom}_{W(k)}(J, L)^\sharp := \mathrm{Hom}_{W(k)}(J, L) + \mathrm{Hom}_{W(k)}(FJ, FL)$$

of the $B(k)$-vector space X. As for an element $x \in \mathrm{Hom}_{W(k)}(J, L)$ we have $p^{-1}FxV = \phi(x) \in \mathrm{Hom}_{W(k)}(FJ, FL)$, the admissible proalgebraic homomorphism of $B(k)$-vector spaces $\Psi : X \to X$ that maps $x \in X$ to $x - p^{-1}FxV = x - \phi(x) \in X$ induces (see Example 3) a homomorphism between prounipotent groups

$$\underline{\Psi_{J,L,+}} : \underline{\mathrm{Hom}_{W(k)}(J, L)} \to \underline{\mathrm{Hom}_{W(k)}(J, L)^\sharp}.$$

Its kernel is the affine proalgebraic group $\underline{\mathrm{Hom}_{\mathbb{E}}(J, L)} = \underline{\mathrm{Hom}(D, E)}$ of Dieudonné module homomorphisms and thus is equal to the kernel of $\underline{\Psi_{J,L}}$.

Moreover, from Equation (14) we get that

$$\mathfrak{b}^\sharp := \mathfrak{b}[1/p] \cap \mathrm{Hom}_{W(k)}(J, L)^\sharp \text{ is equal to } \mathfrak{b} + \phi(\mathfrak{b}_{-1})$$

and $\underline{\Psi_{J,L,+}}$ restricts to a homomorphism between prounipotent groups

$$\underline{\Psi^{\mathbb{B}}_{J,L,+}} : \underline{\mathfrak{b}} \to \underline{\mathfrak{b}^\sharp} \tag{18}$$

whose kernel is the affine proalgebraic group $\underline{\mathrm{Hom}_{\mathbb{J}}(J, L)} = \underline{\mathrm{Hom}_{\mathbb{B}}(D, E)}$.

3.3 The isogeny invariance of $s^{\mathbb{B}}_{D,E}$

As the sequence $(\gamma^{\mathbb{B}}_{D,E}(n))_{n \geq n^{\mathbb{B}}_{D,E}}$ is constant (see Theorem 1 (b)), from Lemma 1 (c) applied to $\underline{\Psi^{\mathbb{B}}_{J,L}}$ we get that for all $n \geq n^{\mathbb{B}}_{D,E}$ we have

$$\gamma^{\mathbb{B}}_{D,E}(n) = \dim(\underline{\mathfrak{b}}/\underline{\Psi^{\mathbb{B}}_{J,L}}(\underline{\mathfrak{b}^\flat})),$$

equivalently, we have

$$s^{\mathbb{B}}_{D,E} = \chi(\underline{\mathfrak{b}}, \underline{\Psi^{\mathbb{B}}_{J,L}}(\underline{\mathfrak{b}^\flat})). \tag{19}$$

The dimension of the k-vector space $\mathfrak{b}/\mathfrak{b}^\flat$ is $d_{\mathfrak{b}}$ (see Equation (14)) and from Equation (15) we get that it is equal to the dimension of the k-vector space $\mathfrak{b}'/(\mathfrak{b}')^\flat$. From this equality and Equation (9) applied with (U_1, U_2, U_3, U_4) equal to $(\underline{\mathfrak{b}'}, \underline{\mathfrak{b}}, \underline{(\mathfrak{b}')^\flat}, \underline{\mathfrak{b}^\flat})$ we get:

$$\chi(\underline{\mathfrak{b}'}, \underline{\mathfrak{b}}) = \chi(\underline{(\mathfrak{b}')^\flat}, \underline{\mathfrak{b}^\flat}). \tag{20}$$

From Equation (20) and Lemma 1 (a) applied with $(\underline{T}, U_1, U_2, U_1', U_2')$ equal to $(\underline{\Psi_{\mathbb{B}}}, \underline{\mathfrak{b}^\flat}, \underline{(\mathfrak{b}')^\flat}, \underline{\mathfrak{b}}, \underline{(\mathfrak{b}')})$ we get directly that $\chi(\underline{\mathfrak{b}'}, \underline{\Psi^{\mathbb{B}}_{J',L'}}(\underline{(\mathfrak{b}')^\flat}))$ is equal to $\chi(\underline{\mathfrak{b}}, \underline{\Psi^{\mathbb{B}}_{J,L}}(\underline{\mathfrak{b}^\flat}))$. From this and Equation (19) and its analogue with (J, L) replaced by (J', L') we get that $s^{\mathbb{B}}_{D,E} = s^{\mathbb{B}}_{D',E'}$ is an isogeny invariant.

3.4 The symmetry of $s^{\mathbb{B}}_{D,E}$

In this subsection we will use Serre duality to prove that $s^{\mathbb{B}}_{D,E} = s^{\mathbb{B}}_{E,D}$.

Fact 4. *The following two inclusions $\underline{\mathfrak{b}^\flat} \subset \underline{\mathfrak{b}}$ and $\underline{\mathfrak{b}} \subset \underline{\mathfrak{b}^\sharp}$ induce a quasi-isomorphism from (17) to (18) viewed as complexes of prounipotent groups and thus for all $m \in \mathbb{N}^*$ they also induce a quasi-isomorphism between the complexes (17) and (18) modulo p^m viewed as complexes of unipotent connected commutative quasi-algebraic groups. In particular, for all $m \in \mathbb{N}^*$ the kernels of the reductions modulo p^m of $\underline{\Psi^{\mathbb{B}}_{J,L}}$ and $\underline{\Psi^{\mathbb{B}}_{J,L,+}}$ have equal dimensions.*

Proof: This is equivalent to the fact that the induced $W(k)$-linear map at the level of cokernels

$$\mathfrak{b}/\mathrm{Im}(\Psi^{\mathbb{B}}_{J,L}) \to \mathfrak{b}^\sharp/\mathrm{Im}(\Psi^{\mathbb{B}}_{J,L,+})$$

is an isomorphism, and thus is equivalent to the following two identities

$$\mathrm{Im}(\Psi^{\mathbb{B}}_{J,L}) = \mathfrak{b} \cap \mathrm{Im}(\Psi^{\mathbb{B}}_{J,L,+}) \quad \text{and} \quad \mathfrak{b} + \mathrm{Im}(\Psi^{\mathbb{B}}_{J,L,+}) = \mathfrak{b}^\sharp.$$

The inclusions '$\subset$' are obvious. We now check the reversed inclusions '$\supset$'.

Let $x \in \mathfrak{b} \cap \mathrm{Im}(\Psi^{\mathbb{B}}_{J,L,+})$. Let $y \in \mathfrak{b}$ be such that $x = \Psi^{\mathbb{B}}_{J,L,+}(y)$; so $x = y - p^{-1}FyV$. Then $p^{-1}FyV = x - y \in \mathfrak{b}$, hence $y \in \mathfrak{b}^\flat$. Thus we have $x = \Psi^{\mathbb{B}}_{J,L}(y) \in \mathrm{Im}(\Psi^{\mathbb{B}}_{J,L})$.

Let $x' \in \mathfrak{b}^\sharp$. Let $(y', w) \in \mathfrak{b} \times \phi(\mathfrak{b}_{-1})$ be such that $x' = y' + w$. Let $u \in \mathfrak{b}_{-1} \subset \mathfrak{b}$ be such that $w = \phi(u) = p^{-1}FuV$. Thus $x' - (y' + u) = w - u \in \mathrm{Im}(\Psi^{\mathbb{B}}_{J,L,+})$. Hence $x' = (y' + u) + (w - u) \in \mathfrak{b} + \mathrm{Im}(\Psi^{\mathbb{B}}_{J,L,+})$.

The reversed inclusions '$\supset$' follow from the last two paragraphs. $\square$

We are now ready to complete the proof of Theorem 1. Let $m \in \mathbb{N}^*$. The Serre dual of (18) modulo p^m is isomorphic to the reduction of

$$\underline{\Psi^{\mathbb{B}}_{L,J}} : \underline{\mathfrak{b}^\flat_{L,J}} \to \underline{\mathfrak{b}_{L,J}} \tag{21}$$

modulo p^m (see the paragraph before Lemma 2); here (21) is the analogue of (17) but with the roles of J and L interchanged. Based on this and Equation (11), we get that $\underline{\mathrm{Hom}_{\mathbb{B}}(E[p^m], D[p^m])} = \mathbf{Hom}_{\mathbb{B}}(E[p^m], D[p^m])^{\mathrm{perf}}$ (i.e., the kernel of the reduction of $\underline{\Psi^{\mathbb{B}}_{L,J}}$ modulo p^m) has the same dimension as the kernel of the reduction of $\underline{\Psi^{\mathbb{B}}_{J,L,+}}$ modulo p^m and thus (see Fact 4) as $\underline{\mathrm{Hom}_{\mathbb{B}}(D[p^m], D[E^m])} = \mathbf{Hom}_{\mathbb{B}}(D[p^m], E[p^m])^{\mathrm{perf}}$ (i.e., as the kernel of the reduction of $\underline{\Psi^{\mathbb{B}}_{J,L}}$ modulo p^m). As this holds for all $m \in \mathbb{N}^*$, we get that $n^{\mathbb{B}}_{D,E} = n^{\mathbb{B}}_{E,D}$ and that $s^{\mathbb{B}}_{D,E} = s^{\mathbb{B}}_{E,D}$.[5] This ends the proof of Theorem 1. $\square$

3.5 A relative context

We recall that $k = \bar{k}$ and L is the (contravariant) Dieudonné module of a BT D over k. We take E to be trivial; so $J = 0$ and we identify $L \oplus J = L$. Thus we have a triple $(L, \phi, \mathbf{G})$ which is an F-crystal with a group over k.

Definition 7. *A pair $(g_1, g_2) \in \mathbf{G}(W(k))^2$ is called Lie compatible if we have $g_1 \mathfrak{g} g_2^{-1} = \mathfrak{g}$ (equivalently, for one $h \in \{g_1 g_2^{-1}, g_2 g_1^{-1}\}$ we have either $h\mathfrak{g} = \mathfrak{g}$ or $\mathfrak{g}h = \mathfrak{g}$).*

For $g \in \mathbf{G}(W(k))$ let D_g be the BT over k of the same dimension d_D and codimension c_D as D whose (contravariant) Dieudonné module is $(L, g\phi, \vartheta g^{-1})$. Thus $D = D_{1_L}$ and (g, g) is Lie compatible. The pair $(1_L, g)$ is Lie compatible if and only if the pair $(g, 1_L)$ (or the pair $(1_L, g^{-1})$) is Lie compatible. If $\mathbf{G} = \mathbf{GL}_L$, then each BT over k of dimension d_D and codimension c_D is isomorphic to D_g for some $g \in \mathbf{G}(W(k))$.

[5] We also get that, referring to Theorem 2 (d), if H is a BT_m (such an H is balanced by Lemma 10 (b) below), then d2 holds.

Example 6. *If* $\mathbf{G}(W(k)) = 1_L + \mathfrak{g}$ *or if* $\mathbf{G}$ *is the group scheme of units of a* $W(k)$*-subalgebra of* $\mathrm{End}_{W(k)}(L)$ *which as a* $W(k)$*-module is a direct summand of* $\mathrm{End}_{W(k)}(L)$*, then all pairs* $(g_1, g_2) \in \mathbf{G}(W(k))^2$ *are Lie compatible.*

We consider the closed subgroup scheme

$$\mathbf{Hom}(D[p^m], D_g[p^m])^{\mathfrak{g}}_{\mathrm{crys}}$$

of $\mathbf{Hom}(D[p^m], D_g[p^m])_{\mathrm{crys}}$ such that for each commutative k-algebra R, its group of R-valued points $\mathbf{Hom}(D[p^m], D_g[p^m])^{\mathfrak{g}}_{\mathrm{crys}}(R)$ is the subgroup

$$\mathbf{Hom}(D[p^m], D_g[p^m])_{\mathrm{crys}}(R) \cap [W_m(R) \otimes_{W_m(k)} (\mathfrak{g}/p^m\mathfrak{g})].$$

The group schemes

$$\mathbf{End}(D[p^m])^{\mathfrak{g}}_{\mathrm{crys}} = \mathbf{Hom}(D[p^m], D[p^m])^{\mathfrak{g}}_{\mathrm{crys}}$$

were first introduced in [GV], Subsect. 5.2.

The following theorem generalizes the particular case of Theorems 1 (b) and (c) in which D and E have the same dimension and codimension.

Theorem 5. *Let* $g \in \mathbf{G}(W(k))$ *be such that the pair* $(1_L, g) \in \mathbf{G}(W(k))^2$ *is Lie compatible. Then the following three properties hold:*

(a) *There exists a smallest non-negative integer* $n^{\mathfrak{g}}_{D,D_g}$ *such that for all integers* $n \geq n^{\mathfrak{g}}_{D,D_g}$ *we have an equality*

$$\dim(\mathbf{Hom}(D[p^n], D_g[p^n])^{\mathfrak{g}}_{\mathrm{crys}}) = \dim(\mathbf{Hom}(D[p^{n^{\mathfrak{g}}_{D,D_g}}], D_g[p^{n^{\mathfrak{g}}_{D,D_g}}])^{\mathfrak{g}}_{\mathrm{crys}}).$$

Moreover, if $n^{\mathfrak{g}}_{D,D_g} > 0$*, then the finite sequence*

$$(\dim(\mathbf{Hom}(D[p^n], D_g[p^n])^{\mathfrak{g}}_{\mathrm{crys}}))_{n \in \{1,\ldots,n^{\mathfrak{g}}_{D,D_g}\}}$$

is strictly increasing.[6]

(b) *If* $s^{\mathfrak{g}}_{D,D_g} := \dim(\mathbf{Hom}(D[p^{n^{\mathfrak{g}}_{D,D_g}}], D_g[p^{n^{\mathfrak{g}}_{D,D_g}}])^{\mathfrak{g}}_{\mathrm{crys}})$*, then* $s^{\mathfrak{g}}_{D,D_g}$ *is a Lie compatible isogeny invariant: for all pairs* $(g_1, g_2) \in \mathbf{G}(W(k))^2$ *which are Lie compatible and such that* $(L, g_1\phi, \mathbf{G})$ *and* $(L, g_2\phi, \mathbf{G})$ *are* $\mathbf{G}$*-isogenous to* $(L, \phi, \mathbf{G})$ *and* $(L, g\phi, \mathbf{G})$ *(respectively), we have* $s^{\mathfrak{g}}_{D,D_g} = s^{\mathfrak{g}}_{D_{g_1},D_{g_2}}$.

[6] If $\mathbf{G} = \mathbf{GL}_L$ part (a) follows from [GV], Subsect. 6.1 as $n^{\mathfrak{g}}_{D,D_g} = n_{D,D_g}$, see Section 1.

(c) *Let* $\mathrm{Tr} : \mathrm{End}_{W(k)}(L) \times \mathrm{End}_{W(k)}(L) \to W(k)$ *be the trace bilinear map that maps a pair* $(x, y) \in \mathrm{End}_{W(k)}(L) \times \mathrm{End}_{W(k)}(L)$ *to the trace of the* $W(k)$*-linear endomorphism* $x \circ y : L \to L$*. We assume that there exists* $\ell \in \mathbb{N}$ *such that* Tr *restricts to* p^ℓ *times a perfect bilinear map* $\mathrm{Tr}_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to W(k)$ *(if* $\ell = 0$*, this forces* $\mathbf{G}$ *to be a reductive group scheme over* $W(k)$*). Then we have the following symmetry properties* $s^{\mathfrak{g}}_{D,D_g} = s^{\mathfrak{g}}_{D_g,D}$ *and* $n^{\mathfrak{g}}_{D,D_g} = n^{\mathfrak{g}}_{D_g,D}$*.*[7]

Proof: Let L_1 and L_2 be L. The (contravariant) Dieudonné module of the BT $D \oplus D_g$ is $(L \oplus L, \phi \oplus g\phi) = (L_1 \oplus L_2, \phi \oplus g\phi)$. Let $\mathfrak{h}$ be the abelian Lie subalgebra of $\mathrm{End}_{W(k)}(L \oplus L) = \mathrm{End}_{W(k)}(L_1 \oplus L_2)$ formed by all those $W(k)$-linear endomorphisms of $L \oplus L = L_1 \oplus L_2$ which annihilate L_1, which map L_2 to L_1, and for which the resulting $W(k)$-linear map from $L_2 = L$ to $L_1 = L$ is an element of $\mathfrak{g}$.

Similar to $\mathbf{B}$, let $\mathbf{H}$ be the closed subgroup scheme of $\mathbf{GL}_{L\oplus L}$ whose Lie algebra is $\mathfrak{h}$ and such that for each commutative k-algebra R, we have

$$\mathbf{H}(R) = 1_{W(R)\otimes_{W(k)}(L\oplus L)} + W(R) \otimes_{W(k)} \mathfrak{h}.$$

We check that the triple $(L \oplus L, \phi \oplus g\phi, \mathbf{H})$ is an F-crystal with a group over k, i.e., Axioms (AX1) and (AX2) hold for it. Axiom (AX2) holds for this triple as one can see by considering the direct sum decomposition

$$L \oplus L = (P^1_1 \oplus P^1_2) \oplus (P^0_1 \oplus P^1_2),$$

where, for $(i, j) \in \{0, 1\} \times \{1, 2\}$, P^i_j is P^i but viewed as a $W(k)$-submodule of L_j. As the pair $(1_L, g) \in \mathbf{G}(W(k))^2$ is Lie compatible, the fact that Axiom (AX1) also holds for this triple follows from the identities

$$\phi(\mathfrak{g}[1/p])g^{-1} = \mathfrak{g}[1/p]g^{-1} = \mathfrak{g}[1/p].$$

For $m \in \mathbb{N}^*$ let $\mathbf{Aut}(D[p^m] \oplus D_g[p^m])^{\mathbf{H}}_{\mathrm{crys}}$ be the group scheme over k defined in [GV], Def. 2 (a). Similar to Equation (16) we have an identification

$$\mathbf{Hom}(D[p^m], D_g[p^m])^{\mathfrak{g}}_{\mathrm{crys}} = \mathbf{Aut}(D[p^m] \oplus D_g[p^m])^{\mathbf{H}}_{\mathrm{crys}} \tag{22}$$

[7] We get a variant of part (c) if we have a direct sum decomposition $(\mathfrak{g}, \phi) = \oplus_{i\in I}(\mathfrak{l}_i, \phi)$ which (without ϕ) is orthogonal with respect to Tr and which has the property that for each $i \in I$ there exists $\ell_i \in \mathbb{N}$ such that the restriction of Tr to $\mathfrak{l}_i$ is p^{ℓ_i} times a perfect bilinear map $\mathrm{Tr}_{\mathfrak{l}_i} : \mathfrak{l}_i \times \mathfrak{l}_i \to W(k)$. E.g., this holds if $\mathbf{G}$ is a reductive group scheme over $\mathrm{Spec}\, W(k)$, the Killing form of the Lie algebra of the derived group scheme of $\mathbf{G}$ is perfect, and the maximal torus of the center of $\mathbf{G}$ is equal to its double centralizer in $\mathbf{GL}_L$.

of affine group schemes over k, which for a commutative k-algebra R maps each $\natural \in \mathbf{Hom}(D[p^m], D_g[p^m])^{\mathfrak{g}}_{\mathrm{crys}}(R)$ to $1_{W(R)\otimes_{W(k)}(L\oplus L)} + \natural$, where $\natural$ is identified with the $W_m(R)$-linear endomorphism of

$$W_m(R) \otimes_{W_m(k)} (L \oplus L) = W_m(R) \otimes_{W_m(k)} (L_1 \oplus L_2)$$

which annihilates $W_m(R) \otimes_{W_m(k)} L_1$ and which maps $W_m(R) \otimes_{W_m(k)} L_2$ to $W_m(R) \otimes_{W_m(k)} L_1$ in the same way as $\natural$ does.

To check part (a) we can assume that there exists $n_0 \in \mathbb{N}^*$ such that we have $\dim(\mathbf{Hom}(D[p^{n_0}], D_g[p^{n_0}])^{\mathfrak{g}}_{\mathrm{crys}}) > 0$. As $\mathfrak{h} \subset \mathrm{End}_{W(k)}(L_1 \oplus L_2)$ is stable under products (we have $\mathfrak{h}^2 = 0$), the condition (i) of [GV], Thm. 6 holds for the triple $(L \oplus L, \phi \oplus g\phi, \mathbf{H})$. Thus part (a) follows from [GV], Prop. 2 (c) and Thm. 6 and from Equation (22) which give that $n^{\mathfrak{g}}_{D,D_g} = n^{\mathbf{H}}_{D\oplus D_g} > 0$ (if $n^{\mathbf{H}}_{D\oplus D_g} = 0$, then loc. cit. implies that n_0 does not exist).

To prove parts (b) and (c) we consider the σ-linear isomorphisms

$$\phi_{D,D_g}, \phi_{D_g,D} : \mathfrak{g}[1/p] \to \mathfrak{g}[1/p]$$

which map $w \in \mathfrak{g}[1/p]$ to $\phi \circ w \circ \phi^{-1} g^{-1}$ and $g\phi \circ h \circ \phi^{-1}$ (respectively). For all $x, y \in \mathrm{End}_{W(k)}(L)$ we have an identity

$$\sigma(Tr(x, y)) = Tr(\phi_{D,D_g}(x), \phi_{D_g,D}(y)).$$

Thus the fact that $\mathrm{Tr}_{\mathfrak{g}}$ is perfect implies that $\mathrm{Tr}_{\mathfrak{g}}$ induces an isomorphism of latticed F-isocrystals from the dual of $(\mathfrak{g}, \phi_{D_g,D})$ to $(\mathfrak{g}, \phi_{D,D_g})$. Based on this, the proofs of parts (b) and (c) are entirely analogous to the proofs of Subsections 3.3 and 3.4, with the roles of (L, J) being replaced by $(L = L_1, L = L_2)$ and with the roles of $\mathfrak{b} = \mathfrak{b}_{J,L} \subset \mathrm{Hom}_{W(k)}(J, L)$ and $\mathfrak{b}_{L,J} \subset \mathrm{Hom}_{W(k)}(L, J)$ being replaced by the Lie subalgebra $\mathfrak{g}$ of $\mathrm{End}_{W(k)}(L) = \mathrm{Hom}_{W(k)}(L_2, L_1)$ and of $\mathrm{End}_{W(k)}(L) = \mathrm{Hom}_{W(k)}(L_1, L_2)$ (respectively). We would only like to add that, with the same notation $\mathrm{Hom}_{W(k)}(J, L)^\flat$ and $\mathrm{Hom}_{W(k)}(J, L)^\sharp$ as in Subsections 3.3 and 3.4 but used under the mentioned replacement of roles, for $\diamond = D, D_g$ let $v_\diamond := 2$ and for $\diamond = D_g, D$ let $v_\diamond := 1$ and we take $\mathfrak{g}^\flat_\diamond$ to be

$$\mathfrak{g} \cap \mathrm{Hom}_{W(k)}(L_{v_\diamond}, L_{3-v_\diamond})^\flat = \{x \in \mathfrak{g} | \phi_\diamond(x) \in \mathrm{Hom}_{W(k)}(L_{v_\diamond}, L_{3-v_\diamond})\} = \mathfrak{g} \cap \phi_\diamond^{-1}(\mathfrak{g})$$

and we take $\mathfrak{g}^\sharp_\diamond$ to be

$$\mathfrak{g}[1/p] \cap \mathrm{Hom}_{W(k)}(L_{v_\diamond}, L_1)^\sharp = \mathfrak{g} + (\mathfrak{g}[1/p] \cap \phi_\diamond(\mathrm{Hom}_{W(k)}(L_{v_\diamond}, L_{3-v_\diamond}))) = \mathfrak{g} + \phi_\diamond(\mathfrak{g}).$$

Note that the dual of $\mathfrak{g}^\sharp_{D,D_g}$ is $\mathfrak{g}^\flat_{D_g,D}$ and the dual of $\mathfrak{g}^\flat_{D,D_g}$ is $\mathfrak{g}^\sharp_{D_g,D}$. $\square$

Taking $g = 1_L$ and $g_2 = g_1$, from Theorem 5 (b) we obtain another generalization of [V2], Thm. 1.2 (e) (see also [GV], Rm. 4.5):

Corollary 2. *Let* $s_D^{\mathfrak{g}} := s_{D,D}^{\mathfrak{g}}$. *Then* s_D *is an isogeny invariant: if* $g_1 \in \mathbf{G}(W(k))$ *is such that* $(L, g_1\phi, \mathbf{G})$ *is* $\mathbf{G}$*-isogenous to* $(L, \phi, \mathbf{G})$, *then* $s_D^{\mathfrak{g}} = s_{D_{g_1}}^{\mathfrak{g}}$.

4 Injective resolutions of rings of fractions

We have $\mathbb{E}_m F = F\mathbb{E}_m$ and $\mathbb{E}_m V = V\mathbb{E}_m$. From this and the identities $FV = VF$ and $p^m = 0$ in $\mathbb{E}_m$ we get that each non-zero element of the union $\mathbb{U}_m := \mathbb{E}_m F \cup \mathbb{E}_m V \cup p\mathbb{E}_m$ is a left and a right zero divisor of $\mathbb{E}_m$. Let

$$\mathbb{S}_m := \mathbb{E}_m \setminus \mathbb{U}_m.$$

For $\bar{z} \in \mathbb{S}_1$ let $d_F(\bar{z}), d_V(\bar{z}) \in \mathbb{N}$ be such that we can write

$$\bar{z} = \sum_{i=0}^{d_F(\bar{z})} \alpha_i F^i + \sum_{j=1}^{d_V(\bar{z})} \beta_j V^j = \sum_{i=1}^{d_F(\bar{z})} \alpha_i F^i + \sum_{j=0}^{d_V(\bar{z})} \beta_j V^j,$$

where $\alpha_0 = \beta_0, \alpha_1, \ldots, \alpha_{d_F(\bar{z})}, \beta_1, \ldots, \beta_{d_V(\bar{z})} \in k$ with $\alpha_{d_F(\bar{z})} \neq 0$, $\beta_{d_V(\bar{z})} \neq 0$. The k-algebra monomorphism $\mathbb{E}_1 \to \mathbb{E}_1/(V) \times \mathbb{E}_1/(F) = k[F] \times k[V]$ maps $\bar{z}$ to a pair $(\bar{z}_F, \bar{z}_V) \in k[F] \times k[V]$ and $d_F(\bar{z})$ and $d_V(\bar{z})$ are the degrees of $\bar{z}_F$ and $\bar{z}_V$ in F and V (respectively).

For $x \in \mathbb{E}_m$ we have $x \in \mathbb{S}_m$ if and only if x modulo p belongs to $\mathbb{S}_1$, if and only if both the images of x in $k[F] = \mathbb{E}_m/(p, V)$ and in $k[V] = \mathbb{E}_m/(p, F)$ are non-zero, and if and only if the left $W_m(k)$-module $\mathbb{E}_m/\mathbb{E}_m x$ (equivalently, the right $W_m(k)$-module $\mathbb{E}_m/x\mathbb{E}_m$) is finitely generated.

Lemma 3. *For* $m \in \mathbb{N}^*$ *the following three properties hold:*

(a) *The subset* $\mathbb{S}_m$ *of* $\mathbb{E}_m$ *is the multiplicative set of regular (equivalently, of left regular or of right regular) elements of* $\mathbb{E}_m$*-modules.*

(b) *The multiplicative set* $\mathbb{S}_m$ *admits calculus of left and right fractions, i.e., the left and right Ore conditions are satisfied.*

(c) *If* $z \in \mathbb{S}_m$ *lifts* $\bar{z} \in \mathbb{S}_1$, *then the* $W_m(k)$*-modules* $\mathbb{E}_m/\mathbb{E}_m z$ *and* $\mathbb{E}_m/z\mathbb{E}_m$ *are free of rank* $d_F(\bar{z}) + d_V(\bar{z})$.

