The classification of p-quasi-healthy henselian regular local rings of dimension 2

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ABSTRACT. Let p be a prime. Let R be a henselian regular local ring of mixed characteristic $(0,p)$ and dimension 2. Let k be the residue field of R, let $C(k)$ be the Cohen ring of k, and let $\widehat{R}$ be the completion of R. We consider an isomorphism $\widehat{R} \simeq C(k)[[x,y]]/(f)$, where $f \in C(k)[[x,y]]$ is a regular parameter. We prove that R is p-quasi-healthy, i.e., each Barsotti–Tate group over the punctured spectrum of R extends to a Barsotti–Tate group over $\mathrm{Spec}(R)$, if and only if f does not belong to the ideal $(p, x^p, y^p, x^{p-1}y^{p-1})$ of $C(k)[[x,y]]$. The 'if' part was proved before by Vasiu–Zink. The 'only if' part is new and generalizes prior counterexamples of Gabber and of Vasiu–Zink to the work of Faltings–Chai. The paper also contains some results in the case when the henselian assumption is weakened and a few general properties of p-quasi-healthy regular rings of arbitrary dimension.

KEY WORDS: Barsotti–Tate group, finite group scheme, regular ring, henselian ring, Breuil module, and cohomology.

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1 Introduction

For a commutative unitary ring J, let $J^\times$ be its group of units. If J is local, then $\mathfrak{m}_J := J \setminus J^\times$ is its maximal ideal.

Let p be a prime. Let R be a *regular* local ring of mixed characteristic $(0,p)$ and dimension at least 2. Let $k := R/\mathfrak{m}_R$ be the residue field of R. Let

$X := \mathrm{Spec}(R)$ and let U be punctured spectrum of R, i.e., the complement in X of the closed point $\mathrm{Spec}(k)$ of X. We recall from [VZ], Def. 2 that R is called *p-quasi-healthy* if each Barsotti–Tate group over U extends to a Barsotti–Tate group over X; from [Gr], Exp. III, Cor. 3.5 applied to coherent sheaves defined by the structure sheaves of truncated Barsotti–Tate groups over X, we get that such an extension is unique up to unique isomorphism. Let $\widehat{R}$ be the completion of R. For basic properties of the Cohen ring $C(k)$ of k we refer to [C], Sect. 6 and [M], Subsect. 29. If k is perfect, then $C(k) = W(k)$ is the ring of p-typical Witt vectors with coefficients in k.

In [VZ], Thm. 3 it is shown that there exist large classes of p-quasi-healthy regular local rings of dimension 2. We prove a converse of loc. cit. that classifies all p-quasi-healthy regular local rings of dimension 2 which are *henselian* as well.

Theorem 1 *We assume that R is of dimension 2. We consider an isomorphism*

$$\widehat{R} \simeq C(k)[[x,y]]/(f),$$

where $C(k)$ is the Cohen ring of k and where $f \in C(k)[[x,y]]$ is a regular parameter. Then the following three properties hold:

(a) *If f does not belong to the ideal $(p, x^p, y^p, x^{p-1}y^{p-1})$ of $C(k)[[x,y]]$, then R is p-quasi-healthy.*

(b) *We assume that R is henselian and that*

$$f \in (p, x^p, y^p, x^{p-1}y^{p-1}) \subset C(k)[[x,y]]$$

(thus the reduction $\bar{f}$ of f modulo p is a non-zero element of the ideal $(x^p, y^p, x^{p-1}y^{p-1})$ of $k[[x,y]]$). Then there exists a homomorphism

$$\gamma : G \to \boldsymbol{\mu}_{\mathbf{p},X}$$

of connected finite flat commutative group schemes annihilated by p over X such that the following three properties hold:

(b.i) *The order of G is p^2 if $f \notin (p, x^p, y^p)$ and is p^3 if $f \in (p, x^p, y^p)$.*

(b.ii) *It is not an epimorphism.*

(b.iii) *Its restriction to U is an epimorphism whose kernel extends to a finite flat group scheme over X which is a form of $(\mathbb{Z}/p\mathbb{Z})_X$ if G has order p^2*

and is a connected truncated Barsotti–Tate group of level 1 *over* X *of height* 2 *and dimension* 1 *if* G *has order* p^3*.*

(c) *We assume that* R *is henselian. Then* R *is* p*-quasi-healthy if and only if* f *does not belong to the ideal* $(p, x^p, y^p, x^{p-1}y^{p-1})$ *of* $C(k)[[x, y]]$*.*

Theorem 1 (a) (hence also the 'if' part of Theorem 1 (c)) is essentially proved in [VZ], Thm. 3. From [VZ], Lem. 27 applied to homomorphisms $\gamma : G \to \boldsymbol{\mu}_{\mathbf{p},X}$ as in Theorem 1 (b) we get that Theorem 1 (b) implies the 'only if' part of Theorem 1 (c). Theorem 1 (b) is an application of the following general result.