Proof: It suffices to prove part (a) for $m = 1$. If $\bar{z} \in \mathbb{S}_1$ is as above, then for each $x = \sum_{i=0}^{c} \alpha'_i F^i + \sum_{j=1}^{d} \beta'_j V^j \in \mathbb{E}_1$ with $\alpha'_0, \ldots \alpha'_c, \beta'_1, \ldots \beta'_d \in k$ such that either $\alpha'_c \neq 0$ or $\beta'_d \neq 0$, the coefficients of either $F^{c+d_F(f)}$ or $V^{d+d_V(f)}$ in $\bar{z}x$ and $x\bar{z}$ are non-zero and thus $x\bar{z} \neq 0$ and $\bar{z}x \neq 0$. Thus part (a) holds.

Based on part (a), part (b) is equivalent to the fact that for each pair $(z,x) \in \mathbb{S}_m \times \mathbb{E}_m$, the intersections $\mathbb{S}_m x \cap \mathbb{E}_m z$ and $x\mathbb{S}_m \cap z\mathbb{E}_m$ are non-empty. We will work work only with $\mathbb{S}_m x \cap \mathbb{E}_m z$ and it suffices to show that each element $x + \mathbb{E}_m z \in O_m := \mathbb{E}_m/\mathbb{E}_m z$ is annihilated by an element $w \in \mathbb{S}_m$.

The ring $\mathbb{E}_m$ is left and right noetherian and moreover $\mathbb{E}_1$ is semiprime, i.e., it is non-zero and its only two-sided nilpotent ideal is 0. Thus the fact that part (b) holds for $m = 1$ follows from [Di], Thm. 3.6.12.

For $m \in \mathbb{N}^*$, from the previous paragraph we get inductively that for $i \in \{1, \ldots, m\}$ there exists $w_i \in \mathbb{S}_m$ such that $(\prod_{j=i}^{1} w_j)x + \mathbb{E}_m z \in p^i O_m$ and thus we can take $w := \prod_{j=m}^{1} w_j$. Therefore part (b) holds.[8]

For part (c), again we work only with $O_m = \mathbb{E}_m/\mathbb{E}_m z$. As the $\mathbb{E}_m$-linear monomorphism $\mathbb{E}_m z \to \mathbb{E}_m$ between flat $W_m(k)$-module is injective modulo p, we have $\mathrm{Tor}_1^{W_m(k)}(k, O_m) = 0$. From this and the nilpotent ideal case of [M], Thm. 22.3 we get that O_m is flat over $W_m(k)$ and thus free. To compute its rank we can assume $m = 1$; so $z = \bar{z}$. Thus part (c) holds as $\{F^0 + \mathbb{E}_1\bar{z}, \ldots, F^{d_F(\bar{z})-1}, V^1 + \mathbb{E}_1\bar{z}, \ldots, V^{d_V(\bar{z})} + \mathbb{E}_1\bar{z}\}$ is a k-basis of O_1. $\square$

Let $\mathbb{K}_m := \mathrm{Frac}(\mathbb{E}_m) = \mathbb{S}_m^{-1}\mathbb{E}_m = \mathbb{E}_m\mathbb{S}_m^{-1}$ be the ring of fractions of $\mathbb{E}_m$, i.e., the (left or right) localization of $\mathbb{E}_m$ with respect to $\mathbb{S}_m$ and let $\mathbb{E}_m \to \mathbb{K}_m$ be the natural inclusion of rings.

We check that the multiplicative set $\mathbb{P}_m := \{(F+V)^i | i \in \mathbb{N}\} \subset \mathbb{S}_m$ also satisfies the left and right Ore conditions. Working as above only with $O_{m,i} := \mathbb{E}_m/\mathbb{E}_m(F+V)^i$, it suffices to show that each element of $O_{m,i}$ is annihilated by an element of $\mathbb{P}_m$, which follows from the fact that F and V and hence $F+V$ are nilpotent on $O_{m,i}$.

Inverting $F+V$ in $\mathbb{E}_m$ we get the product of skew Laurent polynomial rings $\mathbb{P}_m^{-1}\mathbb{E}_m = \mathbb{E}_m\mathbb{P}_m^{-1} = W_m(k)[F, F^{-1}] \times W_m(k)[V, V^{-1}] \subset \mathbb{K}_m$. This gives a product decomposition $\mathbb{K}_m = \mathrm{Frac}(W_m(k)[F]) \times \mathrm{Frac}(W_m(k)[V])$: thus $\mathbb{K}_m$ is flat over $\mathbb{Z}/p^m\mathbb{Z}$ and modulo p it is the product $k(F) \times k(V)$ of two non-commutative division rings.

[8]The existence of $w \in \mathbb{S}_m$ can be proved directly, i.e., without using the abstract result [Di], Thm. 3.6.12, as follows. As the left $\mathbb{E}_m$-module $\mathbb{E}_m/\mathbb{E}_m z$ is finitely generated over $W_m(k)$, it has a composition series whose factors are simple left $\mathbb{E}_m$-modules annihilated by p and it suffices to show that for each simple left $\mathbb{E}_1$-module S and every $y \in S \setminus \{0\}$, there exists $w_y \in \mathbb{S}_1$ such $w_y y = 0$. Each such S is finite dimensional over k and corresponds to a simple object of $\mathcal{F}_{k,1}$. There exist three possibilities: (i) $F = V = 0$ on S and we can take $w_y = F + S$; (ii) $V = 0$ on S, F bijective on S, and we can take w_y to be a generator of the annihilator of y in $k[F]$; (iii) same as (ii) but with F and V interchanged.

Lemma 4. *The following three properties hold:*

(a) *The short exact sequence* $0 \to \mathbb{E}_m \to \mathbb{K}_m \to \mathbb{K}_m/\mathbb{E}_m \to 0$ *is an injective resolution of left (or right)* $\mathbb{E}_m$*-modules.*

(b) *The left (or right)* $\mathbb{E}_m$*-module* $\mathbb{K}_m$ *is the injective hull of the left (or right)* $\mathbb{E}_m$*-module* $\mathbb{E}_m$*.*

(c) *For each left* $\mathbb{E}_m$*-module of finite length* N *we have a natural identity* $\mathrm{Hom}_{\mathbb{E}_m}(N, \mathbb{K}_m/\mathbb{E}_m) = \mathrm{Ext}^1_{\mathbb{E}_m}(N, \mathbb{E}_m)$ *of left* $\mathbb{E}_m$*-modules (of right* $\mathbb{E}_m$*-modules transformed via* ι_m *into left* $\mathbb{E}_m$*-modules).*

Proof: As in the proof of Lemma 3 (c) we argue based on [M], Thm. 22.3 that $\mathbb{K}_m/\mathbb{E}_m$ is flat over either $W_m(k)$ or $\mathbb{Z}/p^m\mathbb{Z}$. To prove part (a) we have to show that for each left $\mathbb{E}_m$-module N, for $\bigstar_m \in \{\mathbb{K}_m, \mathbb{K}_m/\mathbb{E}_m\}$ we have $\mathrm{Ext}^1_{\mathbb{E}_m}(N, \bigstar_m) = 0$. Using the long exact sequence of Ext groups for short exact sequences of the form $0 \to pN \to N \to N/pN \to 0$, we get that we can assume that $pN = 0$, i.e., N is a left $\mathbb{E}_1$-module.

Using the standard free resolution

$$\cdots \xrightarrow{p} \mathbb{E}_m \xrightarrow{p^{m-1}} \mathbb{E}_m \xrightarrow{p} \mathbb{E}_m \to \mathbb{E}_1 \to 0$$

of the left $\mathbb{E}_m$-module $\mathbb{E}_1$ and the fact that $\bigstar_m$ is a flat (hence free) $W_m(k)$-module, we get that $\mathrm{Ext}^i_{\mathbb{E}_m}(\mathbb{E}_1, \bigstar_m) = 0$ for all $i \in \mathbb{N}^*$. From this and [CE], Ch. VI, Prop. 4.1.4 (also [CE], Ch. XVI, Sect. 5) we get that

$$\mathrm{Ext}^1_{\mathbb{E}_m}(N, \bigstar_m) = \mathrm{Ext}^1_{\mathbb{E}_1}(N, \mathrm{Hom}_{\mathbb{E}_m}(\mathbb{E}_1, \bigstar_m)) = \mathrm{Ext}^1_{\mathbb{E}_1}(N, \bigstar_1),$$

where the last equality follows from the flatness of the $W_m(k)$-module $\bigstar_m$. Thus to prove that $\mathrm{Ext}^1_{\mathbb{E}_m}(N, \bigstar_m) = 0$ we can assume that $m = 1$.

As $m = 1$, using the short exact sequence $0 \to VN \to N \to N/VN \to 0$, we can assume that N is annihilated by either F or V. To fix the ideas, say N is a $k[F]$-module. The infinite complex with alternating (right or left) multiplication by F and V maps

$$\cdots \xrightarrow{F} \clubsuit \xrightarrow{V} \clubsuit \xrightarrow{F} \clubsuit \xrightarrow{V} \cdots$$

is exact for $\clubsuit \in \{\mathbb{E}_1, \mathbb{K}_1\}$ and thus also for $\clubsuit = \bigstar_1$. Using the resolution

$$\cdots \xrightarrow{F} \mathbb{E}_1 \xrightarrow{F} \mathbb{E}_1 \xrightarrow{V} k[F] \to 0 \tag{23}$$

(with V and F as right multiplications) of the left $\mathbb{E}_1$-module $k[F]$ we get that

$$\mathrm{Ext}^i_{\mathbb{E}_1}(k[F], \spadesuit) = 0 \quad \forall (i, \spadesuit) \in \mathbb{N}^* \times \{\mathbb{E}_1, \bigstar_1\}.$$

Thus, again by loc. cit., for all $i \in \mathbb{N}^*$ we have

$$\mathrm{Ext}^i_{\mathbb{E}_1}(N, \bigstar_1) = \mathrm{Ext}^i_{k[F]}(N, \mathrm{Hom}_{\mathbb{E}_1}(k[F], \bigstar_1)). \tag{24}$$

As $k[F]$ is a (non-commutative) principal ideal domain, from Baer criterion we get that the injective $k[F]$-modules are the divisible ones. Note that

$$\mathrm{Hom}_{\mathbb{E}_1}(k[F], \bigstar_1) = \{x \in \bigstar_1 | Vx = 0\} = F\bigstar_1 \in \{k(F), k(F)/Fk[F]\} \tag{25}$$

is a divisible $k[F]$-module. Based on the last two sentences, from Equation (24) and (25) we get the identity $\mathrm{Ext}^i_{\mathbb{E}_1}(N, \bigstar_1) = 0$. Taking $i = 1$ we conclude that part (a) holds.

Based on part (a), the injective hall of, say, the left $\mathbb{E}_m$-module $\mathbb{E}_m$ is a left $\mathbb{E}_m$-submodule of $\mathbb{K}_m$ which contains $\mathbb{E}_m$ and which has a direct supplement in $\mathbb{K}_m$. As each non-zero left $\mathbb{E}_m$-submodule of $\mathbb{K}_m$ intersects $\mathbb{E}_m$ non-trivially, this direct supplement is 0, hence part (b) holds.

For part (c), we first remark that we have $\mathrm{Ext}^1_{\mathbb{E}_m}(N, \mathbb{K}_m) = 0$ (see part (a)). Therefore we have an exact complex of left $\mathbb{E}_m$-modules

$$\mathrm{Hom}_{\mathbb{E}_m}(N, \mathbb{K}_m) \to \mathrm{Hom}_{\mathbb{E}_m}(N, \mathbb{K}_m/\mathbb{E}_m) \to \mathrm{Ext}^1_{\mathbb{E}_m}(N, \mathbb{E}_m) \to 0.$$

As N is of finite length we have $\mathrm{Hom}_{\mathbb{E}_m}(N, \mathbb{K}_m) = 0$. Thus the functorial $\mathbb{E}_m$-linear map $\mathrm{Hom}_{\mathbb{E}_m}(N, \mathbb{K}_m/\mathbb{E}_m) \to \mathrm{Ext}^1_{\mathbb{E}_m}(N, \mathbb{E}_m)$ is an isomorphism. $\square$

5 Projective Resolutions

In Subsection 5.1 we present a criterion of when a flat finitely generated left $\mathbb{J}_m$-module is free. Subsection 5.2 characterizes truncated BTs over k in terms of tor and projective dimensions. Subsection 5.3 introduces the main semisimple rings associated to BT_ms over k which are left $\mathbb{B}$-modules and checks that fully projectively balanced implies projectively balanced.

5.1 Flat finitely generated $\mathbb{J}_m$-modules

Let Λ be a left and right possibly non-commutative principal ideal domain. The multiplicative set $\Lambda^\times := \Lambda\backslash\{0\}$ satisfies the left and right Ore conditions (this simple fact also follows from the general result [Di], Thm. 3.6.12) and we have a division ring of fractions $\mathrm{Frac}(\Lambda) := (\Lambda^\times)^{-1}\Lambda = \Lambda(\Lambda^\times)^{-1}$. As in the commutative case, a left Λ-module Θ is flat if and only if for all $x \in \Lambda^\times$ we have $\mathrm{Tor}_1^\Lambda(\Lambda/x\Lambda, \Theta) = 0$, i.e., Θ is torsion free (equivalently, the natural Λ-linear map $\Theta \to \mathrm{Frac}(\Lambda) \otimes_\Lambda \Theta$ is injective). Moreover, if Θ is flat and finitely generated of rank $r \in \mathbb{N}$, then there exists $w \in \Lambda^\times$ such that we have $\Theta \subset \mathrm{Frac}(\Lambda) \otimes_\Lambda \Theta = \mathrm{Frac}(\Lambda)^r \supset \Lambda^r \supset \Theta w$ and it is well-known that Θ is a free left Λ-module. Recall that each Λ-submodule of a free (left or right) Λ-module is free (see [Bo1], Ch. VII, Sect. 3, Thm. 1).

By loc. cit. and the standard Morita equivalence between the categories of Λ-modules and $\mathrm{M}_s(\Lambda)$-modules with $s \in \mathbb{N}^*$, each left ideal of $\mathrm{M}_s(\Lambda)$ is isomorphic to the projective left $\mathrm{M}_s(\Lambda)$-module $\mathrm{J}_s(\Lambda)^d$ for some $d \in \{0, \ldots, s\}$, a submodule of a free left $\mathrm{M}_s(\Lambda)$-module of finite rank is isomorphic to $\mathrm{J}_s(\Lambda)^d$ for some $d \in \mathbb{N}$ and hence it is free if and only if s divides d (if free, its rank is d/s), and every left $\mathrm{M}_s(\Lambda)$-module has projective dimension at most 1.

We will extend these properties to rings

$$\mathbb{X} = \mathbb{X}(k, q) := k[F] \otimes_{\mathbb{F}_p} \mathbb{F}_{p^q} \quad \text{and} \quad \mathbb{V} = \mathbb{V}(k, q) := k[V] \otimes_{\mathbb{F}_p} \mathbb{F}_{p^q}$$

which are not integral domains in general (here $q \in \mathbb{N}^*$).

We consider the $W_m(k)$-algebra

$$\mathbb{Y}_m = \mathbb{Y}_m(k, q) := W_m(k) \otimes_{W_m(\mathbb{F}_p)} W_m(\mathbb{F}_{p^q}).$$

If $\mathbb{B} = \mathrm{M}_s(W(\mathbb{F}_{p^q}))$, then $\mathbb{Y}_m$ is a $W_m(k)$-subalgebra of $\mathbb{J}_m$. The cyclic group $\{\sigma^\ell | \ell \in \mathbb{Z}\}$ acts transitively on the set

$$\mathcal{I} = \mathcal{I}_{k,q}$$

of indecomposable idempotents of $\mathbb{Y}_m$: if $e \in \mathcal{I}$, then $\sigma(e) := (\sigma \otimes 1_{W_m(\mathbb{F}_{p^q})})(e)$. Embedding $\mathbb{F}_{p^q}$ into $\bar{k}$, let $l_1 := k \cap \mathbb{F}_{p^q}$: so $l_1 = \mathbb{F}_{p^{q_1}}$ with $q_1 \in \mathbb{N}$ a divisor of q. Then $\mathcal{I}$ can be identified with the set of indecomposable idempotents of $l_1 \otimes_{\mathbb{F}_p} l_1$ and thus has q_1 elements. For $e \in \mathcal{I}$, let $\bar{e} \in \mathbb{Y}_1$ be the reduction of e modulo p. Thus $k_e := \mathbb{Y}_1 \bar{e}$ is a finite Galois extension of k of degree $\frac{q}{q_1}$ which up to isomorphism does not depend on e. We have a product decomposition

$$\mathbb{Y}_1 = k \otimes_{\mathbb{F}_p} \mathbb{F}_{p^q} = \prod_{e \in \mathcal{I}} k_e \tag{26}$$

of rings and product decompositions into left ideals

$$\mathbb{Y}_m = \bigoplus\nolimits_{e \in \mathcal{I}} \mathbb{Y}_m e,$$
$$\mathbb{X} = k[F] \otimes_{\mathbb{F}_p} \mathbb{F}_{p^q} = \bigoplus\nolimits_{e \in \mathcal{I}} \mathbb{X}\bar{e} \quad \text{and} \quad \mathbb{V} = k[V] \otimes_{\mathbb{F}_p} \mathbb{F}_{p^q} = \bigoplus\nolimits_{e \in \mathcal{I}} \mathbb{V}\bar{e}.$$

Lemma 5. *The following four properties hold:*

(a) *Let $e \in \mathcal{I}$. Then each non-zero submodule $\mathbb{P}$ of the left $\mathbb{X}$-module $\mathbb{X}\bar{e}$ is isomorphic to $\mathbb{X}\bar{e}'$ for some $e' \in \mathcal{I}$.*

(b) *Each $\mathbb{X}$-submodule $\mathbb{O}$ of a finite direct sum $\bigoplus_{i=1}^{j} \mathbb{X}\bar{e}_i$ with each $e_i \in \mathcal{I}$ (equivalently, of a free $\mathbb{X}$-module of finite rank) is projective and isomorphic to a finite direct sum $\bigoplus_{e\in\mathcal{I}}(\mathbb{X}\bar{e})^{d_e}$, where each $d_e \in \mathbb{N}$ is uniquely determined.*

(c) *Referring to (b), $\mathbb{O}$ is a free left $\mathbb{X}$-module if and only if d_e is independent of $e \in \mathcal{I}$ and in such a case it is free of rank d_e.*

(d) *Parts (a) to (c) hold with $\mathbb{X}$ replaced by $\mathbb{V}$.*

Proof: We have $\mathbb{X}\bar{e} = \oplus_{i=0}^{\infty} F^i k_e$. Let $z = \sum_{i=0}^{d} F^i\alpha_i \in \mathbb{P}\setminus\{0\}$ of the smallest degree d in F; we have $\alpha_i \in k_e$. Let $e' := \sigma^d(e) \in \mathcal{I}$. As $F^i\alpha_i = \sigma^i(\bar{e})F^i\alpha_i$, the degree of $z - \bar{e}'z$ is less than d. Thus $z = \bar{e}'z$ and each $i \in \mathbb{N}$ such that $\alpha_i \neq 0$ is congruent to d modulo q. The right multiplication by z is a $\mathbb{X}$-linear map $\mathbb{X}\bar{e}' \to \mathbb{X}\bar{e}$, which by passage to associated graded is seen to be injective and its image $\mathbb{X}z$ is a direct supplement of the degree less than d part of $\mathbb{X}\bar{e}$, which intersects $\mathbb{P}$ trivially. Thus $\mathbb{P} = \mathbb{X}z \simeq \mathbb{X}\bar{e}'$ and part (a) holds. Using filtrations, the first part of (b) follows from (a) by induction on $j \in \mathbb{N}$ while $d_e = \dim_{k_e}(k_e \otimes_{\mathbb{X}} \mathbb{O})$ is uniquely determined. Part (c) follows from part (b). Part (d) is proved similarly to parts (a) to (c). $\square$

As $\mathrm{End}_{\mathbb{X}}(\mathbb{X}\bar{e}) = (\bar{e}\mathbb{X}\bar{e})^{\mathrm{op}} \supset ek_e = k_e$, the left $\mathbb{X}$-module $(\mathbb{X}\bar{e})^{d_e}$ is isomorphic to $X_e \otimes_{k_e} \mathbb{X}\bar{e}$, where X_e is a k_e-vector space of dimension d_e. From this and its analogue for $\mathbb{V}$, based on Lemma 5 (b) and (d) we get:

Corollary 3. *Every finitely generated projective left $\mathbb{X}$-module is isomorphic to a direct sum $\bigoplus_{e\in\mathcal{I}} X_e \otimes_{k_e} \mathbb{X}\bar{e}$ and every finitely generated projective left $\mathbb{V}$-module is isomorphic to a direct sum $\bigoplus_{e\in\mathcal{I}} V_e \otimes_{k_e} \mathbb{V}\bar{e}$, where each X_e and V_e are finite dimensional k_e-vector spaces.*

Proposition 2. *If $\mathbb{B} = \mathrm{M}_s(W(\mathbb{F}_{p^q}))$ with $s, q \in \mathbb{N}^*$ is connected, then the following three properties hold for a flat finitely generated left $\mathbb{J}_m$-module O:*

(a) *The left $\mathbb{J}_m$-module O is isomorphic to a finite direct sum of indecomposable projective left $\mathbb{J}_m$-modules*

$$\bigoplus_{e\in\mathcal{I}} (\mathrm{J}_s(\mathbb{E}_m \otimes_{W_m(k)} \mathbb{Y}_m e))^{d_e}$$

with each $d_e \in \mathbb{N}$ uniquely determined.

(b) *The left $\mathbb{J}_m$-module O is free if and only if d_e is independent of $e \in \mathcal{I}$.*

(c) *The left $\mathbb{E}_m$-module O is free of rank $r_O = sd_O$, where*

$$d_O := \frac{q}{q_1}(\sum_{e\in\mathcal{I}} d_e) \in \mathbb{N}.$$

Proof: As $\mathbb{E}_m$ is left and right noetherian, so is $\mathbb{J}_m$ and hence the left $\mathbb{J}_m$-module O is finitely presented. From [L], Cor. 1.4. we get that O is a projective left $\mathbb{J}_m$-module. As the left $\mathbb{E}_m$-module $\mathbb{J}_m$ is free of finite rank, the left $\mathbb{E}_m$-module O is also projective and finitely generated.[9] Thus we can assume that $m = 1$. Based on the standard Morita equivalence between the categories of left $\mathbb{E}_m \otimes_{\mathbb{Z}_p} W_m(\mathbb{F}_{p^q})$-modules and $\mathbb{J}_m$-modules, we can assume also that $s = 1$, i.e., $\mathbb{B} = W(\mathbb{F}_{p^q})$ and $\mathbb{J}_m = \mathbb{E}_m \otimes_{\mathbb{Z}_p} W_m(\mathbb{F}_{p^q}) = \mathbb{E}_m \otimes_{W_m(k)} \mathbb{Y}_m$.

Tensoring the short exact sequence $0 \to \mathbb{E}_1 \xrightarrow{(\pi_F,\pi_V)} k[F] \times k[V] \to k \to 0$ we get a short exact sequence

$$0 \to \mathbb{J}_1 \xrightarrow{(\pi_F\otimes 1_{\mathbb{F}_{p^q}},\pi_V\otimes 1_{\mathbb{F}_{p^q}})} \mathbb{X} \oplus \mathbb{V} \to \mathbb{Y}_1 \to 0.$$

We consider direct sum decompositions $\mathbb{X} \otimes_{\mathbb{J}_1} O = \bigoplus_{e\in\mathcal{I}} X_e \otimes_{k_e} \mathbb{X}\bar{e}$ and $\mathbb{V} \otimes_{\mathbb{J}_1} O = \bigoplus_{e\in\mathcal{I}} V_e \otimes_{k_e} \mathbb{V}\bar{e}$ (see Corollary 3) and over $\mathbb{Y}_1 = \prod_{e\in\mathcal{I}} k_e$ we get k_e-linear isomorphisms $X_e \simeq V_e$. Let $d_e := \dim_{k_e}(X_e) = \dim_{k_e}(V_e)$. Let $P := \bigoplus_{e\in\mathcal{I}}(\mathbb{E}_m \otimes_{W_m(k)} \mathbb{Y}_m e)^{d_e}$. We get the existence of linear isomorphisms $\mathbb{X} \otimes_{\mathbb{J}_1} P \simeq \mathbb{X} \otimes_{\mathbb{J}_1} O$ and $\mathbb{V} \otimes_{\mathbb{J}_1} P \simeq \mathbb{V} \otimes_{\mathbb{J}_1} O$ and they, due to the existence of the k_e-linear isomorphisms $X_e \simeq V_e$, can be changed so that they coincide over $\mathbb{Y}_1$ and hence, as the inclusion $\mathbb{Y}_1 \subset \mathbb{J}_1$ is a section of the epimorphism $\mathbb{J}_1 \to \mathbb{Y}_1$, they glue together to define a $\mathbb{J}_1$-linear isomorphism $P \simeq O$.

As $d_e = \dim_{k_e}(k_e \otimes_{\mathbb{J}_m} O)$, d_e is uniquely determined. Thus part (a) holds. As $\mathbb{J}_m = \bigoplus_{e\in\mathcal{I}} \mathbb{J}_m e = \bigoplus_{e\in\mathcal{I}} \mathbb{E}_m \otimes_{W_m(k)} \mathbb{Y}_m e$, part (b) follows from part (a).

As $s = 1$ and $[k_e : k] = \frac{q}{q_1}$, part (c) follows from the computation

$$d_O = r_O = \dim_k(\mathbb{Y}_1 \otimes_{\mathbb{J}_m} O) = \sum_{e\in\mathcal{I}} \dim_k(k_e \otimes_{\mathbb{J}_m} O) = \sum_{e\in\mathcal{I}} [k : k_e] d_e. \qquad \square$$

Corollary 4. *The following three properties hold:*

[9] A left $\mathbb{J}_m$-module N is flat (or projective or finitely generated) if and only if as a left $\mathbb{E}_m$-module it is so. The if part follows from the fact that N is a direct summand of $\mathbb{J}_m \otimes_{\mathbb{E}_m} N$, namely $\mathbb{B} \otimes_{\mathbb{Z}_p} \mathbb{B}^{\mathrm{op}}$ acts on $\mathbb{J}_m \otimes_{\mathbb{E}_m} N$ (right multiplication on $\mathbb{J}_m$) and the image of the action of a separability idempotent is canonically N.

(a) *The Grothendieck group $K_0(\mathbb{J}_m)$ is a free abelian group of finite rank on the set of isomorphism classes of indecomposable finitely generated projective left $\mathbb{J}_m$-modules. More precisely, if $\mathbb{B} = \mathrm{M}_s(W(\mathbb{F}_{p^q}))$ with $s, q \in \mathbb{N}^*$ is connected, then $K_0(\mathbb{J}_m)$ is the free abelian group on*

$$\{[\mathrm{J}_s(\mathbb{E}_m \otimes_{W_m(k)} \mathbb{Y}_m e)] | e \in \mathcal{I}\}.$$

(b) *We have a well-defined epimorphism called indecomposable index*

$$\mathrm{index} : K_0(\mathbb{J}_m) \to \mathbb{Z}$$

which maps the isomorphism class of each indecomposable finitely generated projective left $\mathbb{J}_m$-module to 1.

(c) *Let k_1 be a perfect field that contains k. Then the functorial homomorphism $K_0(k/k_1) : K_0(\mathbb{J}_m) \to K_0(\mathbb{J}_m(k_1))$ induced by the inclusion $\mathbb{J}_m = \mathbb{E}_m \otimes_{W_m(\mathbb{F}_p)} \mathbb{B}_m^{\mathrm{op}} \to \mathbb{E}_m(k_1) \otimes_{W_m(\mathbb{F}_p)} \mathbb{B}_m^{\mathrm{op}} = \mathbb{J}_m(k_1)$ is injective (thus a BT_m G over k endowed with a left $\mathbb{B}$-action is projectively balanced if and only if G_{k_1} is so).*

Proof: Parts (a) and (b) follow directly from Proposition 2 (a). If $k = k_e$ (i.e., $q_1 = q$ or $\mathbb{F}_{p^q} \subset k$), then part (c) also follows from part (a). Based on this, to check that part (c) holds, we can assume that k_1 is a finite field extension of k. As the left or right $\mathbb{E}_m(k)$-module $\mathbb{E}_m(k_1)$ is free of rank $[k_1 : k]$, we have a restriction of scalars homomorphism

$$K_0(k_1/k) : K_0(\mathbb{J}_m(k_1)) \to K_0(\mathbb{J}_m)$$

such that $K_0(k_1/k) \circ K_0(k/k_1) = [k_1 : k] 1_{K_0(\mathbb{J}_m)}$. From this and part (a) we get that $K_0(k/k_1)$ is injective. □

5.2 Characterizations of BT_ms over k

Proposition 3. *Let N be a left $\mathbb{E}_m$-module of finite length. Let $r \in \mathbb{N}$ be the minimum number of generators of N as a $W_m(k)$-module (so $r = 0$ if and only if $N = 0$). Then the following three statements are equivalent for N:*

① *It is isomorphic to the (contravariant) Dieudonné module of a BT_m over k of height r.*

② *It has a free resolution $0 \to \mathbb{E}_m^r \to \mathbb{E}_m^r \to N \to 0$.*

③ *It has finite tor dimension.*

Proof: We first check that ① $\Rightarrow$ ②. The $W_m(k)$-module N is free of rank r and there exists $c \in \{0, 1, \ldots, r\}$ (called the codimension of N) and a $W_m(k)$-basis $\{v_1, \ldots, v_r\}$ of N such that by defining

$$(\epsilon_1, \ldots, \epsilon_r) := (v_1, \ldots, v_c, Vv_{c+1}, \ldots, Vv_r)$$

and

$$(\varepsilon_1, \ldots, \varepsilon_r) := (Fv_1, \ldots, Fv_c, v_{c+1}, \ldots, v_r),$$

$\{\epsilon_1, \ldots, \epsilon_r\}$ and $\{\varepsilon_1, \ldots, \varepsilon_r\}$ are $W_m(k)$-bases of N (see [I], proof (b) of Prop. 1.7). Let $g = (g_{i,j})_{1\le i,j\le r} \in \mathbf{GL}_r(W(k))$ be the invertible matrix such that for each integer $i \in \{1, 2, \ldots, r\}$ we have an equality $\epsilon_i = \sum_{j=1}^r g_{ij}\varepsilon_j$.

Let $\{v'_1, \ldots, v'_r\}$ be the standard basis of the left $\mathbb{E}_m$-module $\mathbb{E}_m^r$. Let $\delta : \mathbb{E}_m^r \to N$ be the $\mathbb{E}_m$-linear epimorphism such that for all $i \in \{1, 2, \ldots, r\}$ we have $\delta(v'_i) = v_i$. Let

$$(\zeta_1, \ldots, \zeta_r) := (v'_1, \ldots, v'_c, Vv'_{c+1}, \ldots, Vv'_r),$$

$$(\xi_1, \ldots, \xi_r) := (Fv'_1, \ldots, Fv'_c, v'_{c+1}, \ldots, v'_r).$$

Thus for each $i \in \{1, 2, \ldots, r\}$ we have $\delta(\zeta_i) = \epsilon_i$ and $\delta(\xi_i) = \varepsilon_i$.

Let $\eta : \mathbb{E}_m^r \to \mathbb{E}_m^r$ be the $\mathbb{E}_m$-linear endomorphism such that for all $i \in \{1, 2, \ldots, r\}$ we have $\eta(v'_i) = \zeta_i - \sum_{j=1}^r g_{ij}\xi_j$. We check that the complex

$$0 \to \mathbb{E}_m^r \xrightarrow{\eta} \mathbb{E}_m^r \xrightarrow{\delta} N \to 0 \tag{27}$$

of left $\mathbb{E}_m$-modules is in fact a short exact sequence.

We first check that the induced $\mathbb{E}_m$-linear epimorphism $\omega : \mathrm{Coker}(\eta) \to N$ is an isomorphism. We have two $W_m(k)$-linear sections $N \to \mathrm{Coker}(\omega)$ which for each $i \in \{1, 2, \ldots, r\}$ map ϵ_i to $\zeta_i + \mathrm{Im}(\eta)$ and respectively ε_i to $\xi_i + \mathrm{Im}(\eta)$, they coincide and they commute with F and V as it is easy to see what F does on the elements ϵ_i and ζ_i and V does on the elements ε_i and ξ_i. As $\{\zeta_1, \xi_1, \ldots, \zeta_r, \xi_r\}$ generates $\mathbb{E}_m^r$, the section $N \to \mathrm{Coker}(\omega)$ is onto. Thus ω is an isomorphism.

We are left to check that η is injective. We can assume that $m = 1$, $k = \bar{k}$ and it suffices to show that $1_{\mathbb{K}_1} \otimes \eta$ is injective. We write $1_{\mathbb{K}_1} \otimes \eta = \eta_F \oplus \eta_V$, where $\eta_F : k(F)^r \to k(F)^r$ is $k(F)$-linear and $\eta_V : k(V)^r \to k(V)^r$ is $k(V)$-linear. To check that η_F and η_V are injective it suffices to show that their cokernels are 0, i.e., (see prior paragraph) we have $\mathbb{K}_1 \otimes_{\mathbb{E}_1} N = 0$. But this is so as an object of $\mathcal{F}_{k,m}$ has a composition series whose factors (as $k = \bar{k}$) are isomorphic to $\boldsymbol{\alpha}_p$, $\boldsymbol{\mu}_{p,k}$ or $(\mathbb{Z}/p\mathbb{Z})_k$ and thus have (contravariant) Dieudonné modules which are one dimensional k-vector spaces generated by an element annihilated by $F+V$, $V-1$ or $F-1$ (respectively) which belong to $\mathbb{S}_1$.

As the complex (27) is a short exact sequence we get that ② holds.

Clearly, ② ⇒ ③. We now proof that ③ ⇒ ①. Let $\ell \in \mathbb{N}^*$ be such that the left $\mathbb{E}_m$-module N of finite length has tor dimension at most ℓ. As we have $\mathrm{Tor}_n(\mathbb{E}_m/p\mathbb{E}_m, N) = 0$ for $n > \ell$, by tensoring the periodic free resolution

$$\cdots \xrightarrow{p} \mathbb{E}_m \xrightarrow{p^{m-1}} \mathbb{E}_m \xrightarrow{p} \mathbb{E}_m/p\mathbb{E}_m \to 0$$

of the right $\mathbb{E}_m$-module $\mathbb{E}_m/p\mathbb{E}_m$ we get an exact complex

$$N \xrightarrow{p^{m-1}} N \xrightarrow{p} N.$$

Thus each cyclic $W_m(k)$-module which is a direct summand of N is isomorphic to $W_m(k)$. Hence the finitely generated $W_m(k)$-module N is free, its rank being r.

As the left $\mathbb{E}_m$-module N has tor dimension at most ℓ and is free as a $W_m(k)$-module, for $n > \ell$ we have $\mathrm{Tor}_n^{\mathbb{E}_1}(O, N/pN) = \mathrm{Tor}_n^{\mathbb{E}_m}(O, N) = 0$ for each right $\mathbb{E}_1$-module O (see [CE], Ch. VI, Prop. 4.1.2 or Ch. XVI, Sect. 5). Thus the left $\mathbb{E}_1$-module N/pN has finite tor dimension at most ℓ. An argument as in the previous paragraph but using the free resolution

$$\cdots \xrightarrow{F} \mathbb{E}_1 \xrightarrow{V} \mathbb{E}_1 \xrightarrow{F} \mathbb{E}_1/F\mathbb{E}_1 \to 0$$

of the right $\mathbb{E}_1$-module $\mathbb{E}_1/F\mathbb{E}_1$, shows that we have an exact complex

$$N/pN \xrightarrow{V} N/pN \xrightarrow{F} N/pN.$$

Thus N/pN corresponds to a BT_1 over k. From this and the previous paragraph we get that N corresponds to a BT_m over k of height r. Thus ③ ⇒ ①.□

Proposition 4. *Let N be a left $\mathbb{J}_m$-module of finite length and let $j \in \mathbb{N}$ be its minimum number of generators (so $j = 0$ if and only if $N = 0$). Then the following three statements are equivalent for N:*

① *It is isomorphic to the (contravariant) Dieudonné $\mathbb{J}_m$-module of a BT_m over k endowed with a left $\mathbb{B}$-action.*

② *It has a free resolution $0 \to P_1 \to \mathbb{J}_m^j \to N \to 0$ with P_1 a projective left $\mathbb{J}_m$-module which as a left $\mathbb{E}_m$-module is free of the same rank as $\mathbb{J}_m^j$ (and thus the class $[N] := [\mathbb{J}_m^j] - [P_1] \in K_0(\mathbb{J}_m)$ is well-defined).*

③ *It has finite tor dimension.*

Moreover, if one of these equivalent statements holds, then for all $\ell \in \mathbb{N}^*$ there exists a (contravariant) Dieudonné $\mathbb{J}_{m+\ell}$-module N' of a $\mathrm{BT}_{m+\ell}$ over k endowed with a left $\mathbb{B}$-action such that $N = N'/p^m N'$.

Proof: We prove that ① implies first that N has tor dimension at most 1 (thus it also implies ③) and second that ② holds. We can assume that $\mathbb{B}$ is connected. As N is a free $W_m(k)$-module, as in the proof of Proposition 3 we argue based on [CE], Ch. VI, Prop. 4.1.2 that we can assume that $m = 1$, and we have to show that for each right $\mathbb{J}_1$-module O we have $\mathrm{Tor}_i^{\mathbb{J}_1}(O, N) = 0$ for all integers $i \geq 2$. As in the proof of Lemma 4 (a) we argue that we can assume that either $F = 0$ on O or $V = 0$ on $O = 0$. Say, $OV = 0$, i.e., O is a right $k[F] \otimes_{\mathbb{E}_1} \mathbb{J}_1$-module. Using the standard free resolution of the right $\mathbb{E}_1$-module $k[F]$ (it is similar to (23)) tensored with the flat left $\mathbb{E}_1$-module $\mathbb{J}_1$ and the assumption that N is the (contravariant) Dieudonné module of a BT_1 over k, we get that $\mathrm{Tor}_i^{\mathbb{J}_1}(k[F] \otimes_{\mathbb{E}_1} \mathbb{J}_1, N) = 0$ for $i \in \mathbb{N}^*$. From this and [CE], Ch. VI, Prop. 4.1.2 we get that we have $\mathrm{Tor}_i^{\mathbb{J}_1}(O, N) = \mathrm{Tor}_i^{k[F]\otimes_{\mathbb{E}_1}\mathbb{J}_1}(O, k[F] \otimes_{\mathbb{E}_1} N)$ for all integers $i \geq 2$. As $k[F] \otimes_{\mathbb{E}_1} \mathbb{J}_1 \simeq \mathrm{M}_s(k[F] \otimes_{\mathbb{F}_p} \mathbb{F}_{p^q})$, the left $k[F] \otimes_{\mathbb{E}_1} \mathbb{J}_1$-module $k[F] \otimes_{\mathbb{E}_1} N$ has projective dimensions at most 1 (by the standard Morita equivalence based on which we can assume $s = 1$ and for $s = 1$ see Lemma 5 (b)), and thus we have $\mathrm{Tor}_i^{k[F]\otimes_{\mathbb{E}_1}\mathbb{J}_1}(O, k[F] \otimes_{\mathbb{E}_1} N) = 0$ for all integers $i \geq 2$. We conclude that $\mathrm{Tor}_i^{\mathbb{J}_1}(O, N) = 0$ for all integers $i \geq 2$. Hence ① implies that N has tor dimension at most 1. Thus the kernel P_1 of a $\mathbb{J}_m$-linear epimorphism $\delta : \mathbb{J}_m^j \to N$ is flat. Being also finitely generated (as $\mathbb{J}_m$ is left and right noetherian), P_1 is actually a projective left $\mathbb{J}_m$-module (see Proposition 2 (b)). From Propositions 3 and 2 (a) we get that the left $\mathbb{E}_m$-module P_1 is free of the same rank as $\mathbb{J}_m^j$. Thus ① $\Rightarrow$ ②.

The implications ② $\Rightarrow$ ③ $\Rightarrow$ ① remain as in the proof of Proposition 3.

To check the last part of the proposition, we lift the projective resolution $0 \to P_1 \to \mathbb{J}_m^g \to N \to 0$ to a free resolution $0 \to P_{1,\ell} \to \mathbb{J}_{m+\ell}^g \to N' \to 0$ with $P_{1,\ell}$ as the unique (up to isomorphism) projective left $\mathbb{J}_{m+\ell}$-module whose reduction modulo p^m is P_1. From the snake lemma applied to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_{1,\ell} & \longrightarrow & \mathbb{J}_{m+\ell} & \longrightarrow & N' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p^m} & & \downarrow{\scriptstyle p^m} & & \downarrow{\scriptstyle p^m} & & \\ 0 & \longrightarrow & P_{1,\ell} & \longrightarrow & \mathbb{J}_{m+\ell} & \longrightarrow & N' & \longrightarrow & 0 \end{array}$$

we get that $N = N'/p^m N'$. Thus, as ② ⇒ ①, N' satisfies all requirements.□

Example 7. *If $\mathbb{B} = \mathrm{M}_s(W(\mathbb{F}_{p^q}))$ is connected, then for each indecomposable finitely generated projective left $\mathbb{J}_m$-module P, its rank as a free $\mathbb{E}_m$-module is $\frac{sq}{q_1}$ (see Proposition 2 (c)); we recall that $\mathbb{F}_{p^{q_1}} = k \cap \mathbb{F}_{p^q}$. From this and Proposition 2 we get that if N is such that the equivalent statements of Proposition 4 hold, then $[N] \in K_0(\mathbb{J}_m)$ has indecomposable index* 0 *(see Corollary 4 (b)). Therefore, if $\mathcal{I}$ has one element (i.e., k and $\mathbb{F}_{p^q}$ are linearly disjoint over $\mathbb{F}_p$), then* index : $K_0(\mathbb{J}_m) \to \mathbb{Z}$ *is an isomorphism and hence each object H of $\mathcal{F}^{\mathbb{B}}_{k,m}$ with $\mathbb{D}^{\mathbb{B}}(H) = N$ is projectively balanced.*

Example 8. *We assume that $\mathbb{B} = W(\mathbb{F}_{p^q})$ is connected and commutative and that $q_1 \geq 2$. Let $c, d \in \mathbb{N}^*$ be such that $q_1 \mid (c+d)$ but $q_1 \nmid c$. Let $e_0 \in \mathcal{I}$ be the indecomposable idempotent of $\mathbb{Y}_m$ which defines the epimorphism $k \otimes_{\mathbb{F}_p} \mathbb{F}_{p^{q_1}} \to k$ that maps $a \otimes b$ to ab for all $(a,b) \in k \times \mathbb{F}_{p^{q_1}}$. For $e = \sigma^\ell(e_0) \in \mathcal{I}$ with $\ell \in \{0, \ldots, q_1 - 1\}$, the left $\mathbb{E}_m$-module*

$$M_e := \mathbb{E}_m/(F^c - V^d) \otimes_{\sigma^\ell, W_m(\mathbb{F}_{p^{q_1}})} W_m(\mathbb{F}_{p^q})$$

is a left $\mathbb{J}_m$-module with $j = 1$. We have a short exact sequence

$$0 \to \mathbb{J}_m \sigma^c(e) \xrightarrow{F^c - V^d} \mathbb{J}_m e \to M_e \to 0$$

of left $\mathbb{J}_m$-modules. Thus we have a non-zero element (see Corollary 4 (a))

$$[M_e] = [\mathbb{E}_m \otimes_{W_m(k)} \mathbb{Y}_m e] - [\mathbb{E}_m \otimes_{W_m(k)} \mathbb{Y}_m \sigma^c(e)] \in K_0(\mathbb{J}_m) \simeq \mathbb{Z}^{\mathcal{I}} \simeq \mathbb{Z}^{q_1}$$

and hence no complex $0 \to \mathbb{J}_m \to \mathbb{J}_m \to M_e \to 0$ is a short exact sequence. Thus, in Proposition 4, P_1 is not always free.

5.3 Projectively balanced objects

For a ring A let $J(A)$ be its Jacobson radical and let $\Sigma(A) := A/J(A)$.

For an object G of $\mathcal{F}^{\mathbb{B}}_{k,m}$, the ring scheme $\mathbf{End}(G)$ has $\mathbf{End}(G)^0$ as a two-sided ideal subscheme and the quotient ring scheme

$$\mathbf{C}_G := \mathbf{End}(G)/\mathbf{End}(G)^0 = \mathbf{End}(G)_{\mathrm{red}}/\mathbf{End}(G)^0_{\mathrm{red}}$$

is étale having

$$\mathbf{C}_{G,\mathbb{B}} := \mathbf{End}_{\mathbb{B}}(G)/[\mathbf{End}(G)^0 \cap \mathbf{End}_{\mathbb{B}}(G)]$$

as a subring scheme. We note that $\mathbb{B} \otimes_{\mathbb{Z}_p} \mathbb{B}^{\mathrm{op}}$ acts on the additive group scheme $\mathbf{End}(G)$ and $\mathbf{End}_{\mathbb{B}}(G)$ is the image of (the action of) a separability idempotent and in particular it is a direct factor of $\mathbf{End}(G)$; thus $\mathbf{End}(G)^0 \cap \mathbf{End}_{\mathbb{B}}(G)$ is the identity component of $\mathbf{End}_{\mathbb{B}}(G)$ and hence $\mathbf{C}_{G,\mathbb{B}}$ is the ring of connected components of $\mathbf{End}_{\mathbb{B}}(G)$.

We assume now that G is a BT_m. From [GV], Cor. 6 (b) we get that $\mathbf{Aut}(G)^0(\bar{k}) = 1_G + \mathbf{End}(G)^0(\bar{k})$. From this and the fact that $\mathbf{Aut}(G)^0_{\mathrm{red}}$ is a unipotent group scheme (see [GV], Thm. 5 (a) and Cor. 5), we get that $\mathbf{End}(G)^0(\bar{k})$ acts nilpotently on each $\bar{k}$-vector space which is a $\mathbf{Aut}(G_{\bar{k}})^0$-module, such as on $\bar{k} \otimes_k M/pM$ (we recall that $M = \mathbb{D}^{\mathbb{B}}(G)$). We deduce that $\mathbf{End}(G)^0(\bar{k})$ is a nilpotent two-sided ideal of $\mathbf{End}(G)(\bar{k}) = \mathrm{End}(G_{\bar{k}})$ and thus it is contained in $J(\mathrm{End}(G_{\bar{k}}))$. Thus the $\mathbb{F}_p$-algebra

$$\Sigma_G := \Sigma(\mathrm{End}(G_{\bar{k}})) = \Sigma(\mathrm{Im}(\mathrm{End}(G_{\bar{k}}) \to \mathrm{End}(G[p]_{\bar{k}}))) = \Sigma(\mathbf{C}_G(\bar{k}))$$

is a finite semisimple ring. Similarly, the $\mathbb{F}_p$-algebra

$$\Sigma_{G,\mathbb{B}} := \Sigma((\mathrm{End}_{\mathbb{B}}(G_{\bar{k}})) = \Sigma(\mathrm{Im}(\mathrm{End}_{\mathbb{B}}(G_{\bar{k}}) \to \mathrm{End}_{\mathbb{B}}(G[p]_{\bar{k}}))) = \Sigma(\mathbf{C}_{G,\mathbb{B}}(\bar{k}))$$

is a finite semisimple ring whose center $\mathbb{C}_{G,\mathbb{B}}$ contains the image $\overline{\mathbb{C}}_1$ of the center $\mathbb{C}_1$ of $\mathbb{B}_1$ in $\Sigma_{G,\mathbb{B}}$. We identify $\overline{\mathbb{C}}_1 = \mathrm{Im}(\mathbb{C}_1 \to \mathrm{End}_{\mathbb{B}}(G[p]_{\bar{k}}))$.

As $\Sigma_{G,\mathbb{B}}$ is absolutely semisimple, the epimorphism

$$\mathrm{Im}(\mathrm{End}_{\mathbb{B}}(G_{\bar{k}}) \to \mathrm{End}_{\mathbb{B}}(G[p]_{\bar{k}})) \to \Sigma_{G,\mathbb{B}}$$

has sections extending the inclusion $\overline{\mathbb{C}}_1 \subset \mathrm{Im}(\mathrm{End}_{\mathbb{B}}(G_{\bar{k}}) \to \mathrm{End}_{\mathbb{B}}(G[p]_{\bar{k}}))$ and all such sections are conjugate under inner automorphisms defined by the conjugation with elements in $1 + J(\mathrm{Im}(\mathrm{End}_{\mathbb{B}}(G_{\bar{k}}) \to \mathrm{End}_{\mathbb{B}}(G[p]_{\bar{k}})))$ (see Wedderburn's Principal Theorem of [Bo2], Sect. 13, Cor. 1 of Prop. 7). Thus $G[p]_{\bar{k}}$ has a unique up to isomorphism structure of a left $\mathbb{C}_{G,\mathbb{B}}$-module which extends its structure as a left $\overline{\mathbb{C}}_1$-module.

Corollary 5. *Let G be a BT_m over k endowed with a left $\mathbb{B}$-action. Then the following two properties hold.*

(a) *If G is fully projectively balanced, then it is also projectively balanced.*

(b) *Let k_1 be a perfect field that contains k. Then G is fully projectively balanced if and only if G_{k_1} is fully projectively balanced.*

Proof: Based on Corollary 4 (c) we can assume that k and k_1 are algebraically closed. We have a direct product decomposition $\mathbb{C}_1 = \overline{\mathbb{C}}_1 \times \overline{\mathbb{C}}_1'$ such that $\overline{\mathbb{C}}_1'$ acts trivially on $G[p]$. We assume that the class $[G] \in K_0(\mathbb{E}_1 \otimes_{\mathbb{F}_p} \mathbb{C}_{G,\mathbb{B}})$ is 0 and to prove part (a) we just have to show that the class

$$[G] \in K_0(\mathbb{I}_1) = K_0(\mathbb{E}_1 \otimes_{\mathbb{F}_p} \overline{\mathbb{C}}_1) \times K_0(\mathbb{E}_1 \otimes_{\mathbb{F}_p} \overline{\mathbb{C}}_1^{\perp})$$

is 0. As the component of $[G]$ in $K_0(\mathbb{E}_1 \otimes_{\mathbb{F}_p} \overline{\mathbb{C}}_1^{\perp})$ is 0, it suffices to show that the functorial homomorphism $K_0(\mathbb{E}_1 \otimes_{\mathbb{F}_p} \overline{\mathbb{C}}_1) \to K_0(\mathbb{E}_1 \otimes_{\mathbb{F}_p} \mathbb{C}_{G,\mathbb{B}})$ is injective. To check this we can assume that $\overline{\mathbb{C}}_1$ and $\mathbb{C}_{G,\mathbb{B}}$ are fields and in this case the injectivity is argued as in the proof of Corollary 4 (c) using the restriction of scalar homomorphism $K_0(\mathbb{E}_1 \otimes_{\mathbb{F}_p} \mathbb{C}_{G,\mathbb{B}}) \to K_0(\mathbb{E}_1 \otimes_{\mathbb{F}_p} \overline{\mathbb{C}}_1)$. Thus part (a) holds. Part (b) follows from the fact that $\mathbf{C}_{G,\mathbb{B}}(k) = \mathbf{C}_{G,\mathbb{B}}(k_1)$. $\square$

6 Proof of Theorem 2

To prove Theorem 2 we will use Sections 4 and 5, other homological properties of left $\mathbb{E}_m$-modules and the non-commutative duality over the Cartier–Dieudonné ring $\mathbb{E}_m$ mentioned in Section 1.

Finite parts of left $\mathbb{E}_m$-modules and this sort of non-commutative duality over $\mathbb{E}_m$ for them are presented in Subsection 6.1. The bilinear map (3) is defined in Subsection 6.2. The proof of Theorem 2 (a) is completed in Subsection 6.3. Parts (b), (c) and (d) of Theorem 2 are proved in Subsection 6.4. The section will end with remarks on generalizations (see Subsection 6.5), examples (see Subsection 6.6), and the proof of Proposition 1 (see Subsection 6.7).

6.1 Finite parts of left $\mathbb{E}_m$-modules

Definition 8. *The finite part of a left $\mathbb{E}_m$-module O is its $\mathbb{E}_m$-submodule*

$$\mathrm{Fin}(O) := \mathrm{Ker}(O \to \mathbb{K}_m \otimes_{\mathbb{E}_m} O) = \{x \in O | \exists s \in \mathbb{S}_m \ \text{ such that } \ sx = 0\}.$$

It is easy to see based on Section 4 that we have an identity

$$\mathrm{Fin}(O) = \{x \in O | \mathbb{E}_m x \,\text{is a finitely generated}\, W_m(k)\text{-module}\}.$$

We have 'development at infinity' homomorphisms to skew Laurent series rings

$$\theta_{m,F} : \mathbb{K}_m \to W_m(k)((F^{-1})) \quad \text{and} \quad \theta_{m,V} : \mathbb{K}_m \to W_m(k)((V^{-1}))$$

obtained by mapping V to pF^{-1} and F to pV^{-1} (respectively).

Let $\lambda_m : \mathbb{K}_m/\mathbb{E}_m \to W_m(k)$ be the $W_m(k)$-linear map

$$\lambda_m(f + \mathbb{E}_m) = (\text{constant term of } \theta_{m,F}(f)) - (\text{constant term of } \theta_{m,V}(f)).$$

We view $\mathbb{K}_m/\mathbb{E}_m$ as a $(\mathbb{E}_m, W_m(k))$-bimodule (left $\mathbb{E}_m$-module, right $W_m(k)$-module). Let $\mathrm{Hom}_{W_m(k)}(\mathbb{E}_m, W_m(k))$ be endowed with the structure of a $(\mathbb{E}_m, W_m(k))$-bimodule by the rule $(\alpha\delta y)(x) = \delta(x\alpha)y$, where $\alpha \in \mathbb{E}_m$, $y \in W_m(k)$, $\delta \in \mathrm{Hom}_{W_m(k)}(\mathbb{E}_m, W_m(k))$, and $x \in \mathbb{E}_m$. We define a map of $(\mathbb{E}_m, W_m(k))$-bimodules

$$\tau_m : \mathbb{K}_m/\mathbb{E}_m \to \mathrm{Hom}_{W_m(k)}(\mathbb{E}_m, W_m(k))$$

by the rule $\tau_m(\beta + \mathbb{E}_m)(\alpha) = \lambda_m(\alpha\beta + \mathbb{E}_m)$ with $\alpha \in \mathbb{E}_m$ and $\beta \in \mathbb{K}_m$.

Lemma 6. *The map τ_m of $(\mathbb{E}_m, W_m(k))$-bimodules is injective and its image is the finite part $\mathrm{Fin}(\mathrm{Hom}_{W_m(k)}(\mathbb{E}_m, W_m(k)))$ of $\mathrm{Hom}_{W_m(k)}(\mathbb{E}_m, W_m(k))$ (viewed as a left $\mathbb{E}_m$-module).*

Proof: It suffices to show that for each $z \in \mathbb{S}_m$, the restriction

$$\tau_{m,z} : \{x \in \mathbb{K}_m/\mathbb{E}_m | zx = 0\} \to \{x \in \mathrm{Hom}_{W_m(k)}(\mathbb{E}_m, W_m(k)) | zx = 0\}$$

of τ_m is a bijection of right $W_m(k)$-modules. As we can rewrite this restriction $\tau_{m,z} : z^{-1}\mathbb{E}_m/\mathbb{E}_m \to \mathrm{Hom}_{W_m(k)}(\mathbb{E}_m/\mathbb{E}_m z, W_m(k))$ as a $W_m(k)$-linear map between free right $W_m(k)$-modules of the same rank (see Lemma 3 (c)), it suffices to show that $\tau_{m,z}$ modulo p is an injective k-linear map. Thus to prove the lemma it suffices to show that τ_1 is injective; we can assume that $k = \bar{k}$ and it suffices to show that the restriction of τ_1 to each simple $\mathbb{E}_1$-submodule S of $\mathbb{K}_1/\mathbb{E}_1 = [k(F) \times k(V)]/\mathbb{E}_1$ is injective.

Let $y_0 := (F+V)^{-1}V + \mathbb{E}_1 = (0,1) + \mathbb{E}_1 = (-1,0) + \mathbb{E}_1$, $y_F := (F-1)^{-1} + \mathbb{E}_1$ and $y_V := (V-1)^{-1} + \mathbb{E}_1$; they are non-zero elements of $\mathbb{K}_1/\mathbb{E}_1$. We have $Fy_0 = Vy_0 = Vy_F = Fy_V = 0$ and $Fy_F = y_F$ and $Vy_V = y_V$. Thus $S_0 := ky_0 = y_0k$, $S_{F,[\alpha]} := ky_F\alpha$ and $S_{V,[\alpha]} := ky_V\alpha$ are simple left $\mathbb{E}_1$-modules, where $\alpha \in k$ is non-zero and $[\alpha] := \alpha + \mathbb{F}_p \in k/\mathbb{F}_p$.

We check that S is one of the above simple left $\mathbb{E}_1$-modules. Let

$$x = (x_1, x_2) \in k(F) \times k(V)$$

be such that S is generated by $y := x + \mathbb{E}_1$. If $Fy = Vy = 0$, then $y \in k(F)/Fk[F] \cap k(V)/Vk[V] = ky_0 = S_0$ (see Equation (25) and its analogue for V) and thus $S = S_0$. We assume now that either $Fy \neq 0$ or $Vy \neq 0$. To fix the ideas we can assume that $Fy \neq 0$. By replacing y with Fy we can assume that $Vy = 0$. Thus S is a simple $k[F] = \mathbb{E}_1/(V)$-module and hence (as $k = \bar{k}$) it is a 1-dimensional k-vector space. As $0 \neq Fy \in ky$, by replacing x with a multiple of it by a non-zero element of k we can assume that $Fy = y$. By replacing x with Fx we can assume that $x_2 = 0$. As $Fx - x = (Fx_1 - x_1, 0) \in \mathbb{E}_1$, we can write

$$Fx_1 - x_1 = \sum_{i=1}^{n} F^i \alpha_i = \sum_{i=1}^{n} \alpha_i + (F-1)\Big[\sum_{i=0}^{n-1} F^i \Big(\sum_{j=i+1}^{n} \alpha_j\Big)\Big],$$

where $n \in \mathbb{N}^*$ and $\alpha_1, \ldots, \alpha_n \in k$. By replacing x_1 with a translate of it by an element in $Fk[F]$ we can assume that $x_1 = (F-1)^{-1}\alpha + \alpha$, where $\alpha := \sum_{i=1}^{n} \alpha_i \in k$ is non-zero as $y \neq 0$. Thus $y = y_F \alpha$, i.e., $S = S_{F,[\alpha]}$.

As $(F-1)^{-1} = \sum_{n \geq 1} (F^{-1})^n$, we have $\theta_{1,F}((F-1)^{-1}) = 0$. Similarly, $\theta_{1,V}((V-1)^{-1}) = 0$. Therefore $1 = \lambda_1(y_F) = -\lambda_1(y_V) = -\lambda_1(y_0)$, i.e., τ_1 is non-trivial on all simple left $\mathbb{E}_1$-submodules of $\mathbb{K}_1/\mathbb{E}_1$. $\square$

6.2 The bilinear map $\langle \, , \rangle$

If O is a left $\mathbb{E}_m$-module, we endow

$$O^\vee := \mathrm{Hom}_{W_m(k)}(O, W_m(k))$$

with the left $\mathbb{E}_m$-module structure via the same rules as in the first paragraph of Subsection 3.1, and we endow

$$O^\# := \mathrm{Hom}_{\mathbb{E}_m}(O, \mathbb{E}_m)$$

with the left $\mathbb{E}_m$-module structure given by the rule $(\alpha\delta)(x) = \delta(x)\iota_m(\alpha)$ where $(\alpha, \delta, x) \in \mathbb{E}_m \times O^\# \times O$. Similarly, we endow $\mathrm{Ext}^1_{\mathbb{E}_m}(O, \mathbb{E}_m)$ with the left $\mathbb{E}_m$-module structure given by the rule $\alpha y = r_{\iota_m(\alpha)}(y)$ where $y \in \mathrm{Ext}^1_{\mathbb{E}_m}(O, \mathbb{E}_m)$ and $r_{\iota_m(\alpha)} : \mathrm{Ext}^1_{\mathbb{E}_m}(O, \mathbb{E}_m) \to \mathrm{Ext}^1_{\mathbb{E}_m}(O, \mathbb{E}_m)$ is defined by the right multiplication $\mathbb{E}_m \to \mathbb{E}_m$ by $\iota_m(\alpha)$.

If O is a left $\mathbb{J}_m$-module, then as $\alpha \in \mathbb{E}_m$ commutes with $b \in \mathbb{B}_m^{\mathrm{op}}$ and $\mathbb{J}_m$ is generated by $\mathbb{E}_m = \mathbb{E}_m \otimes_{W_m(\mathbb{F}_p)} W_m(\mathbb{F}_p)$ and $\mathbb{B}_m^{\mathrm{op}} = W_m(\mathbb{F}_p) \otimes_{W_m(\mathbb{F}_p)} \mathbb{B}_m^{\mathrm{op}}$, we get that $O^\vee$, $O^\#$ and $\mathrm{Ext}^1_{\mathbb{E}_m}(O, \mathbb{E}_m)$ are in fact left $\mathbb{J}_m^\vee$-modules, where $\mathbb{J}^\vee := \mathbb{E} \otimes_{\mathbb{Z}_p} \mathbb{B}$ and $\mathbb{J}_m^\vee := \mathbb{J}^\vee / p^m \mathbb{J}^\vee$. If P is a finitely generated projective left $\mathbb{J}_m$-module, then $P^\#$ is a finitely generated projective left $\mathbb{J}_m^\vee$-module.

We recall from Subsection 3.2 that if O is a finitely generated left $\mathbb{J}_m$-module and N is a left $\mathbb{J}_m$-module of finite length, then $\mathrm{Hom}_{\mathbb{J}_m}(O, N)$ has a natural structure of a commutative quasi-algebraic group $\underline{\mathrm{Hom}_{\mathbb{J}_m}(O, N)}$.

We define the bilinear map (3) via the rule

$$\langle [P], [N] \rangle := \dim(\underline{\mathrm{Hom}_{\mathbb{J}_m}(P, N)}) \tag{28}$$

where the classes $[N] \in G_0(\mathbb{J}_m)$ and $[P] \in K_0(\mathbb{J}_m)$ are defined by an N is as above and by a finitely generated projective left $\mathbb{J}_m$-module P.

Fact 5. *If P is a finitely generated projective left $\mathbb{J}_m$-module and N is a left $\mathbb{J}_m$-module of finite length, then the unipotent connected commutative quasi-algebraic groups $\underline{\mathrm{Hom}_{\mathbb{J}_m}(P, N)}$ and $\underline{\mathrm{Hom}_{\mathbb{J}_m^\vee}(P^\#, N^\vee)}$ are naturally Serre dual and therefore we have the duality formula*

$$\langle [P], [N] \rangle = \langle [P^\#], [N^\vee] \rangle$$

(as each $\mathbb{E}_m$-algebra isomorphism $\mathbb{J}_m \to \mathbb{J}_m^\vee$ allows to view left $\mathbb{J}_m^\vee$-modules as left $\mathbb{J}_m$-modules, $\langle\, , \rangle$ denotes also the bilinear map $K_0(\mathbb{J}_m^\vee) \times G_0(\mathbb{J}_m^\vee) \to \mathbb{Z}$).

Proof: We can assume that P is free and that $\mathbb{J}_m = \mathbb{E}_m$ (i.e., $\mathbb{B} = \mathbb{Z}_p$). In this case, the fact follows as Serre duality commutes with finite direct sums and interchanges F and V in the same way as ι_m does. $\square$

6.3 Proof of Theorem 2 (a)

Fact 6. *Let G and H be two objects of $\mathcal{F}_{k,m}^{\mathbb{B}}$ and let $M := \mathbb{D}^{\mathbb{B}}(G)$ and $N := \mathbb{D}^{\mathbb{B}}(H)$. Then the two commutative quasi-algebraic groups $\underline{\mathrm{Hom}_{\mathbb{J}_m}(N, M)}$ and $\underline{\mathrm{Hom}_{\mathbb{B}}(G, H)} = \mathbf{Hom}_{\mathbb{B}}(G, H)^{\mathrm{perf}}$ are isomorphic.*

Proof: The crystalline Dieudonné theory gives a natural evaluation

$$\mathbb{D}_{G,H}^{\mathbb{B}} : \underline{\mathrm{Hom}_{\mathbb{B}}(G, H)} \to \underline{\mathrm{Hom}_{\mathbb{J}_m}(N, M)}$$

which is a morphism of the abelian category $\mathcal{Q}$ with the property that $\mathbb{D}_{G,H}^{\mathbb{B}}(\bar{k})$ is an isomorphism. Thus the fact follows from Fact 1. $\square$

To prove Theorem 2 (a) let G, H, M and N be as in Fact 6 and we assume that G is a BT_m. From Fact 6 applied to both (G, H) and (H, G) we get that Equation (4) is equivalent to the equation

$$\dim(\underline{\mathrm{Hom}_{\mathbb{J}_m}(M, N)}) - \dim(\underline{\mathrm{Hom}_{\mathbb{J}_m}(N, M)}) = \langle [M], [N] \rangle. \tag{29}$$

To prove Equation (29) we consider a projective resolution

$$0 \to P_1 \to P_0 \to M \to 0$$

with P_0, P_1 finitely generated projective left $\mathbb{J}_m$-modules, see Proposition 4. From Equation (11) applied to the homomorphism

$$f : \underline{\mathrm{Hom}_{\mathbb{J}_m}(P_0, N)} \to \underline{\mathrm{Hom}_{\mathbb{J}_m}(P_1, N)}$$

and Fact 5 we get that the kernel $\underline{\mathrm{Hom}_{\mathbb{J}_m}(M, N)}$ of f and the kernel $\underline{\mathrm{Hom}_{\mathbb{J}_m^\vee}(\mathrm{Coker}(P_0^\# \to P_1^\#), N^\vee)}$ of the Serre dual

$$f^* : \underline{\mathrm{Hom}_{\mathbb{J}_m^\vee}(P_1^\#, N^\vee)} \to \underline{\mathrm{Hom}_{\mathbb{J}_m^\vee}(P_0^\#, N^\vee)}$$

of f are related via the formula

$$\dim(\mathrm{Ker}(f)) - \dim(\mathrm{Ker}(f^*)) = \dim(\underline{\mathrm{Hom}_{\mathbb{J}_m}(P_0, N)}) - \dim(\underline{\mathrm{Hom}_{\mathbb{J}_m}(P_1, N)})$$

$$\xlongequal{\text{Equation (28)}} \langle [P_0] - [P_1], [N] \rangle = \langle [M], [N] \rangle.$$

Thus, as $\underline{\mathrm{Hom}_{\mathbb{J}_m^\vee}(M^\vee, N^\vee)}$ and $\underline{\mathrm{Hom}_{\mathbb{J}_m}(N, M)}$ are canonically identified, Equation (29) follows once we show that $\mathrm{Coker}(P_0^\# \to P_1^\#) = \mathrm{Ext}^1_{\mathbb{E}_m}(M, \mathbb{E}_m)$ is isomorphic to $M^\vee$ as left $\mathbb{J}_m^\vee$-modules, equivalently, as left $\mathbb{E}_m$-modules in a natural way which would commute with the actions of $\mathbb{B}_m^{\mathrm{op}}$. But this is a particular case of the following lemma. Thus Theorem 2 (a) holds. $\square$

Lemma 7. *If N is a left $\mathbb{E}_m$-module of finite length, then we have a natural isomorphism $\mathrm{Ext}^1_{\mathbb{E}_m}(N, \mathbb{E}_m) \to N^\vee$ of left $\mathbb{E}_m$-modules.*

Proof: We have identifications between left $\mathbb{E}_m$-modules

$$\mathrm{Ext}^1_{\mathbb{E}_m}(N, \mathbb{E}_m) \xlongequal{\text{Lemma 4 (c)}} \mathrm{Hom}_{\mathbb{E}_m}(N, \mathbb{K}_m/\mathbb{E}_m)$$

$$\xlongequal{\text{Lemma 6}} \mathrm{Hom}_{\mathbb{E}_m}(N, \mathrm{Hom}_{W_m(k)}(\mathbb{E}_m, W_m(k))) = \mathrm{Hom}_{W_m(k)}(N, W_m(k)) = N^\vee$$

where the third identity can be easily check using a presentation of the left $\mathbb{E}_m$-module N of finite length. $\square$

6.4 Proof of Theorems 2 (b) to (d)

We continue to use the notation $M = \mathbb{D}^{\mathbb{B}}(G)$ and $N = \mathbb{D}^{\mathbb{B}}(H)$. To prove Theorems 2 (b) to (d) we can work only with those H which are local-local. We can also assume that $\mathbb{B}$ is connected and, due to the same standard Morita equivalences, commutative. Thus H has a composition series whose simple factors correspond to simple left $\mathbb{J}_m$-modules that are annihilated by F and V and therefore, with the notation of Subsection 5.1, are simple $\mathbb{Y}_1$-modules and hence (see Equation (26)) belong to the set $\{k_e|e \in \mathcal{I}\}$ of simple factors of $\mathbb{Y}_1$. This implies that $G_0^{\text{loc-loc}}(\mathbb{J}_m)$ is the free abelian group on the set $[k_e]$ with $e \in \mathcal{I}$ and therefore we get a canonical isomorphism

$$G_0^{\text{loc-loc}}(\mathbb{J}_m) \to \mathbb{K}_0(\mathbb{J}_m) \tag{30}$$

which maps $[k_e]$ to $[\mathbb{E}_m \otimes_{W_m(k)} \mathbb{Y}_m e]$ for all $e \in \mathcal{I}$.

For two elements $e, e' \in \mathcal{I}$, using the Kronecker δ we have

$$\langle [\mathbb{E}_m \otimes_{W_m(k)} \mathbb{Y}_m e'], [\mathbb{E}_m \otimes_{W_m(k)} \mathbb{Y}_m e] \rangle \xlongequal{\text{Equations (6) and (30)}} \langle [\mathbb{E}_m \otimes_{W_m(k)} \mathbb{Y}_m e'], [k_e] \rangle$$

$$\xlongequal{\text{Equation (28)}} \dim(\underline{e' k_e}) = \dim_k(e' k_e) = [k_e : k]\delta_{ee'} = \frac{q}{q_1}\delta_{ee'}. \tag{31}$$

Thus Theorem 2 (b) holds.

Let $x_0 := \sum_{e \in \mathcal{I}} [k_e] \in G_0^{\text{loc-loc}}(\mathbb{J}_m)$. If $[H] = \sum_{e \in \mathcal{I}} h_e [k_e] \in G_0^{\text{loc-loc}}(\mathbb{J}_m)$ with each $h_e \in \mathbb{N}$, then $[H^{(p)}] = \sum_{e \in \mathcal{I}} h_e [k_{\sigma(e)}]$. As σ acts transitively on $\mathcal{I}$, we get that H is balanced if and only if $[H] \in \mathbb{N}x_0$. Hence, if H is balanced and non-trivial, then for a class $x \in K_0(\mathbb{J}_m)$ we have $\langle x, [H] \rangle = 0$ if and only if $\langle x, x_0 \rangle = 0$ and if and only if $\text{index}(x) = 0$. From this and Theorem 2 (a), as $\text{index}([M]) = 0$ (see Example 7), we get that d1 $\Rightarrow$ d2.

To check that d2 $\Rightarrow$ d1 we can assume that $\mathcal{I}$ has at least 2 elements. If each M_e, $e \in \mathcal{I}$, is as in Example 8, say, with $c = 1$ and $d = q_1 - 1$, then, if the identities $\langle [M_e], [H] \rangle = 0$ hold for all $e \in \mathcal{I}$, from Equation (31) we get that $\frac{q}{q_1}[h_e - h_{\sigma(e)}] = 0$ for all $e \in \mathcal{I}$ and thus $[H] \in \mathbb{N}x_0$. Hence d2 $\Rightarrow$ d1. Thus Theorem 2 (d) holds.

As the ideal $(p) = \text{Ker}(\mathbb{J}_m \to \mathbb{J}_1)$ is nilpotent, the functorial homomorphism $K_0(\mathbb{J}_m) \to \mathbb{K}_0(\mathbb{J}_1)$ is an isomorphism (this general fact which can be deduced from [Bo2], Sect. 9, Cor. 2 of Prop. 7 is as well a direct consequence of Proposition 2 (a)). Thus, referring to Theorem 2 (c), if c1 holds, then $[M] = 0 \in K_0(\mathbb{J}_m)$ and from Theorem 2 (a) we get that c2 holds. If c2 holds, then from Equations (5) and (31) we get that $[M] = 0 \in K_0(\mathbb{J}_m)$ and hence $[G[p]] = [M/pM] \in K_0(\mathbb{J}_1)$ is 0, i.e., c1 holds.

We consider direct sum decompositions $\mathbb{D}^{\mathbb{B}}(\boldsymbol{\alpha}_p(G)) = \oplus_{e\in\mathcal{J}} k_e^{\alpha_p(e)}$ and $\mathbb{D}^{\mathbb{B}}((\boldsymbol{\alpha}_p(G^{\mathrm{t}})^{\mathrm{t}}) = \oplus_{e\in\mathcal{J}} k_e^{\beta_p(e)}$ with $\alpha_p(e), \beta_p(e) \in \mathbb{N}$ for all $e \in \mathcal{I}$. Statement c3 holds if and only if for all $e \in \mathcal{I}$ we have $\alpha_p(e) = \beta_p(e)$. Based on the additivity in each variable of the bilinear map of Theorem 2 (a) and the above part on composition series of each H, statement c3 holds if and only if for all $e \in \mathcal{I}$ we have $\gamma_{\mathbb{B}}(G, H_e) = \gamma_{\mathbb{B}}(H_e, G)$, where H_e is an object of $\mathcal{F}_{k,m}^{\mathbb{B}}$ (unique up to isomorphism) such that $\mathbb{D}^{\mathbb{B}}(H_e) \simeq k_e$. As $\gamma_{\mathbb{B}}(H_e, G) = \frac{q}{q_1}\alpha_e(G)$ and $\gamma_{\mathbb{B}}(G, H_e) = \frac{q}{q_1}\beta_p(e)$ (see Equation (31)), it follows that c2 holds if and only if for all $e \in \mathcal{I}$ we have $\alpha_p(e) = \beta_p(e)$. We conclude that c2 $\Leftrightarrow$ c3. Thus Theorem 2 (c) holds. This ends the proof of Theorem 2. $\square$

6.5 Remarks

(a) Let R be a perfect ring of characteristic p, let $W(R)$ be the ring of p-typical Witt vectors with coefficients in R, and let σ_R be the Frobenius automorphism of R, $W(R)$, and $B(R) := W(R)[1/p]$. Let

$$\mathbb{L}(R) := B(R)_{\sigma_R}[F, F^{-1}] \quad \text{and} \quad \mathbb{E}(R) := W(R)[F, V] \subset \mathbb{L}(R)$$

be defined similarly to $\mathbb{L}(k) = \mathbb{L}$ and $\mathbb{E}(k) = \mathbb{E}$. Let $\mathbb{E}_m(R) := \mathbb{E}(R)/p^m\mathbb{E}(R)$ and $W_m(R) := W(R)/p^mW(R)$. Lemma 7 continues to hold in the context of $\mathbb{E}_m(R)$ and $W_m(R)$ provided the role of N is replaced by the one of a left $\mathbb{E}_m(R)$-module which is finitely presented over $W(R)$ of projective dimension at most one. Denoting $A := \mathbb{E}_m(R)$, the proof is more complicated and is based on a standard resolution of each left A-module O_R by induced modules:

$$\begin{tikzcd}
 & A\otimes O_R^{(\sigma^{-1})} \arrow[r, "\varpi_{11}"] \arrow[ddr, "\varpi_{12}"] & A\otimes O_R \arrow[r, "\delta_{11}"] \arrow[ddr, "\delta_{12}"] & A\otimes O_R^{(\sigma^{-1})} \arrow[dr, "\varsigma_{11}"] & & & \\
\cdots & \bigoplus & \bigoplus & \bigoplus & A\otimes O_R \arrow[r] & O_R \arrow[r] & 0, \\
 & A\otimes O_R^{(\sigma)} \arrow[r, "\varpi_{22}"] \arrow[uur, "\varpi_{21}"] & A\otimes O_R \arrow[r, "\delta_{22}"] \arrow[uur, "\delta_{21}"] & A\otimes O_R^{(\sigma)} \arrow[ur, "\varsigma_{21}"] & & &
\end{tikzcd}$$

where all tensorizations are over $W_m(R)$, $\cdots$ stands for a periodic complex of period 2, and the linear maps are labeled above arrows and are defined by the identities

$$\varsigma_{11}(\alpha\otimes x) = \alpha V\otimes x - \alpha\otimes Vx, \;\; \varsigma_{21}(\alpha\otimes x) = \alpha F\otimes x - \alpha\otimes Fx, \;\; (\alpha, x) \in (A, O_R),$$

$$(\delta_{ij})_{1\le i,j\le 2} = \begin{bmatrix} F\otimes 1 & 1\otimes V \\ 1\otimes F & V\otimes 1 \end{bmatrix}, \quad (\varpi_{ij})_{1\le i,j\le 2} = \begin{bmatrix} V\otimes 1 & -1\otimes V \\ -1\otimes F & F\otimes 1 \end{bmatrix}.$$

The exactness is verified by passing to the associated graded for the degree filtration of the $W_m(R)$-algebra A in which F and V have degree 1.

(b) Lemmas 4, 6 and 7 have analogs over $W_m(k)[F]$ and $W_m(k)[V]$ (instead of over $\mathbb{E}_m$). For instance, working just with F, the role of λ_m is replaced by a $W_m(k)$-linear map $\lambda_{m,F} : \mathrm{Frac}(W_m(k)[F])/W_m(k)[F] \to W_m(k)$ defined by: for $(x,0) \in \mathrm{Frac}(W_m(k)[F]) \times \mathrm{Frac}(W_m(k)[V]) = \mathbb{K}_m$, we have

$$\lambda_{m,F}(x + W_m(k)[F]) = \text{coefficient of } \ F^{-1} \ \text{ in } \ \theta_{m,F}(x,0).$$

6.6 Examples

Example 9. *Equation (5) implies that $s_{D,E} = s_{E,D}$. We recall from [V2], Thm. 1.2 (e) and [GV], Rm. 4.5 that $s_D = s_{D,D}$ is an isogeny invariant. From the last two sentences we get directly that*

$$s_{D,E} = (s_{D,E} + s_{E,D})/2 = (s_{D\oplus E} - s_D - s_E)/2$$

is an isogeny invariant. Based on this and [V2], Thm. 1.2 (c) and (f), one gets that $s_{D,E}$ can be easily computed in terms of the Newton polygons of D and E. For instance, if D is isoclinic of dimension d and codimension c and E is isoclinic of dimension f and codimension e, then $s_D = cd$, $s_E = ef$, and

$$s_{D\oplus E} = (c+e)(d+f) - |cf - de|$$

(see [V2], Thm. 1.2 (c) and (f)) and therefore we have

$$s_{D,E} = \min\{cf, de\}.$$

Example 10. *Let $n \in \mathbb{N}^*$ and $\ell \in \mathbb{N}$, let $m := n+\ell$, and let $\ell' \in \{2\ell-1, 2\ell\}$ if $\ell \geq 1$ and let $\ell' := 0$ if $\ell = 0$. Let G be such that $M = \mathbb{D}(G) = \mathbb{E}_n/(F-V)$; so if $k = \bar{k}$, then $G = \mathcal{S}[p^n]$, where $\mathcal{S}$ is a supersingular elliptic curve over k. Let H be such that $N = \mathbb{D}(H) = \mathbb{E}_m/(F,V)^{2n+\ell'}$. Thus p^{m-1} does not annihilate H. We have canonical identifications of quasi-algebraic groups*

$$\underline{\mathrm{Hom}_{\mathbb{E}_m}(N,M)} = \underline{M} \quad \text{and} \quad \underline{\mathrm{Hom}_{\mathbb{E}_m}(M,N)} = \underline{(F,V)^{2n+\ell'-1}/(F,V)^{2n+\ell'}} \tag{32}$$

(see the notation of Subsections 2.4 and 3.2). From this and Fact 6 we get

$$\gamma(H,G) = \mathrm{length}_{W_m(k)}((F,V)^{2n+\ell'-1}/(F,V)^{2n+\ell'}) = 2n + \ell',$$

$$\gamma(G,H) = \mathrm{length}_{W_m(k)}(M) = 2n.$$

Thus $\gamma(G,H)-\gamma(H,G) = -\ell'$. *From Cartier duality we get that* $\gamma(G^{\mathrm{t}},H^{\mathrm{t}})-\gamma(H^{\mathrm{t}},G^{\mathrm{t}}) = \ell'$.

If $\ell' = 2\ell$, *then we have* $\gamma(H,G) = \frac{m}{n}\gamma(G,H)$ *and* $\gamma(H^{\mathrm{t}},G^{\mathrm{t}}) = \frac{n}{m}\gamma(G^{\mathrm{t}},H^{\mathrm{t}})$.

If $\ell = 0$ *and* $m = n \geq 2$, *then the quasi-algebraic groups of Equation (32) are not isomorphic.*[10]

Let now $\mathbb{B}$ *be a finite flat separable* $\mathbb{Z}_p$*-algebra. For* $\star \in \{\frac{m}{n},\frac{n}{m}\}$, *we want to get examples of a* BT_n $\tilde{G}$ *over* k *and of a finite commutative group scheme* $\tilde{H}$ *over* k *which are endowed with a left* $\mathbb{B}$*-action and such that we have* $\gamma_{\mathbb{B}}(\tilde{H},\tilde{G}) = \star\gamma_{\mathbb{B}}(\tilde{G},\tilde{H})$, *equivalently, we have* $\gamma_{\mathbb{B}^{\mathrm{op}}}(\tilde{H}^{\mathrm{t}},\tilde{G}^{\mathrm{t}}) = \star^{-1}\gamma_{\mathbb{B}^{\mathrm{op}}}(\tilde{G}^{\mathrm{t}},\tilde{H}^{\mathrm{t}})$. *As* $\mathbb{B}$ *and* $\mathbb{B}^{\mathrm{op}}$ *are isomorphic, it suffices to find examples with* $\star = \frac{m}{n}$. *Using direct sum decompositions indexed by the simple factors of* $\mathbb{B}_1$, *we can assume that* $\mathbb{B} = M_s(W(\mathbb{F}_{p^q}))$ *is connected. If* $\tilde{G}$ *and* $\tilde{H}$ *are left* $W(\mathbb{F}_{p^q})$*-modules such that we have* $\gamma_{W(\mathbb{F}_{p^q})}(\tilde{H},\tilde{G}) = \frac{m}{n}\gamma_{W(\mathbb{F}_{p^q})}(\tilde{G},\tilde{H})$, *then for* $s \in \mathbb{N}^*$, $\tilde{G}^s$ *and* $\tilde{H}^s$ *are left* $\mathrm{M}_s(W(\mathbb{F}_{p^q})$*-modules such that we have an identity* $\gamma_{M_s(W(\mathbb{F}_{p^q}))}(\tilde{H}^s,\tilde{G}^s) = \frac{m}{n}\gamma_{M_s(W(\mathbb{F}_{p^q}))}(\tilde{G}^s,\tilde{H}^s)$. *Thus we can assume that* $\mathbb{B} = W(\mathbb{F}_{p^q})$ *is also commutative. Let* G *and* H *be as above with* $\ell' = 2\ell$ *and* $\gamma(H,G) = \frac{m}{n}\gamma(G,H)$. *Let* $\tilde{G} := G^q$ *and* $\tilde{H} := H^q$ *be the left* $\mathbb{B}$*-modules defined by composing a* $\mathbb{Z}_p$*-algebra monomorphism* $\mathbb{B} \to \mathrm{M}_q(\mathbb{Z}_p)$ *with* $\mathrm{M}_q(\mathbb{Z}_p) \xrightarrow{\mathrm{M}_q(\iota_G)} \mathrm{M}_q(\mathrm{End}(G)) = \mathrm{End}(\tilde{G})$ *and* $\mathrm{M}_q(\mathbb{Z}_p) \xrightarrow{\mathrm{M}_q(\iota_H)} \mathrm{M}_q(\mathrm{End}(H)) = \mathrm{End}(\tilde{H})$, *where* $\iota_G : \mathbb{Z}_p \to \mathrm{End}(G)$ *and* $\iota_H : \mathbb{Z}_p \to \mathrm{End}(H)$ *are the unique ring homomorphisms. As* $\mathbb{B}$ *is its own centralizer in* $M_q(\mathbb{Z}_p)$, *we have canonical identifications*

$$\mathbf{Hom}_{\mathbb{B}}(\tilde{G},\tilde{H}) = \mathbb{B}\otimes_{\mathbb{Z}_p}\mathbf{Hom}(G,H) \ \text{ and } \ \mathbf{Hom}_{\mathbb{B}}(\tilde{H},\tilde{G}) = \mathbb{B}\otimes_{\mathbb{Z}_p}\mathbf{Hom}(H,G)$$

from which we get that we have $\gamma_{\mathbb{B}}(\tilde{H},\tilde{G}) = \frac{m}{n}\gamma_{\mathbb{B}}(\tilde{G},\tilde{H})$.

6.7 Proof of Proposition 1

The part (c) and the optimality part of Proposition 1 follow from Example 10 (which makes sense over $\mathbb{F}_p$). Part (b) also follows from the computations

[10] Based on this and the fact that for the Grothendieck group $K_0(\mathcal{U})$ of the category $\mathcal{U}$ of unipotent connected commutative quasi-algebraic groups over k the dimension homomorphism $\dim : K_0(\mathcal{U}) \to \mathbb{Z}$ is an isomorphism (this can be deduced easily based on [SGA3-2], Exp. XVII, Sect. 3, Thm. 3.5), we get that Theorem 2 cannot be improved within $\mathcal{U}$ and this motivates why its refinement, Theorem 3, works with Lie algebras.

of Example 10 by taking the same G and by taking H such that N is a direct sum of copies of $\mathbb{E}_{n+1}/(F,V)^{2n+1}$ (so H is balanced).

If the inequality $\gamma_{\mathbb{B}}(G,H) \leq \frac{m}{n}\gamma_{\mathbb{B}}(H,G)$ always holds, then by replacing in it the triple $(G,H,\mathbb{B})$ by the triple $(G^{\mathrm{t}},H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}})$ and using Cartier duality we get that the inequality $\frac{n}{m}\gamma_{\mathbb{B}}(H,G) \leq \gamma_{\mathbb{B}}(G,H)$ also always holds. Thus it suffices to show that the inequality $\gamma_{\mathbb{B}}(G,H) \leq \frac{m}{n}\gamma_{\mathbb{B}}(H,G)$ holds.

Let G' be a BT_m over k endowed with a left $\mathbb{B}$-action such that we have an identification $G = G'[p^n]$ of left $\mathbb{B}$-modules, see Proposition 4. The commutative affine group scheme $\Upsilon := \mathbf{Hom}_{\mathbb{B}}(H,G')$ is of finite type over k and is annihilated by p^m. As $G = G'[p^n]$, we can identity $\mathbf{Hom}_{\mathbb{B}}(H,G) = \Upsilon[p^n]$ and thus we have $\gamma_{\mathbb{B}}(H,G) = \dim(\Upsilon[p^n])$. The exact complex $G' \to G \to 0$ induces a closed embedding $\mathbf{Hom}_{\mathbb{B}}(G,H) \to \mathbf{Hom}_{\mathbb{B}}(G',H)$ and thus we have $\gamma_{\mathbb{B}}(G,H) \leq \gamma_{\mathbb{B}}(G',H)$. Moreover, $\dim(\Upsilon) \leq \frac{m}{n}\dim(\Upsilon[p^n])$ (see Lemma 8 below) and $\gamma_{\mathbb{B}}(G',H) = \dim(\Upsilon)$ (see Theorems 2 (a) and (c)). Combing the inequalities of the last three sentences we get that that $\gamma_{\mathbb{B}}(G,H) \leq \frac{m}{n}\gamma_{\mathbb{B}}(H,G)$. Thus Inequalities (7) hold. $\square$

Lemma 8. *Let $m > n > 0$ be integers. Let Υ be a commutative group scheme of finite type over k annihilated by p^m. Then the dimension of its subgroup scheme $\Upsilon[p^n] = \mathrm{Ker}(p^n : \Upsilon \to \Upsilon)$ is at least equal to $\frac{n}{m}\dim(\Upsilon)$.*

Proof: For $i \in \{0,\ldots,m-1\}$ let $d_i := \dim(\Upsilon[p^{i+1}]/\Upsilon[p^i])$. Then we have $\dim(\Upsilon[p^n]) = \sum_{j=0}^{n-1} d_j$ and $\dim(\Upsilon) = \sum_{i=0}^{m-1} d_i$. Thus the difference

$$m\dim(\Upsilon[p^n]) - n\dim(\Upsilon) = (m-n)\sum_{j=0}^{n-1} d_j - n\sum_{i=n}^{m-1} d_i$$

is the sum of $n(m-n)$ expressions of the form $d_j - d_i$ with $0 \leq j < n \leq i \leq m-1$. But for $0 \leq j < i \leq m-1$ the multiplication by p^{i-j} induces a monomorphism $\Upsilon[p^{i+1}]/\Upsilon[p^i] \to \Upsilon[p^{j+1}]/\Upsilon[p^j]$ and therefore the inequality $d_j - d_i \geq 0$ holds. The lemma follows from the last two sentences. $\square$

7 On (fully) BT-balanced objects

After recalling and complementing Deligne's filtrations introduced in [De] (see Subsection 7.1) and presenting some general properties of lattices (see Subsection 7.2), we will use them to provide different characterizations of fully BT-balanced objects (see Subsection 7.3). Applications of Subsection 7.3 to balanced objects are gathered in Subsection 7.4.

7.1 Deligne's filtrations

Let $\mathcal{O}$ be an object of an abelian category $\mathcal{C}$. An increasing filtration $(\mathcal{O}_j)_{j\in\mathbb{Z}}$ of $\mathcal{O}$ by subojects of $\mathcal{O}$ in $\mathcal{C}$ is called finite if there exist integers j_1, j_2 such that $\mathcal{O}_{j_1} = 0$ and $\mathcal{O}_{j_2} = \mathcal{O}$. We also call $(\mathcal{O}_j)_{j\in\{j_1,\ldots,j_2\}}$ an increasing finite filtration of $\mathcal{O}$. For $j \in \mathbb{Z}$, let $\mathrm{gr}_j(\mathcal{O}) := \mathcal{O}_j/\mathcal{O}_{j-1}$.

Let $\mathcal{N} : \mathcal{O} \to \mathcal{O}$ be a nilpotent endomorphism of $\mathcal{C}$. We will also denote by $\mathcal{N}$ the different morphisms induced by it between objects of $\mathcal{C}$ obtained from $\mathcal{O}$ by taking repeatedly subojects and quotient objects. Let $n \in \mathbb{N}^*$ be the index of nilpotency of $\mathcal{N}$ (so $\mathcal{N}^n = 0$ but $\mathcal{N}^{n-1} \neq 0$ if $\mathcal{O} \neq 0$).

We recall (see [De], Prop. (1.6.1)) that there exists a unique finite increasing filtration $(\mathcal{O}_j)_{j\in\mathbb{Z}}$ of $\mathcal{O}$, to be called Deligne's filtration of $\mathcal{O}$ defined by $\mathcal{N}$, which satisfies the following two conditions:

(i) For all $j \in \mathbb{Z}$ we have $\mathcal{N}(\mathcal{O}_j) \subset \mathcal{O}_{j-2}$.

(ii) For all $j \in \mathbb{N}$, $\mathcal{N}^j$ induces an isomorphism $\mathrm{gr}_j(\mathcal{O}) \xrightarrow{\mathcal{N}^j} \mathrm{gr}_{-j}(\mathcal{O})$.

Loc. cit. gives formulas $\mathcal{O}_{-n} = 0$, $\mathcal{O}_{-n+1} = \mathrm{Im}(\mathcal{N}^{n-1})$, $\mathcal{O}_{n-2} = \mathrm{Ker}(\mathcal{N}^{n-1})$, $\mathcal{O}_{n-1} = \mathcal{O}$, and for $n \geq 3$ and $-n+1 \leq j \leq n-2$ the recursive rule

$$\mathcal{O}_j/\mathrm{Im}(\mathcal{N}^{n-1}) = (\mathrm{Ker}(\mathcal{N}^{n-1})/\mathrm{Im}(\mathcal{N}^{n-1}))_j.$$

We recall (see [De], Def. (1.6.3)) that for $j \in \mathbb{Z}$,

$$\mathcal{P}_j := \mathrm{Ker}(\mathrm{gr}_j(\mathcal{O}) \xrightarrow{\mathcal{N}} \mathrm{gr}_{j-2}(\mathcal{O}))$$

is called the *j-th primitive part* of $\mathcal{N} : \mathcal{O} \to \mathcal{O}$. If $j \leq -n$, then clearly we have $\mathcal{P}_j = 0$. If $j > 0$, then as the composite $\mathrm{gr}_j(\mathcal{O}) \xrightarrow{\mathcal{N}} \mathrm{gr}_{j-2}(\mathcal{O}) \xrightarrow{\mathcal{N}} \cdots \xrightarrow{\mathcal{N}} \mathrm{gr}_{-j}(\mathcal{O})$ is an isomorphism (see property (ii)), we get that $\mathcal{P}_j = 0$. Moreover, for all $j \in \mathbb{Z}$ we have an isomorphism (see [De], Eq. (1.6.4.1))

$$\mathrm{gr}_j(\mathcal{O}) \simeq \bigoplus_{i\geq|j|, i\equiv j \ mod\ 2} \mathcal{P}_{-i}. \tag{33}$$

Proposition 5. *With $\mathcal{C}$, $\mathcal{N} : \mathcal{O} \to \mathcal{O}$, and $n \in \mathbb{N}^*$ as above, the following seven properties hold:*

(a) *For each $j \in \mathbb{Z}$ we have*

$$\mathcal{O}_j = \sum_{i=\max\{0,-j\}}^{\infty} \mathrm{Im}(\mathcal{N}^i) \cap \mathrm{Ker}(\mathcal{N}^{i+j+1}) = \sum_{i=\max\{0,-j\}}^{n} \mathrm{Im}(\mathcal{N}^i) \cap \mathrm{Ker}(\mathcal{N}^{i+j+1}).$$

(b) *If $\mathcal{N}' : \mathcal{O}' \to \mathcal{O}'$ is another nilpotent endomorphism of the category $\mathcal{C}$ and if $\delta : \mathcal{O} \to \mathcal{O}'$ is a morphism of $\mathcal{C}$ such that $\delta \circ \mathcal{N} = \mathcal{N}' \circ \delta$, then for all $j \in \mathbb{Z}$, $\delta(\mathcal{O}_j)$ is a subobject of $\mathcal{O}'_j$ (to be denoted as $\delta(\mathcal{O}_j) \subset \mathcal{O}'_j$).*

(c) *For Deligne's filtration $(\mathcal{O}_{n-2,j})_{j\in\mathbb{Z}}$ of $\mathcal{O}_{n-2}$ defined by $\mathcal{N}$ the following two types of identities hold:*

- $\forall j \in \{-n+2, -n+4, \ldots, n-4, n-2\}$ *we have* $\mathcal{O}_{n-2,j} = \mathcal{O}_j$;
- $\forall j \in \{-n+3, -n+5, \ldots, n-5, n-3\}$ *we have* $\mathcal{O}_j = \mathcal{O}_{n-2,j} + \mathrm{Im}(\mathcal{N}^{\frac{n-1-j}{2}})$.

(d) *For each $j \in \mathbb{N}$ we have a monomorphism $\mathcal{O}_{-j} \subset \mathrm{Im}(\mathcal{N}^j)$ and an identity $\mathrm{Ker}(\mathcal{N}) \cap \mathcal{O}_{-j} = \mathrm{Ker}(\mathcal{N}) \cap \mathrm{Im}(\mathcal{N}^j)$ and thus $\mathrm{Ker}(\mathcal{N}) \subset \mathcal{O}_0$.*

(e) *For each $j \in \mathbb{N}$ we have isomorphisms*

$$\mathcal{P}_{-j} \simeq [\mathrm{Im}(\mathcal{N}^j) \cap \mathrm{Ker}(\mathcal{N})]/[\mathrm{Im}(\mathcal{N}^{j+1}) \cap \mathrm{Ker}(\mathcal{N})]$$
$$\simeq \mathrm{Ker}(\mathcal{N} : \mathrm{Im}(\mathcal{N}^j)/\mathrm{Im}(\mathcal{N}^{j+1}) \to \mathrm{Im}(\mathcal{N}^{j+1})/\mathrm{Im}(\mathcal{N}^{j+2}))$$
$$\simeq \mathrm{Coker}(\mathcal{N} : \mathrm{Ker}(\mathcal{N}^{j+2})/\mathrm{Ker}(\mathcal{N}^{j+1}) \to \mathrm{Ker}(\mathcal{N}^{j+1})/\mathrm{Ker}(\mathcal{N}^j)).$$

(f) *If $n \geq 2$, then for each $j \in \{0, \ldots, n-2\}$, the $-j$-th primitive part $\mathcal{P}_{n-2,-j}$ of $\mathcal{N} : \mathcal{O}_{n-2} \to \mathcal{O}_{n-2}$ satisfies the rules:*

- *If $j \in \{0, \ldots, n-3\}$, we have $\mathcal{P}_{n-2,-j} = \mathcal{P}_{-j}$.*
- *We have a short exact sequence $0 \to \mathcal{P}_{-n+1} \to \mathcal{P}_{n-2,-n+2} \to \mathcal{P}_{-n+2} \to 0$.*

(g) *For a full subcategory $\mathcal{C}'$ of $\mathcal{C}$ whose objects are stable under direct summands and finite direct sums, the following two statements are equivalent:*

g1 *For all $j \in \mathbb{Z}$, $\mathrm{gr}_j(\mathcal{O})$ is an object of $\mathcal{C}'$.*

g2 *For all $j \in \mathbb{N}$, $\mathcal{P}_{-j}$ is an object of $\mathcal{C}'$.*

Proof: The formulas of part (a) of subbjects $\mathcal{O}_j$ of $\mathcal{O}$ clearly define an increasing filtration of $\mathcal{O}$ which is finite as $\mathcal{O}_{-n} = 0$ and $\mathcal{O}_{n-1} = \mathcal{O}$. By inspection condition (i) holds and for $j \geq 0$ we have $\mathcal{N}^j\mathcal{O}_j = \mathcal{O}_{-j}$. Condition (ii) is equivalent to the statement that for each $j \in \mathbb{N}$ the right vertical arrow of the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{j-1} \arrow[r] \arrow[d, "\mathcal{N}^j"] & \mathcal{O}_j \arrow[r] \arrow[d, "\mathcal{N}^j"] & \mathrm{gr}_j(\mathcal{O}) \arrow[r] \arrow[d, "\mathcal{N}^j"] & 0 \\
0 \arrow[r] & \mathcal{O}_{-j-1} \arrow[r] & \mathcal{O}_{-j} \arrow[r] & \mathrm{gr}_{-j}(\mathcal{O}) \arrow[r] & 0
\end{tikzcd} \tag{34}$$

is an isomorphism. To check this we can assume that $j > 0$. As $j > 0$, we have $\mathcal{N}^j\mathcal{O}_j = \mathcal{O}_{-j}$ and $\mathcal{N}^{j+1}\mathcal{O}_{j+1} = \mathcal{O}_{-j-1}$, thus $\mathcal{N}^j\mathcal{O}_{j-1} = \mathcal{O}_{-j-1}$. Referring to Diagram (34) whose left and central vertical arrows are epimorphisms, the snake lemma gives that its right vertical arrow is an isomorphism if and only if the kernel of the left vertical arrow maps isomorphically onto the kernel of the central vertical arrow which follows from the inclusions $\mathrm{Ker}(\mathcal{N}^j : \mathcal{O}_j \to \mathcal{O}_{-j}) \subset \mathrm{Ker}(\mathcal{N}^j) = \mathrm{Im}(\mathcal{N}^0) \cap \mathrm{Ker}(\mathcal{N}^{0+(j-1)+1}) \subset \mathcal{O}_{j-1}$. Thus part (a) holds. Part (b) follows from part (a).

To prove part (c), we can assume that $n \geq 2$ and we first remark that the index of nilpotency of the endomorphism of $\mathcal{O}_{n-2} = \mathrm{Ker}(\mathcal{N}^{n-1})$ induced by $\mathcal{N}$ is $n-1$. For $i = 0$ we have

$$\mathrm{Im}(\mathcal{N}^0 : \mathcal{O}_{n-2} \to \mathcal{O}_{n-2}) = \mathrm{Im}(\mathcal{N}^0) \cap \mathrm{Ker}(\mathcal{N}^{n-1}),$$

for $0 < i < n-1$ and $i+1 \leq 2i+j+1 \leq n-1$ we have

$$\mathrm{Im}(\mathcal{N}^i) \cap \mathrm{Ker}(\mathcal{N}^{i+j+1}) = \mathrm{Im}(\mathcal{N}^i : \mathcal{O}_{n-2} \to \mathcal{O}_{n-2}) \cap \mathrm{Ker}(\mathcal{N}^{i+j+1}),$$

for $0 < i < n-1$ and $2i+j+1 > n$ we have

$$\mathrm{Im}(\mathcal{N}^i) \cap \mathrm{Ker}(\mathcal{N}^{i+j+1}) \subset \mathrm{Im}(\mathcal{N}^{i-1} : \mathcal{O}_{n-2} \to \mathcal{O}_{n-2}) \cap \mathrm{Ker}(\mathcal{N}^{i+j}),$$

and for $0 < i < n-1$ and $2i+j+1 = n$ (so $i = \frac{n-1-j}{2}$) we have

$$\mathrm{Im}(\mathcal{N}^i) \cap \mathrm{Ker}(\mathcal{N}^{i+j+1}) = \mathrm{Im}(\mathcal{N}^i) = \mathrm{Im}(\mathcal{N}^{\frac{n-1-j}{2}}).$$

As $\mathrm{Im}(\mathcal{N}^{n-1}) = \mathcal{N}^{n-2}\mathcal{N}\mathcal{O} \subset \mathcal{N}^{n-2}\mathcal{O}_{n-2} = \mathcal{O}_{n-2,-n+2} \subset \mathcal{O}_{n-2,j}$, then based on the last two sentences and part (b), for $j \in \{-n+2, \dots, n+2\}$ such that $n-1-j$ is odd, we compute

$$\mathcal{O}_j = \mathrm{Im}(\mathcal{N}^{n-1}) \cap \mathrm{Ker}(\mathcal{N}^{n+j}) + \sum_{-j \leq i < n-1} \mathrm{Im}(\mathcal{N}^i) \cap \mathrm{Ker}(\mathcal{N}^{i+j+1})$$

$$\subset \mathcal{O}_{n-2,j} + \sum_{-j \leq i < n-1} \mathrm{Im}(\mathcal{N}^i : \mathcal{O}_{n-2} \to \mathcal{O}_{n-2}) \cap \mathrm{Ker}(\mathcal{N}^{i+j+1}) = \mathcal{O}_{n-2,j}.$$

As from part (c) we get that $\mathcal{O}_{n-2,j} \subset \mathcal{O}_j$ we conclude that $\mathcal{O}_j = \mathcal{O}_{n-2,j}$ if $n-1-j$ is odd. If $n-1-j$ is even, then an entirely similar computation gives that $\mathcal{O}_j = \mathcal{O}_{n-2,j} + \mathrm{Im}(\mathcal{N}^{\frac{n-1-j}{2}})$. Thus part (c) holds.

To check parts (d) and (e), let $j \in \mathbb{N}$. As

$$\mathcal{O}_{-j} = \sum_{i=j}^{n-1} \mathrm{Im}(\mathcal{N}^i) \cap \mathrm{Ker}(\mathcal{N}^{i-j+1})$$

(see part (a)) we get that $\mathrm{Im}(\mathcal{N}^j) \cap \mathrm{Ker}(\mathcal{N}) \subset \mathcal{O}_{-j} \subset \mathrm{Im}(\mathcal{N}^j)$. This implies that (d) holds.

To compute $\mathcal{P}_{-j}$ we use the snake lemma in the context of the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{-j-1} \arrow[r] \arrow[d, "\mathcal{N}"] & \mathcal{O}_{-j} \arrow[r] \arrow[d, "\mathcal{N}"] & \mathrm{gr}_{-j}(\mathcal{O}) \arrow[r] \arrow[d, "\mathcal{N}"] & 0 \\ 0 \arrow[r] & \mathcal{O}_{-j-2} \arrow[r] & \mathcal{O}_{-j-2} \arrow[r] & \mathrm{gr}_{-j-2}(\mathcal{O}) \arrow[r] & 0. \end{tikzcd} \tag{35}$$

As above we argue that the left and central vertical arrows of Diagram (35) are epimorphisms and from part (d) we get that the first isomorphism of part (e) holds. The third isomorphism of part (e) follows from the well-known Fact 7 recalled below and the second isomorphism of part (e) follows again from the snake lemma applied to the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Im}(\mathcal{N}^{j+1}) \arrow[r] \arrow[d, "\mathcal{N}"] & \mathrm{Im}(\mathcal{N}^{j}) \arrow[r] \arrow[d, "\mathcal{N}"] & \mathrm{Im}(\mathcal{N}^{j})/\mathrm{Im}(\mathcal{N}^{j+1}) \arrow[r] \arrow[d, "\mathcal{N}"] & 0 \\ 0 \arrow[r] & \mathrm{Im}(\mathcal{N}^{j+2}) \arrow[r] & \mathrm{Im}(\mathcal{N}^{j+1}) \arrow[r] & \mathrm{Im}(\mathcal{N}^{j+1})/\mathrm{Im}(\mathcal{N}^{j+2}) \arrow[r] & 0 \end{tikzcd}$$

whose left and central vertical arrow are epimorphisms.

Part (f) follows from either part (c) or part (e). E.g., using the first isomorphism of part (e), as $n \geq 2$ we have $\mathrm{Ker}(\mathcal{N}) = \mathrm{Ker}(\mathcal{O}_{n-2} \xrightarrow{\mathcal{N}} \mathcal{O}_{n-2})$, if $0 \leq j \leq n-3$, then $\mathrm{Im}(\mathcal{N}^j) \cap \mathrm{Ker}(\mathcal{N})$ is $\mathrm{Im}(\mathcal{O}_{n-2} \xrightarrow{\mathcal{N}^j} \mathcal{O}_{n-2}) \cap \mathrm{Ker}(\mathcal{N})$ and hence $\mathcal{P}_{n-2,-j} = \mathcal{P}_{-j}$, and if $j = n-2 \geq 0$, then the three terms

$$\mathcal{P}_{n-2,-n+2} \simeq \mathrm{Im}(\mathcal{N}^{n-2}) \cap \mathrm{Ker}(\mathcal{N}),$$

$$\mathcal{P}_{-n+1} = \mathcal{O}_{-n+1}/\mathcal{O}_{-n} \simeq \mathcal{O}_{-n+1} = \mathrm{Im}(\mathcal{N}^{n-1}) = \mathrm{Im}(\mathcal{N}^{n-1}) \cap \mathrm{Ker}(\mathcal{N}),$$
$$\text{and} \quad \mathcal{P}_{-n+2} \simeq [\mathrm{Im}(\mathcal{N}^{n-2}) \cap \mathrm{Ker}(\mathcal{N})]/[\mathrm{Im}(\mathcal{N}^{n-1}) \cap \mathrm{Ker}(\mathcal{N})]$$

can be put together in a short exact sequence as in part (f).

Part (g) follows from Equation (33). □

Fact 7. *For an endomorphism $\eta : \mathcal{O} \to \mathcal{O}$ of $\mathcal{C}$ and $i \in \mathbb{N}$, the monomorphism*

$$\mathrm{Ker}(\eta^{i+2})/\mathrm{Ker}(\eta^{i+1}) \xrightarrow{\eta} \mathrm{Ker}(\eta^{i+1})/\mathrm{Ker}(\eta^{i})$$

has a cokernel isomorphic to the kernel of the epimorphism

$$\mathrm{Im}(\eta^{i})/\mathrm{Im}(\eta^{i+1}) \xrightarrow{\eta} \mathrm{Im}(\eta^{i+1})/\mathrm{Im}(\eta^{i+2}).$$

7.2 Lattices

We recall that a lattice is a partially ordered set $(\mathcal{L}, \preceq)$ such that for all $x, y \in \mathcal{L}$, their infimum $x \wedge y$ and supremum $x \vee y$ exist in $\mathcal{L}$. The lattice $(\mathcal{L}, \preceq)$ is called distributive if we have $x \wedge (y_1 \vee y_2) = (x \wedge y_1) \vee (x \wedge y_2)$ for all $x, y_1, y_2 \in \mathcal{L}$ (equivalently, we have $x \vee (y_1 \wedge y_2) = (x \vee y_1) \wedge (x \vee y_2)$ for all $x, y_1, y_2 \in \mathcal{L}$), and it is called modular if we have $y_1 \vee (x \wedge y_2) = (y_1 \vee x) \wedge y_2$ for all $y_1, x, y_2 \in \mathcal{L}$ with $y_1 \preceq y_2$ (this condition is self-dual). For instance, the lattice of subobjects of the object $\mathcal{O}$ of the abelian category $\mathcal{C}$ is modular. Also, if $(\mathcal{L}, \preceq)$ is modular, the sublattice generated by arbitrary two chains $\cdots \preceq x_j \preceq x_{j+1} \preceq \cdots$ and $\cdots \preceq y_j \preceq y_{j+1} \preceq \cdots$ in $\mathcal{L}$, is distributive.

We recall that a family of increasing filtrations $(\mathcal{O}_j^{(i)})_{j \in \mathbb{Z}}$ of $\mathcal{O}$ indexed by $i \in \{1, \ldots, m\}$ is called *compatible*, if the sublattice of the lattice of subobjects of $\mathcal{O}$ they generate is distributive (see [Sa], Def. 1.1.13 and Rm. 1.2.15).

Suppose we have two finite increasing filtrations $(\mathcal{O}_j^{(1)})_{j \in \mathbb{Z}}$ and $(\mathcal{O}_j^{(2)})_{j \in \mathbb{Z}}$ of $\mathcal{O}$. For $i \in \{1, 2\}$ let $\mathrm{gr}_j^{(i)}(\mathcal{O}) := \mathcal{O}_j^{(i)} / \mathcal{O}_{j-1}^{(i)}$ and let $c_i, d_i \in \mathbb{Z}$ with $c_i \leq d_i$ be such that $\mathrm{gr}_j^{(i)}(\mathcal{O}) = 0$ if $j \notin \{c_i, c_i + 1, \ldots, d_i\}$. Let $\mathcal{L}^{(1)(2)}$ be the distributive lattice of subobjects of $\mathcal{O}$ generated by all subobjects $\mathcal{O}_j^{(i)}$ with $(i, j) \in \{1, 2\} \times \mathbb{Z}$. By Zassenhaus lemma, for all $j_1, j_2 \in \mathbb{Z}$ we have

$$\mathrm{gr}_{j_1}^{(1)}(\mathrm{gr}_{j_2}^{(2)}(\mathcal{O})) \simeq \mathrm{gr}_{j_2}^{(2)}(\mathrm{gr}_{j_1}^{(1)}(\mathcal{O})) \simeq \mathcal{O}_{j_1}^{(1)} \cap \mathcal{O}_{j_2}^{(2)} / [(\mathcal{O}_{j_1-1}^{(1)} \cap \mathcal{O}_{j_2}^{(2)}) + (\mathcal{O}_{j_1}^{(1)} \cap \mathcal{O}_{j_2-1}^{(2)})].$$

Let $\mathcal{A} := \{c_1, \ldots, d_1\} \times \{c_2, \ldots, d_2\}$ be endowed with the product $\leq$ order: for $(x_1, y_1), (x_2, y_2) \in \mathcal{A}$ we have $(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2$. Let $(\mathcal{L}_{\mathcal{A}}, \subset)$ be the lattice of *initial segments* of $\mathcal{A}$: a subset $\mathcal{A}' \subset \mathcal{A}$ is an initial segment (i.e., we have $\mathcal{A}' \in \mathcal{L}_{\mathcal{A}}$) if and only if

$$(x \leq y \in \mathcal{A}') \Rightarrow (x \in \mathcal{A}').$$

We have a surjection $\psi : \mathcal{L}_{\mathcal{A}} \to \mathcal{L}^{(1)(2)}$ which is a lattice homomorphism defined for all $n \in \mathbb{N}^*$, $j_{1,1}, \ldots, j_{1,n} \in \{c_1, \ldots, d_1\}$ and $j_{2,1}, \ldots, j_{2,n} \in \{c_2, \ldots, d_2\}$ by the rule

$$\psi(\cup_{\ell=1}^n \{c_1, c_1 + 1, \ldots, j_{1,\ell}\} \times \{c_2, c_2 + 1, \ldots, j_{2,\ell}\}) := \sum_{\ell=1}^n O_{j_{1,\ell}}^{(1)} \cap O_{j_{2,\ell}}^{(2)}. \quad (36)$$

If $\mathcal{A}', \mathcal{A}'' \in \mathcal{L}_{\mathcal{A}}$ are such that $\mathcal{A}' \subset \mathcal{A}''$ and $\mathcal{A}'' \setminus \mathcal{A}' = \{(j_1, j_2)\}$, then we have

$$\psi(\mathcal{A}'')/\psi(\mathcal{A}') \simeq \mathrm{gr}_{j_1}^{(1)}(\mathrm{gr}_{j_2}^{(2)}(\mathcal{O})) \quad (37)$$

via the natural monomorphism $\mathcal{O}^{(1)}_{j_1} \cap \mathcal{O}^{(2)}_{j_2} \subset \psi(\mathcal{A}'')$.

We will apply the previous paragraph in the content of the two natural finite filtrations associated to a nilpotent endomorphism $\mathcal{N} : \mathcal{O} \to \mathcal{O}$ of index of nilpotency n: for $j \in \mathbb{N}^*$, let $\mathcal{O}^{(1)}_j := \mathrm{Ker}(\mathcal{N}^j)$ and $\mathcal{O}^{(2)}_j := \mathcal{O}$, and for $j \in \mathbb{Z} \setminus \mathbb{N}^*$ let $\mathcal{O}^{(1)}_j := 0$ and $\mathcal{O}^{(2)}_j := \mathrm{Im}(\mathcal{N}^{-j})$. We can take, say, $c_1 = c_2 = -n$ and $d_1 = d_2 = n$. As these two finite filtrations depend only on $\mathcal{N}$, we call $\mathcal{L}_{\mathcal{N}} := \mathcal{L}^{(1)(2)}$ the *subobject lattice defined by* $\mathcal{O} \xrightarrow{\mathcal{N}} \mathcal{O}$. Deligne's pieces $\mathcal{O}_j$ and $\mathrm{gr}_j(\mathcal{O})$ (see Subsection 7.1) are given by the following formulas (see Proposition 5 (a) and Equations (36) and (37))

$$\mathcal{O}_j = \psi(\{(i_1, i_2) | -n \leq i_1, i_2 \leq n, i_1 + i_2 \leq j + 1\}), \tag{38}$$

$$\mathrm{gr}_j(\mathcal{O}) = \oplus_{i \in \{-n,\ldots,n\}, -n \leq j+1-i \leq n} \mathrm{gr}^{(1)}_i(\mathrm{gr}^{(2)}_{j+1-i}(\mathcal{O})). \tag{39}$$

From Zassenhaus lemma and Proposition 5 (e) we get

$$\mathrm{gr}^{(1)}_1(\mathrm{gr}^{(2)}_j(\mathcal{O})) = \mathcal{P}_j \quad \text{if} \quad j \leq 0. \tag{40}$$

If $i \geq 2$ and $j \leq 0$ we check that we have an isomorphism

$$\mathrm{gr}^{(1)}_i(\mathrm{gr}^{(2)}_j(\mathcal{O})) \xrightarrow{\mathcal{N}} \mathrm{gr}^{(1)}_{i-1}(\mathrm{gr}^{(2)}_{j-1}(\mathcal{O})).$$

The epimorphism part follows from Zassenhaus lemma and the monomorphism part follows from Equation (39) $i + j \geq 2$ as $\mathrm{gr}_{i+j-1}(\mathcal{O}) \xrightarrow{\mathcal{N}} \mathrm{gr}_{i+j-3}(\mathcal{O})$ is a monomorphism (its kernel being $\mathcal{P}_{i+j-1} = 0$, see Subsection 7.1) and from Equations (39) and (40) if $i + j \leq 1$ as we have

$$\mathcal{P}_{i+j-1} = \mathrm{gr}^{(1)}_1(\mathrm{gr}^{(2)}_{i+j-1}(\mathcal{O})) \subset \mathrm{Ker}(\mathrm{gr}_{i+j-1}(\mathcal{O}) \xrightarrow{\mathcal{N}} \mathrm{gr}_{i+j-3}(\mathcal{O}))$$

and from the definition of $\mathcal{P}_{i+j-1}$ we get that the last inclusion is an identity.

For integers $1 \leq i \leq n$, $-n \leq j \leq 0$, we have (see last two paragraphs)

$$\mathrm{gr}^{(1)}_i(\mathrm{gr}^{(2)}_j(\mathcal{O})) = \mathcal{P}_{j-i+1}, \tag{41}$$

and for all other $(i, j) \in \mathbb{Z}^2$ we have

$$\mathrm{gr}^{(1)}_i(\mathrm{gr}^{(2)}_j(\mathcal{O})) = 0. \tag{42}$$

In particular, Equation (39) coincides with [De], Eq. (1.6.4.1), i.e., with Equation (33).

Corollary 6. *With $\mathcal{C}$, $\mathcal{N} : \mathcal{O} \to \mathcal{O}$, and $n \in \mathbb{N}^*$ as above, let $\mathcal{C}_1$ be a full subcategory of the abelian category $\mathcal{C}$ which is abelian. Let $\mathcal{C}_2$ be a full subcategory of $\mathcal{C}_1$ which contains the zero object and such that if $\mathcal{T}$ is an object of $\mathcal{C}_1$ which has a finite increasing filtration whose consecutive quotients are objects of $\mathcal{C}_2$, then $\mathcal{T}$ itself is an object of $\mathcal{C}_2$. If for each $j \in \{1, \ldots, n\}$, $\mathrm{gr}_j^{(1)}(\mathcal{O})$ is an object of $\mathcal{C}_1$ (equivalently, for each $j \in \{-n+1, \ldots, 0\}$, $\mathrm{gr}_j^{(2)}(\mathcal{O})$ is an object of $\mathcal{C}_1$), then the following five statements are equivalent:*

① *For all $j \in \{1, \ldots, n\}$, $\mathrm{gr}_j^{(1)}(\mathcal{O})$ is an object of $\mathcal{C}_2$.*

② *For all $j \in \{-n+1, \ldots, 0\}$, $\mathrm{gr}_j^{(2)}(\mathcal{O})$ is an object of $\mathcal{C}_2$.*

③ *For all $j \in \{-n, \ldots, n-1\}$, $\mathrm{gr}_j(\mathcal{O})$ is an object of $\mathcal{C}_2$.*

④ *For all elements $\mathcal{T}_1 \subset \mathcal{T}_2$ of $\mathcal{L}_{\mathcal{N}}$ such that $\mathcal{T}_2/\mathcal{T}_1$ is an object of $\mathcal{C}_1$, $\mathcal{T}_2/\mathcal{T}_1$ is in fact an object of $\mathcal{C}_2$.*

⑤ *For all $j \in \{0, \ldots, n-1\}$, $\mathcal{P}_{-j}$ is an object of $\mathcal{C}_2$.*

Proof: For $d \in \{1, 2, 3, 4\}$, we check that ⓓ $\Leftrightarrow$ ⑤. The case $d = 1$ gets reduced to $d = 2$ by Zassenhaus lemma. If $d = 2$, then the equivalence follows from Equations (41) and (42) applied with j fixed and i varying (see hypotheses). If $d = 3$ (resp. $d = 4$), then the equivalence follows from Equations (41), (42) and (39) (resp. and (37) and the hypothesis on $\mathcal{C}_2$). $\square$

7.3 Characterizations of fully BT-balanced objects

We first recall the following well-known fact:

Fact 8. *Let $0 \to \mathcal{B}_1 \to \mathcal{B}_2 \to \mathcal{B}_3 \to 0$ be a short exact sequence of* fppf *$\mathbb{Z}/p\mathbb{Z}$-sheaves over some scheme X. If i, j, ℓ are distinct elements of $\{1, 2, 3\}$ such that $\mathcal{B}_i$ and $\mathcal{B}_j$ are BT_1s over X, then $\mathcal{B}_\ell$ is also a BT_1 over X.*

For a module $\mathcal{W}$ over a discrete valuation ring $\mathcal{D}$ of uniformizer π and residue field $\mathcal{K}$ and for all $i \in \mathbb{N}$, let $\overline{\mathcal{W}} := \mathcal{W}/\pi\mathcal{W}$ and let $\overline{\mathcal{W}}_i$ be the imagine of $\mathcal{W}[\pi^i] := \{x \in \mathcal{W} | \pi^i x = 0\}$ in $\overline{\mathcal{W}}$. We get an increasing (possibly non-exhaustive) filtration $(\overline{\mathcal{W}}_i)_{i \geq 0}$ of the $\mathcal{K}$-vector space $\overline{\mathcal{W}}$. For $i \in \mathbb{N}^*$, let $\mathrm{gr}_i(\overline{\mathcal{W}}) := \overline{\mathcal{W}}_i/(\overline{\mathcal{W}}_{i-1})$. Recall that $\mathcal{W}_{\mathrm{tors}} = \cup_{i \in \mathbb{N}} \mathcal{W}[\pi^i]$ and we define

$$\mathrm{gr}_\infty(\overline{\mathcal{W}}) := \overline{\mathcal{W}}/(\cup_{i \geq 0} \overline{\mathcal{W}}_i) \simeq (\mathcal{W}/\mathcal{W}_{\mathrm{tors}})/\pi(\mathcal{W}/\mathcal{W}_{\mathrm{tors}}).$$

Lemma 9. *A $\mathcal{D}$-linear map $\Omega : \mathcal{W} \to \mathcal{V}$ between two finitely generated $\mathcal{D}$-modules is left invertible if and only if for all $i \in \mathbb{N}^* \cup \{\infty\}$ the $\mathcal{K}$-linear map $\mathrm{gr}_i(\Omega) : \mathrm{gr}_i(\overline{\mathcal{W}}) \to \mathrm{gr}_i(\overline{\mathcal{V}})$ is injective .*

Proof: The condition is clearly necessary. For sufficiency, let $n \in \mathbb{N}^*$ be such that we have two direct sum decompositions $\mathcal{W} = \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_n \oplus \mathcal{W}_\infty$ and $\mathcal{V} = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_n \oplus \mathcal{W}_\infty$, where for $i \in \{1, \ldots, n, \infty\}$, $\mathcal{W}_i$ and $\mathcal{V}_i$ are free $\mathcal{D}/\pi^i\mathcal{D}$-modules, with $\mathcal{D}/\pi^\infty\mathcal{D} := \mathcal{D}$. So Ω is given by $\mathcal{D}$-linear maps $\Omega_{j,i} : \mathcal{W}_i \to \mathcal{V}_j$ with $i, j \in \{1, \ldots, n, \infty\}$. For $j > i$, we have $\Omega_{j,i}(\mathcal{W}_i) \to \pi\mathcal{V}_j$, i.e., Ω modulo π is 'upper-triangular'. As $\mathrm{gr}_i(\Omega)$ is injective, Ω_{ii} modulo π is injective, thus maps a $\mathcal{K}$-basis of $\overline{\mathcal{W}_i}$ to a subset of a $\mathcal{K}$-basis of $\overline{\mathcal{V}}_i$ and hence it is left invertible. This implies that there exists a $\mathcal{D}$-linear map $\Phi_{ii} : \mathcal{V}_i \to \mathcal{W}_i$ such that $\Phi_{ii} \circ \Omega_{ii} = 1_{\mathcal{W}_i}$. Let $\Phi := \Phi_{11} \oplus \cdots \oplus \Phi_{nn} \oplus \Phi_{\infty\infty} : \mathcal{V} \to \mathcal{W}$. The reduction of $\Phi \circ \Omega : \mathcal{W} \to \mathcal{V}$ modulo π is 'upper-triangular' with identity functions on the diagonal, hence invertible. Thus $\Phi \circ \Omega$ is surjective by Nakayama lemma, hence bijective (as its reductions modulo powers of π are so by reasons of length). As $\Phi \circ \Omega$ is an isomorphism, Ω is left invertible. $\square$

Theorem 6. *Let G be an object of $\mathcal{F}_k$. Let $m \in \mathbb{N}^*$ be the smallest integer such that p^m annihilates G. Then the following five statements are equivalent:*

① *The object G is fully* BT*-balanced in the sense of Definition 2 (f).*

② *For each $i \in \{0, \ldots, m-1\}$, $p^iG/p^{i+1}G$ is a* BT_1*.*

③ *For each $i \in \{0, \ldots, m-1\}$, $G[p^{i+1}]/G[p^i] \simeq G[p] \cap p^iG$ is a* BT_1*.*

④ *If $(G_j)_{j\in\{-m,\ldots,m-1\}}$ is Deligne's finite increasing filtration of G defined by the nilpotent endomorphism $p : G \to G$, then for all $j \in \{-m+1, \ldots, m-1\}$ (equivalently, for all $j \in \{-m+1, \ldots, 0\}$ or all $j \in \{0, \ldots, m-1\}$), G_{j+1}/G_j is a* BT_1*.*

⑤ *Let $\mathcal{L}_G$ be the subobject lattice defined by $G \xrightarrow{p} G$ (see Subsection 7.2).*[11] *If H_1 and H_2 are in $\mathcal{L}_G$ and H_1 is a subgroup of H_2 such that H_2/H_1 is annihilated by p, then H_2/H_1 is a* BT_1*.*

Proof: The fact that the last four statements are equivalent follows from Corollary 6 applied with $(\mathcal{C}, \mathcal{C}_1) = (\mathcal{F}_k, \mathcal{F}_{k,1})$, with $\mathcal{C}_2$ as the full subcategory of $\mathcal{F}_{k,1}$ formed by objects which are BT_1s, and with $\mathcal{O} \xrightarrow{\mathcal{N}} \mathcal{O}$ being $G \xrightarrow{p} G$ (the hypotheses of Corollary 6 hold, see Fact 8).[12]

[11] We have $\{G_j | -m \leq j \leq m-1\} \subset \mathcal{L}_G$, see either Proposition 5 (a) or Subsection 7.2. But for $m \geq 3$, in general $\mathcal{L}_G$ has more than $2m$ elements as in the case of $\oplus_{i=1}^m D[p^i]$ with D a non-trivial BT over k.

[12] The equivalence ② $\Leftrightarrow$ ③ can be simply checked as follows. If ② holds, then from Fact 8 we get that for $i \in \{0, \ldots, m-1\}$ the kernel $\mathrm{Ker}(p : p^iG/p^{i+1}G) \to p^{i+1}G/p^{i+2}G)$ is a BT_1 over k and by a decreasing induction on $i \in \{m-1, \ldots, 0\}$ we get, based on Fact 8

Let $\mathcal{F}_{W(k)}$ and $\mathcal{F}_{B(k)}$ be the categories of finite flat commutative group schemes annihilated by a power of p over $\operatorname{Spec} W(k)$ and $\operatorname{Spec} B(k)$ (respectively). Let $\mathcal{F}_{W(k)}^{\mathrm{conn}}$ be the full subcategory of $\mathcal{F}_{W(k)}$ defined by objects whose fibers over k are connected. The category $\mathcal{F}_{B(k)}$ is abelian. If $p > 2$, the category $\mathcal{F}_{W(k)}$ is abelian, and the two pullback functors $\mathcal{F}_{W(k)} \to \mathcal{F}_k$ and $\mathcal{F}_{W(k)} \to \mathcal{F}_{B(k)}$ are exact and faithful. For all primes p, the category $\mathcal{F}_{W(k)}^{\mathrm{conn}}$ is abelian and the two pullback functors $\mathcal{F}_{W(k)}^{\mathrm{conn}} \to \mathcal{F}_k$ and $\mathcal{F}_{W(k)}^{\mathrm{conn}} \to \mathcal{F}_{B(k)}$ are exact and faithful. The last two sentences follow from [R], Thm. 3.3.3 and Rm. 3.3.5 as the generic fiber functors are fully faithful and exact.[13]

To check that ① ⇒ ⑤ we can assume that $G = G^0$ and we consider a finite flat commutative group scheme G^{lift} over $\operatorname{Spec} W(k)$ whose reduction modulo p is G. As the category $\mathcal{F}_{W(k)}^{\mathrm{conn}}$ is abelian, we have a smallest lattice $\mathcal{L}_G^{\mathrm{lift}}$ of flat closed subgroup schemes of G^{lift} with respect to the subgroup scheme partial order which contains each p^iG^{lift} and $G^{\mathrm{lift}}[p^i]$ with $i \in \{0, \ldots, m-1\}$. The pullback functor $\mathcal{F}_{W(k)}^{\mathrm{conn}} \to \mathcal{F}_k$ induces a lattice isomorphism $\mathcal{L}_G^{\mathrm{lift}} \to \mathcal{L}_G$. Thus, if $H_1^{\mathrm{lift}}, H_2^{\mathrm{lift}} \in \mathcal{L}_G^{\mathrm{lift}}$ lift H_1 and H_2 (respectively), then H_1^{lift} is a closed subgroup scheme of H_2^{lift}, p annihilates $H_2^{\mathrm{lift}}/H_1^{\mathrm{lift}}$ and hence $H_2^{\mathrm{lift}}/H_1^{\mathrm{lift}}$ is a BT_1 over $\operatorname{Spec} W(k)$. Thus H_2/H_1 is a BT_1 over k. Hence ① ⇒ ⑤.

To check that ③ ⇒ ①, we can assume that G is connected and we embed it into a connected BT D over k. Let $E := D/G$ and let L, J and M be as usual the (contravariant) Dieudonné modules of D, E and G (respectively). We get a short exact sequence of left $\mathbb{E}$-modules $0 \to J \to L \to M \to 0$.

We will use the notation before Lemma 9 with the triple $(\mathcal{D}, \pi, \mathcal{W})$ equal to $(W(k), p, M)$, $M = \mathbb{D}(G)$. So $\overline{M} = M/pM$, $\overline{L} = L/pL$, etc. As $\overline{L}_{-1} := \operatorname{Ker}(\overline{L} \to \overline{M})$ is $\mathbb{D}(D[p]/G[p])$ with $D[p]/G[p]$ a BT_1, there exists a k-basis $\{\bar{x}_{01}, \ldots, \bar{x}_{0d_0}, \bar{y}_{01}, \ldots, y_{0c_0}\}$ of $\overline{L}_{-1}$ such that $\{V\bar{x}_{01}, \ldots, V\bar{x}_{0d_0}\}$ is a k-basis of $\operatorname{Ker}(\overline{L}_{-1} \xrightarrow{F} \overline{L}_{-1}) = V\overline{L}_{-1}$. As J surjects onto $\overline{L}_{-1}$ let $(x'_{01}, \ldots, x'_{0d_0}, y'_{01}, \ldots, y'_{0,c_0}) \in J^{c_0+d_0}$ that lifts $(\bar{x}_{01}, \ldots, \bar{x}_{0d_0}, \bar{y}_{01}, \ldots, y_{0c_0})$.

For $i \in \{1, \ldots, m\}$ we proceed similarly. As ③ holds for G, we have $\operatorname{Ker}(\overline{M}_i \xrightarrow{F} \overline{M}_i) = V\overline{M}_i = \operatorname{Im}(VM[p^i] \to \overline{M})$. As $\overline{M}_i = \operatorname{Im}(M[p^i] \to \overline{M})$ is $\mathbb{D}(G[p]/(G[p] \cap p^iG))$, we lift the elements of a k-basis of the quotient $\operatorname{Ker}(\overline{M}_i \xrightarrow{F} \overline{M}_i)/\operatorname{Ker}(\overline{M}_{i-1} \xrightarrow{F} \overline{M}_{i-1})$ to elements $V\bar{x}_{i1}, \ldots, V\bar{x}_{id_i}$ of

and on Fact 7 applied to the endomorphism $p : G \to G$, that $p^iG/p^{i+1}G$ is a BT_1 over k, i.e., ③ holds. The implication ③ ⇒ ② is argued similarly, or it follows via Cartier duality from the implication ② ⇒ ③ applied to G^{t}.

[13] If $p = 2$, strictly speaking, [R], Rm. 3.3.5 is stated for the local-local context but the arguments of it work for the connected context.

$\mathrm{Ker}(\overline{M}_i \xrightarrow{F} \overline{M}_i)$ with $\{x_{i1}, \ldots, x_{id_i}\} \subset \bar{M}_i$ and we enlarge this set to a k-basis $\{\bar{x}_{i1}, \ldots, \bar{x}_{id_i}, \bar{y}_{i1}, \ldots, \bar{y}_{ic_i}\}$ of a direct supplement of $\overline{M}_{i-1}$ in $\overline{M}_{i-1}$. If $(x_{i1}, \ldots, x_{id_i}, y_{i1}, \ldots, y_{ic_i}) \in M[p^i]^{c_i+d_i}$ lifts $(\bar{x}_{i1}, \ldots, \bar{x}_{id_i}, \bar{y}_{i1}, \ldots, \bar{y}_{ic_i})$, then we use these x_{ij_i} elements (resp. y_{ij_i} elements) with $i \in \{1, \ldots, m\}$ and $j_i \in \{1, \ldots, d_i\}$ (resp. $j_i \in \{1, \ldots, c_i\}$) to get a $W(k)$-linear map $\Omega_x : \oplus_{i=1}^m W_i(k)^{d_i} \to M$ (resp. $\Omega_y : \oplus_{i=1}^m W_i(k)^{c_i} \to M$). From Lemma 9, Ω_x, $V\Omega_x$, Ω_y are injective with images direct summands O_x, VO_x and O_y of M such that $M = O_x \oplus O_y$. If $x'_{i,j_i} \in L$ (resp. $y'_{i,j_i} \in L$) maps to x_{ij_i} (resp. y_{ij_i}) for all $i \in \{1, \ldots, m\}$ and $j_i \in \{1, \ldots, d_i\}$ (resp. $j_i \in \{1, \ldots, c_i\}$), then Lemma 9 gives that $\{x'_{ij_i} | 0 \le i \le m, 1 \le j_i \le d_i\} \cup \{y'_{ij_i} | 0 \le i \le m, 1 \le j_i \le c_i\}$ is a $W(k)$-basis of L such that $P_L^1 := \oplus_{i=0}^m \oplus_{j_i=1}^{d_i} W(k)Vx'_{ij_i}$ is a direct summand of L. If

$$J' := \bigoplus_{i=0}^{m} (\oplus_{j_i=1}^{d_i} p^i W(k) x'_{ij_i} \oplus_{j_i=1}^{d_i} p^i W(k) y'_{ij_i}),$$

then $P_{J'}^1 := \oplus_{i=0}^m (\oplus_{j_i=1}^{d_i} p^i W(k) V x'_{ij_i})$ is a direct summand of J', we have an inclusion $J' \subset J$ and the resulting epimorphism $L/J' \to L/J = M$ is an isomorphism as $M = O_x \oplus O_y$. Thus $J' = J$ and we get an inclusion $(J, P_J^1, F, V) \to (L, P_L^1, F, V)$ between filtered Dieudonné F-crystals over k. From the Grothendieck–Messing deformation theory[14] (for $p = 2$ see also [V3], Lem. 2.2.5 and Prop. 2.2.6) we get that it corresponds to a homomorphism $D^{\mathrm{lift}} \to E^{\mathrm{lift}}$ of BTs over $\mathrm{Spec}\, W(k)$ whose kernel is the lift G^{lift} of G. Thus ③ ⇒ ①. $\square$

Example 11. *If $\mathcal{S}$ is a supersingular elliptic curve over k, for $j \in \mathbb{N}^*$ let $H_j := \mathcal{S}[F^j] = \mathrm{Ker}(\mathrm{Frob}_{\mathcal{S}}^j : \mathcal{S} \to \mathcal{S}^{(p^j)})$. So $H_1 \simeq \boldsymbol{\alpha}_p$ and $H_3/H_1 \simeq H_2 = \mathcal{S}[p]$ is a* BT$_1$*. We check that $H := H_3 \oplus \boldsymbol{\alpha}_p$ is* BT*-balanced. Let $\delta : H \to H_3/H_1$ be the epimorphism which extends the projection $H_3 \to H_3/H_1$ and satisfies $\delta(\boldsymbol{\alpha}_p) = H_2/H_1$. The closed embedding $H_2 \to H_3$ and the epimorphism $H_2 \to \boldsymbol{\alpha}_p$ induced by -1 times the inverse $H_2/H_1 \to \boldsymbol{\alpha}_p$ of the isomorphism $\boldsymbol{\alpha}_p \to H_2/H_1$ defined by δ give a closed embedding $\delta_2 : H_2 \to H$ with* $\mathrm{Im}(\delta_2) = \mathrm{Ker}(\delta)$*. The resulting short exact sequence $0 \to H_2 \xrightarrow{\delta_2} H \xrightarrow{\delta} H_3/H_1 \to 0$ proves that H is* BT*-balanced. As the kernel of the natural epimorphism $H \to (H_3/H_2) \oplus \boldsymbol{\alpha}_p \simeq \boldsymbol{\alpha}_p^2$ is H_2, we get that the notion* BT*-balanced is*

[14] Grothendieck's statement for divided power thickening $S \to S'$ of schemes where p is locally nilpotent in [G2], p. 111: "Alors, ce foncteur est une équivalence de catégories si S' est à puissances divisées nilpotentes ou si on se borne à des groupes G, G' qui sont infinitésimaux ou ind-unipotents."

not stable under taking either quotients or (via Cartier duality) kernels of epimorphisms. We have $\boldsymbol{\alpha}_p(H) \cap \boldsymbol{\beta}_p(H) = \boldsymbol{\alpha}_p$, $a_H = a_{H^{\mathrm{t}}} = 2$ and H has no non-trivial BT_1 *subgroup invariant under* $\mathrm{Aut}(H)$.

An object of $\mathcal{F}_{k,1}$ is a BT_1 if and only if it is fully BT-balanced and if and only if it is BT-balanced (see Fact 8). But for $m \geq 2$, the three notions are distinct: BT_m implies fully BT-balanced which implies BT-balanced. The notion fully BT-balanced object of $\mathcal{F}_{k,m}$ is stable under taking direct summands and direct sums but not under extensions (see Example 11). The notion BT-balanced object of $\mathcal{F}_{k,m}$ is not stable under taking direct summands (see Example 11) but is stable under extensions.

7.4 Applications to balanced objects

We recall that H is an object of $\mathcal{F}_{k,m}$. We endow H with a left action by ∇, where ∇ is a group or a ring or a monoid or a monoid scheme over k. In practice, ∇ will be a reduced monoid or $\mathbb{B}$ or another finite flat separable $\mathbb{Z}_p$-algebra naturally associated to H and $\mathbb{B}$. We say that the left ∇-module H is BT-balanced if it has an increasing separated and exhaustive filtration invariant under ∇ and such that each consecutive quotient of it is a BT_1 endowed with a left ∇-action.

We say that the left ∇-module H is balanced if H and $H^{(p)}$ have the same class in the corresponding Grothendieck group $G_0(\nabla)$; it suffices to check this over a perfect field extension of k, such as $\bar{k}$. Thus H is balanced if $H_{\bar{k}}$ and its left action are defined over $\mathbb{F}_p$. If H is étale (resp. of multiplicative type), then H is balanced as the Frobenius (resp. Verschiebung) homomorphism $F_H : H \to H^{(p)}$ (resp. $V_H : H^{(p)} \to H$) is an isomorphism.

If H_{mult} is the multiplicative type part of H, then the usual filtration $H_{\mathrm{mult}} \subset H^0 \subset H$ is invariant under ∇ even in the case when ∇ is a non-reduced monoid group scheme over k. This implies that H is balanced if and only if H^0/H_{mult} is so and we have a 'product decomposition' isomorphism

$$G_0(\nabla) \simeq G_0^{\text{loc-loc}}(\nabla) \times G_0^{\text{ét}}(\nabla) \times G_0^{\text{mult}}(\nabla).$$

Moreover, $G_0^{\text{loc-loc}}(\nabla)$ is the Grothendieck group $G_0(\mathrm{Rep}(\nabla))$ of the abelian category of finite dimensional k-linear representations of ∇: the isomorphism

$$G_0^{\text{loc-loc}}(\nabla) \to G_0(\mathrm{Rep}(\nabla))$$

maps $[H]$ to either

$$\sum_{i\geq 0}[\mathrm{Lie}(H/\mathrm{Ker}(F_H^i))] = \sum_{i\geq 0}[\mathrm{Lie}(\mathrm{Im}(F_H^i))] \quad \text{or}$$

$$\sum_{j\in\mathbb{Z}}[\mathrm{Lie}((\boldsymbol{\alpha}_p(G)_j)/(\boldsymbol{\alpha}_p(G)_{j-1}))] \quad \text{or} \quad \sum_{j\in\mathbb{Z}}[\mathrm{Lie}((\boldsymbol{\beta}_p(G)_j)/(\boldsymbol{\beta}_p(G)_{j-1}))].$$

Considering the Frobenius homomorphism $F_H : H \to H^{(p)}$, we get that

$$H \text{ is balanced} \Longleftrightarrow [\mathrm{Lie}(\mathrm{Ker}(F_H))] = [\mathrm{Lie}(\mathrm{Coker}(F_H))] \in G_0(\mathrm{Rep}(\nabla)).$$

Lemma 10. *The following two properties hold:*

(a) *If G is fully* BT*-balanced and endowed with a left action by ∇, then the left ∇-module G is* BT*-balanced as well as balanced.*

(b) *If the left ∇-module H is* BT*-balanced, then it is also balanced.*

Proof: As $G[p^i]$ is invariant under ∇ for all $i \in \mathbb{N}$, based on Theorem 6, to prove part (a) we can assume that G is a BT_1. As we have short exact sequences of left ∇-modules

$$0 \to \mathrm{Ker}(F_G) \to G \to \mathrm{Ker}(V_G) \to 0, \quad 0 \to \mathrm{Ker}(V_G) \to G^{(p)} \to \mathrm{Ker}(F_G) \to 0,$$

we have $[G] - [G^{(p)}] = 0 \in G_0(\nabla)$. To prove part (b) we can assume that H is a BT_1 and this case follows from part (a). □

8 Proof of Theorem 4

We first proof the following fact:

Fact 9. *If K is a finitely generated field extension of $\bar{k}$, then we have*

$$\cap_{n\in\mathbb{N}} K^{p^n} = \bar{k}.$$

Proof: This follows from facts in [E], p. 239. Alternatively, if there exists $x \in (\cap_{n\in\mathbb{N}} K^{p^n}) \setminus \bar{k}$, then we have strict inclusions

$$\bar{k} \subset \bar{k}(x) \subset \bar{k}(x^{1/p}) \subset \bar{k}(x^{1/p^2}) \subset \cdots \subset K$$

and so the subfield $\cup_{n\in\mathbb{N}} k(\bar{x}^{1/p^n})$ of K is not a finitely generated field extension of $\bar{k}$ which contradicts [Bo1], Ch. 5, Sect. 14, Cor. 3 of Prop. 17. □

To prove Theorem 4, we first remark that by a compactness argument there exists an element $\tau \in \mathrm{Gal}(\mathbb{F}_p) = \mathrm{Gal}(\bar{k}/\mathbb{F}_p)$ such that we have an identity $f = \tau \circ h \in \mathrm{Maps}(Y(\bar{k}), \bar{k})$. If f, h are constant on the irreducible components of $Y_{\bar{k}}$, then τ is determined up to an open subgroup of $\mathrm{Gal}(\mathbb{F}_p)$ and hence, as $\mathbb{Z}$ is dense in $\mathrm{Gal}(\mathbb{F}_p) = \widehat{\mathbb{Z}}$, we can choose it to be in $W(\bar{k}/\mathbb{F}_p)$.

Thus, by replacing k by a finite field extension of it, we can assume that Y is geometrically irreducible and that f and h are not constant, and we have to show that $f = \tau \circ h$ on closed points of $Y_{\bar{k}}$ forces τ to be in $W(\bar{k}/\mathbb{F}_p)$.

By shrinking we can assume that Y is affine and smooth and (based also on Fact 9 by taking p-th power roots) that $f, h \in R \setminus R^p$, with $\mathrm{Spec}\, R = Y_{\bar{k}}$, define smooth morphism $\nu_f : Y_{\bar{k}} \to \mathbb{A}^1_{\bar{k}}$ and $\nu_h : Y_{\bar{k}} \to \mathbb{A}^1_{\bar{k}}$ (respectively). Let $\mathbb{A}_f := \mathrm{Im}(\nu_f)$ and $\mathbb{A}_h := \mathrm{Im}(\nu_h)$. For $x, y \in Y(\bar{k})$ we have $f(x) = f(y)$ if and only if $h(x) = h(y)$, and thus $Y_{\bar{k}} \times_{\mathbb{A}_f} Y_{\bar{k}} = Y_{\bar{k}} \times_{\mathbb{A}_h} Y_{\bar{k}}$. By descent theory (see [SGA1], Exp. VIII, Cor. 5.3), $\mathbb{A}_f$ and $\mathbb{A}_h$ are the categorical quotients of the equivalence relations $Y_{\bar{k}} \times_{\mathbb{A}_f} Y_{\bar{k}}$ and $Y_{\bar{k}} \times_{\mathbb{A}_h} Y_{\bar{k}}$ (respectively) on $Y_{\bar{k}}$. Thus there exists an isomorphism $\varphi : \mathbb{A}_f \to \mathbb{A}_h$ over $\bar{k}$ such that we have $\tau(\nu_f(x)) = \nu_h(x) = \varphi(\nu_f(x))$ for all $x \in Y_{\bar{k}}(\bar{k})$; it extends to an automorphism $\omega \in \mathrm{Aut}(\mathbb{P}^1_{\bar{k}}) = \mathbf{PGL}_2(\bar{k})$. For $\alpha \in \mathbb{A}_f(\bar{k}) \subset \bar{k}$ we have $\tau(\alpha) = \omega(\alpha)$ and from the below fact we get that $\tau \in W(\bar{k}/\mathbb{F}_p)$. $\square$

Fact 10. *If $(\tau, \omega) \in \mathrm{Gal}(\mathbb{F}_p) \times \mathbf{PGL}_2(\bar{k})$ is such that we have $\tau(\alpha) = \omega(\alpha)$ for all but a finite number of $\alpha \in \mathbb{P}^1(\bar{k})$, then (τ, ω) is the identity element.*

Proof: It suffices to prove that τ is the identity element. As ω has finite order m, we have $\tau^m(\alpha) = \alpha$ for all but a finite number of $\alpha \in \mathbb{P}^1(\bar{k})$. As such elements α generate $\bar{k}$, we have $\tau^m = 1_{\bar{k}}$. Thus $\tau = 1_{\bar{k}}$ (as $\widehat{\mathbb{Z}}$ is torsion free). $\square$

9 Proof of Theorem 3

Let $S(\mathbf{M})$ be the set (of representatives) of isomorphism classes of finite dimensional simple left $\mathbf{M}$-modules. The abelian group $K_0(\mathbf{M})$ is canonically identified with the free abelian group on $S(\mathbf{M})$. For a $\mathbf{M}$-module Z we define inductively $Z^{(\sigma^0)} := Z$ and $Z^{(\sigma^i)} := (Z^{(\sigma^{i-1})})^{(\sigma)}$ for $i \in \mathbb{N}^*$.

As $\mathrm{Frob}_{\mathbf{M}} : \mathbf{M} \to \mathbf{M}^{(p)}$ is schematically dominant, the functor $Z \mapsto Z^{(\sigma)}$ is fully faithful and induces a bijection between $\mathbf{M}$-submodules of Z and $\mathbf{M}$-submodules of $Z^{(\sigma)}$. In particular, $Z^{(\sigma)}$ is simple if and only if Z is, and the rule $[Z] \to [Z^{(\sigma)}]$ defines an injective self map $\sigma : S(\mathbf{M}) \to S(\mathbf{M})$. We get a partition of $S(\mathbf{M})$ into orbits of the action of σ on $S(\mathbf{M})$: two distinct elements $[Z_1], [Z_2] \in S(\mathbf{M})$ belong to the same orbit if and only if there exists $i \in \mathbb{N}^*$ such that either $[Z_1] = [Z_2^{(\sigma^i)}]$ or $[Z_1^{(\sigma^i)}] = [Z_2]$.

Let $O(\mathbf{M})$ be the set of orbits of the action of σ on $S(\mathbf{M})$. The abelian group $K_0(\mathbf{M})/I_0(\mathbf{M})$ is canonically identified with the free abelian group on $O(\mathbf{M})$. For $i \in \{1, 2\}$ we write

$$[\mathbf{L}_1^\vee] = \sum_{[Z]\in S(\mathbf{M})} n_{1,[Z]}[Z] \in K_0(\mathbf{M}) \quad \text{and} \quad [\mathbf{L}_2] = \sum_{[Z]\in S(\mathbf{M})} n_{2,[Z]}[Z] \in K_0(\mathbf{M}),$$

where each $n_{i,[Z]} \in \mathbb{N}$ and all but a finite number of the integers $n_{i,[Z]}$ are 0. Let $O(\mathbf{M}, G, H)$ be the smallest finite subset of $O(\mathbf{M})$ such that for each orbit $o \in O(\mathbf{M}) \setminus O(\mathbf{M}, G, H)$ and for every $[Z] \in o$ we have $n_{1,[Z]} = n_{2,[Z]} = 0$.

Theorem 3 is equivalent to the statement: for each orbit $o \in O(\mathbf{M}, G, H)$ we have an identity

$$\sum_{[Z]\in o} n_{1,[Z]} = \sum_{[Z]\in o} n_{2,[Z]}. \tag{43}$$

The Galois group $\mathrm{Gal}(k) := \mathrm{Gal}(\bar{k}/k)$ acts on the finite set $\pi_0(\mathbf{M}_{\bar{k}})$ of connected components of the pullback $\mathbf{M}_{\bar{k}}$ of $\mathbf{M}$ to $\bar{k}$, so it defines a finite étale scheme $\pi(\mathbf{M})$ which has a monoid structure such that the natural morphism $\varrho_{\mathbf{M}} : \mathbf{M} \to \pi(\mathbf{M})$ is a homomorphism of monoids.

Lemma 11. *For a semisimple left $\mathbf{M}$-module Z of finite dimension we have the following two disjoint possibilities:*

(a) *If the image of $\mathbf{M}(\bar{k})$ in $\mathbf{End}(Z)(\bar{k})$ is finite, then there exists a smallest $n_Z \in \mathbb{N}^*$ such that the left $\mathbf{M}$-modules Z and $Z^{(\sigma^{n_Z})}$ are isomorphic.*

(b) *If the image of $\mathbf{M}(\bar{k})$ in $\mathbf{End}(Z)(\bar{k})$ is infinite, then there exists a smallest $n_Z \in \mathbb{N}^*$ such that for each integer $m \geq n_Z$ there exists no left $\mathbf{M}$-module W such that Z and $W^{(\sigma^m)}$ are isomorphic.*

Proof: Case (a) occurs if and only if the matrix coefficients of ϱ_Z in $\mathcal{O}(\mathbf{M})$ (relative to a k-basis of Z) come from $\mathcal{O}(\pi(\mathbf{M}))$, i.e., the representation ϱ_Z defining Z (see Section 1) factors through $\varrho_{\mathbf{M}}$, inducing a representation from $\pi(\mathbf{M})$ to the multiplicative monoid scheme over k associated to the ring scheme $\mathbf{End}(Z)$. A representation of $\pi(\mathbf{M})$ is the same as a representation of the monoid's corresponding k-algebra $(\bar{k}[\pi_0(\mathbf{M}_{\bar{k}})])^{\mathrm{Gal}(k)}$, which is a finite dimensional k-vector space and thus which has a finite number of isomorphism classes of semisimple representations of a given dimension.

As $\pi(\mathbf{M}) = \pi(\mathbf{M}^{(p)})$, the functor $\dagger \mapsto \dagger^{(\sigma)}$ is an autoequivalence of the category of $\pi(\mathbf{M})$-modules of finite dimension and therefore part (a) follows from the finiteness part of the previous paragraph.

Case (b) occurs if the matrix coefficients of ϱ_Z in $\mathcal{O}(\mathbf{M})$ (relative to a k-basis of Z) do not come from $\mathcal{O}(\pi(\mathbf{M}))$, which can be checked over $\bar{k}$. On some irreducible component $\mathbf{C}$ of $\mathbf{M}_{\bar{k}}$ some matrix coefficient is not constant, hence by Fact 9 it is not in $\bar{k}(\mathbf{C})^{p^n}$ for some $n \in \mathbb{N}^*$, where $\bar{k}(\mathbf{C})$ is the field of rational functions on $\mathbf{C}$. As the functor $W \mapsto W^{(\sigma)}$ is fully faithful, for $m \in \mathbb{N}$ a $\mathbf{M}$-module W such that $[Z] = [W^{(\sigma^m)}] \in K_0(\mathbf{M})$ is unique up to isomorphisms if it exists and it exists if and only if the matrix coefficients come from $\mathbf{M}^{(p^m)}$, and hence they are p^m-th powers in the function field of every irreducible component of $\mathbf{M}_{\bar{k}}$. Thus, if $m \geq n$, then W does not exist, i.e., we have $1 \leq n_Z \leq n$. $\square$

9.1 Step 1: reduction to the case of absolutely simple M-modules

If k' is a perfect field extension of k, then we have an extension of scalars homomorphism

$$e_{k'/k} : K_0(\mathbf{M})/I_0(\mathbf{M}) \to K_0(\mathbf{M}_{k'})/I_0(\mathbf{M}_{k'}).$$

If the extension $k \to k'$ is finite, then we have a restriction of scalars homomorphism

$$r_{k'/k} : K_0(\mathbf{M}_{k'})/I_0(\mathbf{M}_{k'}) \to K_0(\mathbf{M})/I_0(\mathbf{M})$$

such that the endomorphism $r_{k'/k} \circ e_{k'/k}$ is the multiplication by $[k' : k]$, and thus, as the free abelian group $K_0(\mathbf{M})/I_0(\mathbf{M})$ is torsion free, $e_{k'/k}$ is injective. This implies that $e_{\bar{k}/k}$ is also injective.

As G is fully projectively balanced if and only if $G_{k'}$ is fully projectively balanced and H is fully balanced if and only if $H_{k'}$ is fully balanced (see Corollary 5 (b) and Section 7), in what follows we will often replace k by k' or by some perfect subfield of it.

Lemma 12. *To prove that Equation (43) holds we can assume that for every $[Z] \in o \in O(\mathbf{M}, G, H)$, the simple left $\mathbf{M}$-module Z is absolutely simple.*

Proof: Let k' be a finite Galois extension of k such that each simple factor of a composition series of either the left $\mathbf{M}_{k'}$-module $k' \otimes_k \mathbf{L}_1$ or of the left

$\mathbf{M}_{k'}$-module $k'\otimes_k \mathbf{L}_2$ is absolutely simple. To prove the lemma it suffices to show that if Equation (43) holds in the case when $O(\mathbf{M},G,H)$ is replaced by $O(\mathbf{M}_{k'},G_{k'},H_{k'})$, then Equation (43) holds as well.

If $[Z]\in o\in O(\mathbf{M},G,H)$, then $k'\otimes_k Z$ is a direct sum of absolutely simple left $\mathbf{M}_{k'}$-modules. It is well-known that we can write

$$k'\otimes_k Z=\oplus_{j=1}^{\kappa_Z} m_Z Z'_j,$$

where $\kappa_Z, m_Z\in\mathbb{N}^*$ and where the left $\mathbf{M}_{k'}$-modules Z'_j are absolutely simple and not pairwise isomorphic and such that the Galois group $\mathrm{Gal}(k'/k)$ acts transitively on the set $\{Z'_1,\ldots,Z'_{\kappa_Z}\}$ (via its natural action on the first factor of $k'\otimes_k Z$). We consider the orbit $o'\in O(\mathbf{M}_{k'},G_{k'},H_{k'})$ such that $[Z'_1]\in o'$. Let $I_{Z'_1}$ be the non-empty subset of $\{1,\ldots,\kappa_Z\}$ formed by all those elements j such that $[Z'_j]\in o'$ and let $\varpi_Z\in\mathbb{N}^*$ be the number of elements of $I_{Z'_1}$. It is easy to see that κ_Z, m_Z, and ϖ_Z depend only on the orbit o and not on the choice of Z with the property that $[Z]\in o$. Thus we can define $\kappa_o:=\kappa_Z$, $m_o:=m_Z$, and $\varpi_o:=\varpi_Z$.

If $[\tilde{Z}]\in\tilde{o}\in O(\mathbf{M},G,H)\setminus\{o\}$ and if we similarly write

$$k'\otimes_k \tilde{Z}=\oplus_{j=1}^{\kappa_{\tilde{Z}}} m_{\tilde{Z}}\tilde{Z}'_j,$$

then for all $j\in\{1,\ldots,\kappa_Z\}$ and $\tilde{j}\in\{1,\ldots,\kappa_{\tilde{Z}}\}$ the orbits in $O(\mathbf{M}_{k'},G_{k'},H_{k'})$ to which Z'_j and $\tilde{Z}'_{\tilde{j}}$ belong are distinct. This is so as for all $i,\tilde{i}\in\mathbb{N}$, the k-vector space $\mathrm{Hom}_{\mathbf{M}}(Z^{(\sigma^i)},\tilde{Z}^{(\sigma^{\tilde{i}})})$ is non-zero if and only if the k'-vector space $\mathrm{Hom}_{\mathbf{M}'_k}((k'\otimes_k Z)^{(\sigma^i)},(k'\otimes_k\tilde{Z})^{(\sigma^{\tilde{i}})})$ is non-zero. Alternatively, this follows from the injectivity of $e_{k'/k}$.

We write $[(k'\otimes_k\mathbf{L}_1)^\vee]=\sum_{[Z']\in S(\mathbf{M}_{k'})} n_{1,[Z']}[Z']\in K_0(\mathbf{M}_{k'})$ as well as $[k'\otimes_k\mathbf{L}_2]=\sum_{[Z']\in S(\mathbf{M}_{k'})} n_{2,[Z']}[Z']\in K_0(\mathbf{M}_{k'})$, where each $n_{i,[Z']}\in\mathbb{N}$ and all but a finite number of the integers $n_{i,[Z']}$ are zero. Varying o, we get that o' can be an arbitrary element of $O(\mathbf{M}_{k'},G_{k'},H_{k'})$. Thus, based on the last two paragraphs we get that for $i\in\{1,2\}$ we have

$$\sum_{[Z']\in o'} n_{i,[Z']}=m_o\varpi_o\sum_{[Z]\in o} n_{i,[Z]}. \tag{44}$$

As we have assumed that Equation (43) holds in the case when $O(\mathbf{M},G,H)$ is replaced by $O(\mathbf{M}_{k'},G_{k'},H_{k'})$, the equality $\sum_{[Z']\in o'} n_{1,[Z']}=\sum_{[Z']\in o'} n_{2,[Z']}$ holds. From this and Equation (44) we get that Equation (43) holds. $\square$

9.2 Step 2: reduction to the case of a finite field

In this subsection we show that to prove that Equation (43) holds for all orbits $o \in O(\mathbf{M}, G, H)$ we can assume that k is a finite field. Based on Lemma 12 we can assume that each simple factor of a composition series of either $\mathbf{L}_1$ or $\mathbf{L}_2$ is absolutely simple. Based on Lemma 12 we can also assume that the finite étale monoid scheme $\pi(\mathbf{M})$ is constant and thus the pullback of a constant monoid π over $\mathbb{F}_p$. Thus also the finite étale monoid scheme $\mathbf{C}_G$ introduced in Subsection 5.3 is constant. Moreover, as $e_{\bar{k}/k}$ is injective, we can always replace k by $\bar{k}$, or in connection to what follows consider geometric fibers instead of fibers.

Let $\mathcal{R}$ be a finitely generated $\mathbb{F}_p$-subalgebra of k such that the following six properties hold for it:

(i) There exist finite flat commutative group scheme $\mathcal{G}$ and $\mathcal{H}$ over $\operatorname{Spec} \mathcal{R}$ annihilated by p^m with $\mathcal{G}$ as a BT_m and such that $G = \mathcal{G}_k$ and $H = \mathcal{H}_k$ and the ring homomorphisms $\rho_G : \mathbb{B} \to \mathrm{End}(G)$ and $\rho_H : \mathbb{B} \to \mathrm{End}(H)$ factor through ring homomorphisms $\rho_{\mathcal{G}} : \mathbb{B} \to \mathrm{End}(\mathcal{G})$ and $\rho_{\mathcal{H}} : \mathbb{B} \to \mathrm{End}(\mathcal{H})$ (respectively); let $\mathbf{End}_{\mathbb{B}}(\mathcal{G})$, $\mathbf{End}_{\mathbb{B}}(\mathcal{H})$, $\mathbf{Hom}_{\mathbb{B}}(\mathcal{G}, \mathcal{H})$ and $\mathbf{Hom}_{\mathbb{B}}(\mathcal{H}, \mathcal{G})$ be the closed subgroup schemes of $\mathbf{End}(\mathcal{G})$, $\mathbf{End}(\mathcal{H})$, $\mathbf{Hom}(\mathcal{G}, \mathcal{H})$ and $\mathbf{Hom}(\mathcal{H}, \mathcal{G})$ (respectively) whose valued points are those which commute with the $\mathbb{B}$-actions and whose pull backs to k are $\mathbf{End}_{\mathbb{B}}(G)$, $\mathbf{End}_{\mathbb{B}}(H)$, $\mathbf{Hom}_{\mathbb{B}}(G, H)$ and $\mathbf{Hom}_{\mathbb{B}}(H, G)$ (respectively).

(ii) The reduced scheme $\mathbf{End}_{\mathbb{B}}(\mathcal{G})_{\mathrm{red}} \times_{\operatorname{Spec} \mathcal{R}} \mathbf{End}_{\mathbb{B}}(\mathcal{H})^{\mathrm{op}}_{\mathrm{red}}$ is a smooth subgroup scheme of $\mathbf{End}(\mathcal{G}) \times_{\operatorname{Spec} \mathcal{R}} \mathbf{End}(\mathcal{H})^{\mathrm{op}}$; we will denote by $\boldsymbol{\mathcal{M}}$ the multiplicative monoid scheme over $\mathcal{R}$ associated to the reduced ring scheme $\mathbf{End}_{\mathbb{B}}(\mathcal{G})_{\mathrm{red}} \times_{\operatorname{Spec} \mathcal{R}} \mathbf{End}_{\mathbb{B}}(\mathcal{H})^{\mathrm{op}}_{\mathrm{red}}$.

(iii) The reduced scheme $\mathbf{Hom}_{\mathbb{B}}(\mathcal{G}, \mathcal{H})_{\mathrm{red}} \times_{\operatorname{Spec} \mathcal{R}} \mathbf{Hom}_{\mathbb{B}}(\mathcal{H}, \mathcal{G})_{\mathrm{red}}$ is a smooth subgroup scheme of $\mathbf{Hom}(\mathcal{G}, \mathcal{H}) \times_{\operatorname{Spec} \mathcal{R}} \mathbf{Hom}(\mathcal{H}, \mathcal{G})$; let $\boldsymbol{\mathcal{L}}_1$ and $\boldsymbol{\mathcal{L}}_2$ be the Lie algebras over $\mathcal{R}$ of $\mathbf{Hom}_{\mathbb{B}}(\mathcal{G}, \mathcal{H})_{\mathrm{red}}$ and $\mathbf{Hom}_{\mathbb{B}}(\mathcal{H}, \mathcal{G})_{\mathrm{red}}$ (respectively) and let $\boldsymbol{\mathcal{L}}_1^{\vee} := \mathrm{Hom}_{\mathcal{R}}(\boldsymbol{\mathcal{L}}_1, \mathcal{R})$ be the dual of the $\mathcal{R}$-module $\boldsymbol{\mathcal{L}}_1$.

(iv) If G is fully projectively balanced, and if $\mathcal{G}^0$ denotes the open subgroup scheme of $\mathcal{G}$ whose fibers are connected components of identity elements (see [SGA3-1], Exp. VI_{B}, Sect.3, Thm. 3.10), then the quotient group scheme $\mathcal{G}/\mathcal{G}^0$ is a constant finite étale group scheme over $\operatorname{Spec} \mathcal{R}$ and each geometric fiber of $\mathcal{G}$ is fully projectively balanced. If H is fully balanced, then all geometric fibers of $\mathcal{H}$ are fully balanced.

(v) The left $\boldsymbol{\mathcal{M}}$-modules $\boldsymbol{\mathcal{L}}_1^{\vee}$ and $\boldsymbol{\mathcal{L}}_2$ have a composition series whose factors $\mathcal{Z}_{1,1},\ldots,\mathcal{Z}_{1,\delta_1}$ and $\mathcal{Z}_{2,1},\ldots,\mathcal{Z}_{1,\delta_2}$ (respectively) have absolutely simple fibers and are defined by free $\mathcal{R}$-modules.

(vi) For each $\jmath \in \{(1,1),\ldots,(1,\delta_1),(2,1),\ldots,(2,\delta_2)\}$, there exists $n_\jmath \in \mathbb{N}$ such that either $\mathcal{Z}_\jmath$ comes from a representation of $\pi_{\mathbb{F}_{p^{n_\jmath+1}}}$ or it is of the form $W_\jmath^{(\sigma^{n_\jmath})}$ where $W_\jmath$ is a representation of $\boldsymbol{\mathcal{M}}$ on a free $\mathcal{R}$-module which on all geometric fibers does not come from $\boldsymbol{\mathcal{M}}^{(p)}$. Moreover, for distinct $\jmath_1,\jmath_2 \in \{(1,1),\ldots,(1,\delta_1),(2,1),\ldots,(2,\delta_2)\}$ such that $W_{\jmath_1}$ and $W_{\jmath_2}$ are defined, either they are isomorphic or, in every (geometric) fiber, they are non-isomorphic.

The existence of $\mathcal{R}$ such that properties (i) to (vi) hold is a standard piece of algebraic geometry (see Lemma 11 in connection to (vi), with n_j taken to be $n_{\mathcal{Z}_j}-1$).

One sees by inspection that the assertion to be shown (i.e., Equation (43)) holds for the generic geometric fiber (i.e., over $\bar{k}$) is equivalent to the assertion for one geometric fiber, so by replacing $\mathcal{R}$ by a finite field quotient of it we can assume that k is finite.

9.3 Step 3: reduction to the case of abstract monoids

As k is finite, we can take $(\mathcal{R},\boldsymbol{\mathcal{L}}_1,\boldsymbol{\mathcal{L}}_2) = (k,\mathbf{L}_1,\mathbf{L}_2)$ and thus (see property (iv) of Subsection 9.2) $\mathbf{L}_1^\vee$ and $\mathbf{L}_2$ have composition series whose factors $\mathcal{Z}_{1,1},\ldots,\mathcal{Z}_{1,\delta_1}$ and $\mathcal{Z}_{2,1},\ldots,\mathcal{Z}_{1,\delta_2}$ (respectively) are absolutely simple.

To prove that Equation (43) holds we can replace the finite field k by a finite field extension of it (see Lemma 12), and as such we will check that we can also assume that the following two properties also hold:

(i) for each element $\jmath \in \{(1,1),\ldots,(1,\delta_1),(2,1),\ldots,(2,\delta_2)\}$, $\mathcal{Z}_\jmath$ is an absolutely simple left $\mathbf{M}(k)$-module;

(ii) for all distinct elements $\jmath_1,\jmath_2 \in \{(1,1),\ldots,(1,\delta_1),(2,1),\ldots,(2,\delta_2)\}$, $[\mathcal{Z}_{\jmath_1}]$ and $[\mathcal{Z}_{\jmath_2}]$ belong to the same orbit $o \in O(\mathbf{M},G,H)$ if and only if $\mathcal{Z}_{\jmath_1}$ and $\mathcal{Z}_{\jmath_2}$ belong to the same orbit of the natural (analogous) action of σ on the set of isomorphism classes of finite dimensional simple left $\mathbf{M}(k)$-modules.

Let $\mathrm{Tr}_Z : \mathbf{M}(\bar{k}) \to \bar{k}$ be the trace function defined by a $\mathbf{M}$-module Z. If $s \in \mathbb{N}^*$ and $W_1,\ldots,W_s$ are simple non-isomorphic $\mathbf{M}_{\bar{k}}$-modules, then they are also simple non-isomorphic $\mathbf{M}(\bar{k})$-modules, so by the Jacobson density theorem ([Bo2], Sect. 3, Thm. 1 and Sect. 5, Thm. 3) the $\bar{k}$-algebra homomorphism

$$\Pi : \bar{k}[\mathbf{M}(\bar{k})] \to \prod_{i=1}^{s} \mathrm{End}_{\bar{k}}(W_i)$$

is surjective (see also [Bo2], Sect. 5, Cor. 1 of Prop. 4). As each such Π is defined over a finite field extension of k, we can assume that the property (i) holds. As Π is surjective, the set $\{\mathrm{Tr}_{W_1}, \ldots, \mathrm{Tr}_{W_s}\} \subset \mathrm{Maps}(\mathbf{M}(\bar{k}), \bar{k})$ has s elements and is linearly independent. As $\mathrm{Tr}_{Z^{(\sigma)}} = \mathrm{Tr}_Z^p$, arguing with trace functions, the fact that we can assume that the property (ii) holds follows from Theorem 4 applied to $\mathbf{M}$ and to pairs of trace functions $\mathrm{Tr}_{\mathcal{Z}_\jmath} \in \mathcal{O}(\mathbf{M}_{\bar{k}})$ indexed by $\jmath \in \{(1,1), \ldots, (1,\delta_1), (2,1), \ldots, (2,\delta_2)\}$.

Let the groups $K_0(\mathbf{M}(k))$, $I_0(\mathbf{M}(k))$, $K_0(\mathbf{M}(k))/I_0(\mathbf{M}(k))$ and the subset $O(\mathbf{M}(k), G, H) \subset O(\mathbf{M}(k))$ be analogues to the groups $K_0(\mathbf{M})$, $I_0(\mathbf{M})$, $K_0(\mathbf{M})/I_0(\mathbf{M})$ and the subset $O(\mathbf{M}, G, H) \subset O(\mathbf{M})$ (respectively) but working in the category of finite dimensional k-vector spaces which are left modules over the abstract monoid $\mathbf{M}(k)$. We have an injective pullback map

$$O(\mathbf{M}, G, H) \to O(\mathbf{M}(k), G, H)$$

(see properties (i) and (ii)). Thus to prove that Equation (43) holds it suffices to show that it holds in the case when $O(\mathbf{M}, G, H)$ is replaced by the analogous $O(\mathbf{M}(k), G, H)$. Therefore to prove that Equation (43) holds we can assume that k is a finite field and we are viewing $\mathbf{L}_1^\vee$ and $\mathbf{L}_2$ as left modules over the abstract monoid $\mathbf{M}(k) = \mathrm{End}_{\mathbb{B}}(G) \times \mathrm{End}_{\mathbb{B}}(H)^{\mathrm{op}} = \mathrm{End}_{\mathbb{B}}(G) \times \mathrm{End}_{\mathbb{B}^{\mathrm{op}}}(H^{\mathrm{t}})$ which is $\mathbf{End}_{\mathbb{B}}(G)_{\mathrm{red}}(k) \times \mathbf{End}_{\mathbb{B}}(H)^{\mathrm{op}}_{\mathrm{red}}(k) = \mathbf{End}_{\mathbb{B}}(G)_{\mathrm{red}}(k) \times \mathbf{End}_{\mathbb{B}^{\mathrm{op}}}(H^{\mathrm{t}})_{\mathrm{red}}(k)$; to differentiate between module structures we will denote $\mathrm{A}_{G,\mathbb{B}} := \mathrm{End}_{\mathbb{B}}(G)$ and $\mathrm{A}_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}} := \mathrm{End}_{\mathbb{B}^{\mathrm{op}}}(H^{\mathrm{t}})$ viewed as finite $\mathbb{Z}/p^m\mathbb{Z}$-algebras. Let $\overline{\mathrm{A}}_{G,\mathbb{B}}$ and $\overline{\mathrm{A}}_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}}$ be the reductions modulo p of $\mathrm{A}_{G,\mathbb{B}}$ and $\mathrm{A}_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}}$ (respectively); they are finite dimensional $\mathbb{F}_p$-algebras.

The left modules $\mathbf{L}_1^\vee$ and $\mathbf{L}_2$ over the abstract monoid $\{1_G\} \times \mathrm{End}_{\mathbb{B}^{\mathrm{op}}}(H^{\mathrm{t}})$ are actually left $\mathrm{A}_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}}$-modules and for each $\delta_G \in \mathrm{End}_{\mathbb{B}}(G)$ the multiplication by $(\delta_G, 1_{H^{\mathrm{t}}})$ on $\mathbf{L}_1^\vee$ and $\mathbf{L}_2$ are $\mathrm{A}_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}}$-linear maps. Similarly, the left modules $\mathbf{L}_1^\vee$ and $\mathbf{L}_2$ over the abstract monoid $\mathrm{End}_{\mathbb{B}}(G) \times \{1_{H^{\mathrm{t}}}\}$ are actually left $\mathrm{A}_{G,\mathbb{B}}$-modules and for each $\delta_H \in \mathrm{End}_{\mathbb{B}}(H)^{\mathrm{op}} = \mathrm{End}_{\mathbb{B}^{\mathrm{op}}}(H^{\mathrm{t}})$ the multiplication by $(1_G, \delta_H)$ on $\mathbf{L}_1^\vee$ and $\mathbf{L}_2$ are $\mathrm{A}_{G,\mathbb{B}}$-linear maps. From the last two sentences we get that $\mathbf{L}_1^\vee$ and $\mathbf{L}_2$ are left $k \otimes_{\mathbb{F}_p} (\overline{\mathrm{A}}_{G,\mathbb{B}} \otimes_{\mathbb{F}_p} \overline{\mathrm{A}}_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}})$-modules.

9.4 Step 4: applying Theorem 2

We will use the notation of Subsection 5.3. As $\mathrm{A}_{G,\mathbb{B}}$ is artinian, $J(\mathrm{A}_{G,\mathbb{B}})$ is the largest two-sided nilpotent ideal of $\mathrm{A}_{G,\mathbb{B}}$ and $\Sigma_{G,\mathbb{B}} := \Sigma(\mathrm{A}_{G,\mathbb{B}})$ is semisimple (see [Bo2], Sect. 10, Prop. 1) and (due to property (iv) of Subsection 9.3) it is the same semisimple ring defined in Subsection 5.3. We know that $\Sigma_{G,\mathbb{B}}$ is a finite product of matrix rings over finite fields, see Wedderburn's Theorem (see [Bo2], Sect 7, Thm. 1 and Sect. 18, Thm. 2). As $\Sigma_{G,\mathbb{B}}$ and its analog $\Sigma_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}}$ for H^{t} are absolutely semisimple, the epimorphisms $\overline{\mathrm{A}}_{G,\mathbb{B}} \to \Sigma_{G,\mathbb{B}}$ and $\overline{\mathrm{A}}_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}} \to \Sigma_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}}$ have sections (see Wedderburn's Principal Theorem of [Bo2], Sect. 13, Cor. 1 b) of Prop. 7). Thus the orthogonal sequences of all indecomposable central idempotents of $\Sigma_{G,\mathbb{B}}$ and $\Sigma_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}}$ (i.e., those central idempotents whose images are the simple factors) lift to orthogonal sequences of indecomposable idempotents of $\overline{\mathrm{A}}_{G,\mathbb{B}}$ and $\overline{\mathrm{A}}_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}}$ (respectively) whose sums are 1 and which at their turn lift to orthogonal sequences of indecomposable idempotents of $\mathrm{A}_{G,\mathbb{B}}$ and $\mathrm{A}_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}}$ (respectively) whose sums is 1, see [J], Ch. III, Sect. 8, Props. 2 to 5 (see also [Bo2], Sect. 9, Cor. 2 of Prop. 7 and Exc. 18 (a) and [J], Ch. III, Sect. 7, Prop. 2). The existence of such orthogonal sequences of indecomposable idempotents of $\mathrm{A}_{G,\mathbb{B}}$ and $\mathrm{A}_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}}$ also follows from the following general result by applying it, say for G, with $(B, A) = (\mathrm{A}_{G,\mathbb{B}}, \Sigma_{G,\mathbb{B}})$, and by considering the orthogonal sequence of all indecomposable central idempotents of $\tilde{\Sigma}_{G,\mathbb{B}}$.

Proposition 6. *Let B be a finite ring with* 1 *annihilated by p^m. Let A be a semisimple subring of $\Sigma(B)$ (so A is a finite product of matrix rings over finite fields). Let $\tilde{A}$ be a finite product of matrix rings over étale $W_m(\mathbb{F}_p)$-algebras such that $\tilde{A}/p\tilde{A} = A$. Then the composite ring homomorphism $\tilde{A} \to A = \tilde{A}/p\tilde{A} \to \Sigma(B)$ lifts to a ring homomorphism $\tilde{A} \to B$.*

Proof: Using induction on $m \in \mathbb{N}^*$, the case $m = 1$ follows from Wedderburn's Principal Theorem. For $m \geq 2$, the passage from $m-1$ to m goes as follows. Let $\tilde{A} \to B/p^{m-1}B$ be a ring homomorphism inducing $\tilde{A} \to A = \tilde{A}/p\tilde{A} \to \Sigma(B)$. Pulling it back via the ring epimorphism $B \to B/p^{m-1}B$ we get a ring homomorphism $\tilde{A}^+ \to B$ and a short exact

$$0 \to p^{m-1}B \to \tilde{A}^+ \to \tilde{A} \to 0.$$

As $\tilde{A}$ is a flat $W_m(\mathbb{F}_p)$-module, by tensoring this short exact sequence with $\mathbb{F}_p$ over $W_m(\mathbb{F}_p)$ we get a short exact sequence

$$0 \to p^{m-1}B \to \tilde{A}^+/p\tilde{A}^+ \to A \to 0$$

which (by Wedderburn's Principal Theorem) has a section $s_A : A \to \tilde{A}^+/p\tilde{A}^+$. The inverse image $\tilde{A}'$ of $s_A(A)$ in $\tilde{A}^+$ is a subring of $\tilde{A}^+$ with the same number of elements as $\tilde{A}$ and is endowed with a ring homomorphism $\tilde{A}' \to \tilde{A}$ whose reduction modulo p is onto. This implies that $\tilde{A}' \to \tilde{A}$ is an isomorphism and by composing its inverse $\tilde{A} \to \tilde{A}'$ with the composite ring homomorphism $\tilde{A}' \to \tilde{A}^+ \to B$ we get the desired ring homomorphism $\tilde{A} \to B$. $\square$

To prove that Theorem 3 holds, i.e., Equation (43) holds when $O(\mathbf{M}, H, G)$ is replaced by $O(\mathbf{M}(k), G, H)$, using orthogonal sequences of indecomposable idempotents of $\mathrm{A}_{G,\mathbb{B}}$ and $\mathrm{A}_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}}$ which induce direct sum decompositions of G, H, $\mathbf{L}_1^{\vee}$ and $\mathbf{L}_2$ preserved by $\mathbb{B}$ and a direct sum decomposition of $G[p]$ preserved by $\mathbb{C}_{G,\mathbb{B}}$, we can reduce to the case when $\Sigma_{G,\mathbb{B}}$ and $\Sigma_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}}$ are simple $\mathbb{F}_p$-algebras and thus matrix algebras $\mathrm{M}_{r_1}(l_1)$ and $\mathrm{M}_{r_2}(l_2)$ (respectively) over finite fields l_1 and l_2 (respectively).

Using the mentioned sections of the epimorphisms $\overline{\mathrm{A}}_{G,\mathbb{B}} \to \Sigma_{G,\mathbb{B}}$ and $\overline{\mathrm{A}}_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}} \to \Sigma_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}}$, the left $k\otimes_{\mathbb{F}_p}(\overline{\mathrm{A}}_{G,\mathbb{B}}\otimes_{\mathbb{F}_p}\overline{\mathrm{A}}_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}})$-modules $\mathbf{L}_1^{\vee}$ and $\mathbf{L}_2$ become left $k \otimes_{\mathbb{F}_p} (\Sigma_{G,\mathbb{B}} \otimes_{\mathbb{F}_p} \Sigma_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}})$-module and their composition series do not depend on which one of the two left modules structures are used. In other words, in what follows we can work with $k \otimes_{\mathbb{F}_p} (\Sigma_{G,\mathbb{B}} \otimes_{\mathbb{F}_p} \Sigma_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}})$ instead of $k \otimes_{\mathbb{F}_p} (\overline{\mathrm{A}}_{G,\mathbb{B}} \otimes_{\mathbb{F}_p} \overline{\mathrm{A}}_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}})$ and each $\mathcal{Z}_\jmath$ is a simple left $k \otimes_{\mathbb{F}_p} (\Sigma_{G,\mathbb{B}} \otimes_{\mathbb{F}_p} \Sigma_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}})$-module; here $\jmath \in \{(1,1),\ldots,(1,\delta_1),(2,1),\ldots,(2,\delta_2)\}$.

The irreducible representations over $\bar{k}$ of a simple finite dimensional $\mathbb{F}_p$-algebra are in the same orbit of the natural action of σ, so the problem is to compare dimensions of k-vector subspaces of $\mathbf{L}_1^{\vee}$ and $\mathbf{L}_2$ that correspond to (i.e., are cut out by) simple factors of $\Sigma_{G,\mathbb{B}} \otimes_{\mathbb{F}_p} \Sigma_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}}$.

We embed l_1 and l_2 in $\bar{k}$. Let $l_0 := l_1 \cap l_2$. Let Γ be the multiplicative group of non-zero elements of l_0: so $p \nmid |\Gamma|$. The simple factors of

$$\mathrm{M}_{n_1}(l_1) \otimes_{\mathbb{F}_p} \mathrm{M}_{n_2}(l_2) = \mathrm{M}_{n_1 n_2}(l_1 \otimes_{\mathbb{F}_p} l_2)$$

correspond to those of $l_1 \otimes_{\mathbb{F}_p} l_2$ and hence to those of $l_0 \otimes_{\mathbb{F}_p} l_0$. The diagonal idempotent of $l_0 \otimes_{\mathbb{F}_p} l_0$ is

$$\frac{1}{|\Gamma|} \sum_{b\in\Gamma} b \otimes b^{-1}.$$

In view of the structure of $\Sigma_{G,\mathbb{B}} \otimes_{\mathbb{F}_p} \Sigma_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}}$, for $a \in \mathrm{Aut}(l_0) \leqslant \mathrm{Aut}(\Gamma)$ we have to compare dimensions of pieces cut out by idempotents of the form

$$\frac{1}{|\Gamma|} \sum_{b\in\Gamma} b \otimes a^{-1}(b).$$

The inclusions $l_0 \to \overline{\mathrm{A}}_{G,\mathbb{B}}$ and $l_0 \to \overline{\mathrm{A}}_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}}$ lift to ring homomorphisms $W_m(l_0) \to \mathrm{A}_{G,\mathbb{B}}$ and $W_m(l_0) \to \mathrm{A}_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}}$ (respectively), see Proposition 6.[15]

For $a \in \mathrm{Aut}(l_0)$, let $W(l_0)$ act on G and H_a^t via the homomorphism $W_m(l_0) \to \mathrm{A}_{G,\mathbb{B}}$ and via the composite of $W_m(a)^{-1} : W_m(l_0) \to W_m(l_0)$ with the homomorphism $W_m(l_0) \to \mathrm{A}_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}}$. The role of the lower right index a in connection to H_a^{t} or H_a, here and in what follows, is to keep track of the composite with $W_m(a)^{-1}$; for instance, $\mathbf{L}_{i,a} := \mathrm{Lie}(\mathbf{Hom}_{\mathbb{B}}(G, H_a)_{\mathrm{red}})$ with $i \in \{1,2\}$ is a Γ-module. Hence G and H_a are left $\mathbb{B} \otimes_{\mathbb{Z}_p} W(l_0)$-modules and thus also left $\mathbb{B}[\Gamma]$-modules, where $\mathbb{B}[\Gamma] := \mathbb{B} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Gamma]$. We have

$$\mathbf{L}_{1,a}^{\Gamma} = \mathrm{Lie}(\mathbf{Hom}_{\mathbb{B}[\Gamma]}(G, H_a)_{\mathrm{red}}) = \mathrm{Lie}(\mathbf{Hom}_{\mathbb{B}\otimes_{\mathbb{Z}_p} W(l_0)}(G, H_a)_{\mathrm{red}})$$

and

$$\mathbf{L}_{2,a}^{\Gamma} = \mathrm{Lie}(\mathbf{Hom}_{\mathbb{B}[\Gamma]}(H_a, G)_{\mathrm{red}}) = \mathrm{Lie}(\mathbf{Hom}_{\mathbb{B}\otimes_{\mathbb{Z}_p} W(l_0)}(H_a, G)_{\mathrm{red}}).$$

Therefore the analogue of Equation (43) for the case when $O(\mathbf{M}, G, H)$ is replaced by $O(\mathbf{M}(k), G, H)$ becomes the identities indexed by $a \in \mathrm{Aut}(l_0)$

$$\dim_k((\mathbf{L}_{1,a}^{\Gamma})^{\vee}) = \dim_k(\mathbf{L}_{2,a}^{\Gamma})$$

which follow from Equation (5) applied with $\mathbb{B}$ replaced by $\mathbb{B} \otimes_{\mathbb{Z}_p} W(l_0)$ as $\dim_k((\mathbf{L}_{1,a}^{\Gamma})^{\vee}) = \gamma_{\mathbb{B}\otimes_{\mathbb{Z}_p} W(l_0)}(G, H_a)$ and $\dim_k(\mathbf{L}_{2,a}^{\Gamma}) = \gamma_{\mathbb{B}\otimes_{\mathbb{Z}_p} W(l_0)}(H_a, G)$. We would only add that if G is fully projectively balanced, then the left $\mathbb{B} \otimes_{\mathbb{Z}_p} W(l_0)$-module G is projectively balanced and thus Theorem 2 (b) applies and if H is fully balanced, then for all $a \in \mathrm{Aut}(l_0)$ the left $\mathbb{B}\otimes_{\mathbb{Z}_p} W(l_0)$-module H_a is balanced and thus Theorem 2 (c) applies. This proves Theorem 3. □

9.5 Remark

From Subsection 5.3 we get that $\mathbf{End}(G)^0_{\mathrm{red}}$ acts trivially on each $\mathcal{Z}_\jmath$ with $\jmath \in \{(1,1), \ldots, (1,\delta_1), (2,1), \ldots, (2,\delta_2)\}$. Thus $\mathcal{Z}_\jmath$ is a left $\mathbf{C}_G$-module and hence also a left $\mathbf{C}_{G,\mathbb{B}}$-module.

[15] A simpler way to argue this is as follows. If $\overline{x}_0$ is a generator of the field extension $\mathbb{F}_p \to l_0$ and, working only with G, if $x_0 \in \mathrm{A}_{G,\mathbb{B}}$ lifts $\overline{x_0}$, let B_0 be the (commutative) subring of $\mathrm{A}_{G,\mathbb{B}}$ generated by x_0. We have $B_0/(B_0 \cap p\mathrm{A}_{G,\mathbb{B}}) = l_0$. As $(B_0 \cap p\mathrm{A}_{G,\mathbb{B}})$ is a nilpotent ideal of B_0 and the monomorphism $W_m(\mathbb{F}_p) \to W_m(l_0)$ is étale, there exists a unique ring homomorphism $W_m(l_0) \to B_0$ which induces the identity on residue fields. By composing it with the inclusion $B_0 \to \mathrm{A}_{G,\mathbb{B}}$ we get a lifting $W_m(l_0) \to \mathrm{A}_{G,\mathbb{B}}$.

If $H[p]$ is a BT_1 (for instance, if H is fully BT-balanced), then as we have a restriction homomorphism $\mathrm{End}(H)^0_{\mathrm{red}} \to \mathrm{End}(H[p])^0_{\mathrm{red}}$, as in Subsection 5.3, based on the fact that $1_{H[p]} + \mathrm{End}(H[p])^0_{\mathrm{red}} = \mathrm{Aut}(H[p])^0_{\mathrm{red}}$ is unipotent, we argue that $\mathbf{End}(G)^0_{\mathrm{red}} \times_k \mathbf{End}(H^{\mathrm{t}})^0_{\mathrm{red}}$ acts trivially on each $\mathcal{Z}_\jmath$ and therefore $\mathcal{Z}_\jmath$ is a left $\mathbf{C}_{G,\mathbb{B}}$-module as well as a left $\mathbf{C}_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}}$-module; thus in such a case the above four steps could be easily combined with $\mathrm{A}_{G,\mathbb{B}}$ and $\mathrm{A}_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}}$ being replaced by the étale ring schemes $\mathbf{C}_{G,\mathbb{B}}$ and $\mathbf{C}_{H^{\mathrm{t}},\mathbb{B}^{\mathrm{op}}}$.

Acknowledgement. The second author would like to thank SUNY at Binghamton, U.S.A., Mathematisches Forschungsinstitut Oberwolfach, Germany, I.H.E.S., Bures-sur-Yvette, France, and I.I.S.E.R., Thiruvananthapuram, India for good working conditions.

References

[BD] M. Boyarchenko and V. Drinfeld, *Character sheaves on unipotent groups in positive characteristic: foundations,* Selecta Math. (N.S.) **20** (2014), no. 1, 125–235

[Be] L. Begueri, *Dualité sur un corps local à corps résiduel algébriquement clos,* Mém. Soc. Math. France (N.S.) 1980/81, no. **4**

[Bo1] N. Bourbaki, *Éléments de mathématique. Algèbre. Chapitres 4 à 7,* Springer-Verlag Berlin Heidelberg, 2007

[Bo2] N. Bourbaki, *Éléments de mathématique. Algèbre. Chapitre 8: Modules et anneaux semi-simples,* Springer-Verlag Berlin Heidelberg, 2012

[CE] H. Cartan and S. Eilenberg, *Homological algebra,* Princeton University Press, Princeton, N. J., 1956

[De] P. Deligne, *La conjecture de Weil. II,* Publ. Math. Inst. Hautes Études Sci. No. **52** (1980), 137–252

[DG] M. Demazure and P. Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs. Avec un appendice Corps de classes local par Michiel Hazewinkel,* Masson & Cie, Éditeur, Paris, North-Holland Publishing Co., Amsterdam, 1970

[Di] J. Dixmier, *Algèbres enveloppantes,* Cahiers Scientifiques, Fasc. **XXXVII**. Gauthier-Villars Éditeur, Paris-Brussels-Montreal, Que., 1974

[E] H. P. Epp, *Eliminating wild ramification,* Invent. Math. **19** (1973), 235–249

[G1] A. Grothendieck, *Eléments de géométrie algébrique. I. Le langage des schémas,* Publ. Math. Inst. Hautes Études Sci. No. **4** (1960)

[G2] A. Grothendieck, *Groupes de Barsotti–Tate et cristaux de Dieudonné,* Séminaire de Mathématiques Supérieures, No. **45** (Été, 1970), Les Presses de l'Université de Montréal, Montreal, Que., 1974

[GV] O. Gabber and A. Vasiu, *Dimensions of group schemes of automorphisms of truncated Barsotti–Tate groups,* Int. Math. Res. Not. IMRN **2013**, no. 18, 4285–4333

[I] L. Illusie, *Déformations de groupes de Barsotti–Tate (d'après A. Grothendieck),* Séminaire sur les pinceaux arithmétiques: la conjecture de Mordell (Paris, 1983/84), 151–198, Astérisque **127**, Soc. Math. de France, Paris, 1985

[J] N. Jacobson, *Structure of rings,* Amer. Math. Soc. Coll. Publ., Vol. **37**, Revised edition, American Mathematical Society, Providence, R.I. 1964

[LNV] E. Lau, M.-H. Nicole, and A. Vasiu, *Stratifications of Newton polygon strata and Traverso's conjectures for p-divisible groups,* Ann. of Math. (2) **178** (2013), no. 3, 789–834

[L] D. Lazard, *Autour de la platitude,* Bull. Soc. Math. France **97** (1969), 81–128

[M] H. Matsumura, *Commutative ring theory. Second edition,* Cambidge Stud. Adv. Math., **8**, Cambridge University Press, Cambridge, 1989

[R] M. Raynaud, *Schémas en groupes de type* $(p, \cdots, p)$, Bull. Soc. Math. France **102** (1974), 241–280

[Sa] M. Saito, *Modules de Hodge polarisables,* Publ. Res. Inst. Math. Sci. **24** (1988), no. 6, 849–995 (1989)

[Se1] J.-P. Serre, *Groupes proalgébriques,* Publ. Math. Inst. Hautes Études Sci. No. **7**, 1960

[Se2] J.-P. Serre, *Trees,* Springer-Verlag, Berlin-New York, 1980

[SGA1] *Revêtements étales et groupe fondamental. Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1),* Dirigé par Alexandre Grothendieck. Augmenté de deux exposés de M. Raynaud, Lecture Notes in Math., Vol. **224**, Springer-Verlag, Berlin-New York, 1971

[SGA3-1] *Schémas en groupes. I: Propriétés générales des schémas en groupes, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3),* Dirigé par M. Demazure et A. Grothendieck, Lecture Notes in Math., Vol. **151**, Springer-Verlag, Berlin-New York 1970

[SGA3-2] *Schémas en groupes. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3),* Dirigé par M. Demazure et A. Grothendieck, Lecture Notes in Math., Vol. **152**, Springer-Verlag, Berlin-New York 1970

[SY] T. Suzuki and M. Yoshida, *A refinement of the local class field theory of Serre and Hazewinkel,* Algebraic number theory and related topics 2010, 163–191, RIMS Kôkyûroku Bessatsu, **B32**, 2012

[V1] A. Vasiu, *Crystalline boundedness principle,* Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 2, 245–300

[V2] A. Vasiu, *Level m stratifications of versal deformations of p-divisible groups,* J. Alg. Geom. **17** (2008), no. 4, 599–641

[V3] A. Vasiu, *A motivic conjecture of Milne,* J. Reine Angew. Math. **685** (2013), 181–247

[V4] A. Vasiu, *Good reductions of Shimura varieties of Hodge type in arbitrary mixed characteristic. Part I,* Math. Nachr. **293** (2020), no. 12, 2399–2448

Ofer Gabber, E-mail: gabber@ihes.fr
Address: IHÉS, Le Bois-Marie, 35, Route de Chartres,
F-91440 Bures-sur-Yvette, France.

Adrian Vasiu, E-mail: adrian@math.binghamton.edu
Address: Department of Mathematical Sciences, Binghamton University,
P. O. Box 6000, Binghamton, New York 13902-6000, U.S.A.