Proposition 1 *Let* $\mathcal{R}$ *be a henselian local ring whose residue field has characteristic* p*. We assume that there exist elements* $x, y \in \mathfrak{m}_{\mathcal{R}}$*,* $a, b \in \mathcal{R}$ *and* $c \in \{0\} \cup \mathcal{R}^\times$ *such that we have an identity*

$$p + ax^p + by^p + cx^{p-1}y^{p-1} = 0.$$

Then there exists a homomorphism $\delta : \mathcal{G} \to \boldsymbol{\mu}_{\mathbf{p},\mathcal{X}}$ *of connected finite flat commutative group schemes annihilated by* p *over* $\mathcal{X} := \mathrm{Spec}(\mathcal{R})$ *such that the following three properties hold:*

(i) *The order of* $\mathcal{G}$ *is* p^2 *if* $c \in \mathcal{R}^\times$ *and is* p^3 *if* $c = 0$*.*

(ii) *Tts restriction to* $\mathrm{Spec}(\mathcal{R}) \setminus \mathrm{Spec}(\mathcal{R}/(x, y))$ *is an epimorphism but at every point of* $\mathrm{Spec}(\mathcal{R}/(x, y))$ *it is not an epimorphism.*

(iiii) *The kernel of this restriction extends to a finite flat group scheme over* $\mathcal{X}$ *which is a connected truncated Barsotti–Tate group of level* 1 *over* $\mathcal{X}$ *of height* 2 *and dimension* 1 *if* $c = 0$*, is a form of* $(\mathbb{Z}/p\mathbb{Z})_{\mathcal{X}}$ *if* $c \in \mathcal{R}^\times$*, and it is* $(\mathbb{Z}/p\mathbb{Z})_{\mathcal{X}}$ *if and only if* c *is the* $(p-1)$*-th power of a unit of* $\mathcal{R}$*.*

If R is strictly henselian and if $\mathrm{Spec}(R/pR)$ is not irreducible but all its irreducible components have multiplicities divisible by $p - 1$ (e.g., if k is separably closed, R is complete, and $f = p - x^{p-1}y^{p-1}$), then Theorem 1 (b) is proved in [Ga1].

In the proof of [VZ], Thm. 28 it is shown that a variant of the first part of Theorem 1 (b) holds provided $R = \widehat{R}$ is of dimension 2, k is perfect, and one of the following three conditions hold:

① The p-th power of a non-zero element of $\mathfrak{m}_{k[[x,y]]}$ divides $\bar{f}$.

② There exists a regular sequence u, v in $k[[x, y]]$ such that $u^{p-1}v^{p-1}$ divides $\bar{f}$.

③ The element $h := p - f$ belongs to the ideal (x, y) of $W(k)[[x, y]]$ and there exist $\bar{a}, \bar{b}, \bar{c} \in k[[x, y]]$ such that $\bar{h} = -\bar{f} = (\bar{a}x^p + \bar{b}y^p + \bar{c}x^{p-1}y^{p-1})\bar{c}$.

If ② or ③ (resp. ①) holds, then loc. cit. constructs a homomorphism $G \to H$ with G connected of order p^2 (resp. of order p^3) and with H connected of order p which in general is not $\boldsymbol{\mu}_{\mathbf{p},X}$. If ③ holds and $\bar{c} \notin k[[x, y]]^\times$, i.e., and $-\bar{h} = \bar{f} = (\bar{a}_1x^p + \bar{b}_1y^p)(\bar{a}_2x + \bar{b}_2y)$ with $\bar{a}_1, \bar{a}_2, \bar{b}_1, \bar{b}_2 \in k[[x, y]]$ is a particular type of the elements one gets in Theorem 1 (b), then H is not a form of $\boldsymbol{\mu}_{\mathbf{p},X}$ and the kernel of the restriction of the homomorphism $G \to H$ to U extends to a connected finite flat group scheme over X of order p; thus the homomorphism $G \to H$ is unrelated to the γs of Theorem 1 (b).

Section 2 is of algebraic nature and provides the necessary computations with matrices of small (up to 3×3) sizes that are required to prove Proposition 1 in the particular case when $\mathcal{R}$ is regular complete and x, y is part of a regular system of parameters of $\mathcal{R}$ using the language of Breuil modules over a suitable frame associated to a ring of formal power series in as many indeterminates as the dimension of $\mathcal{R}$ and with coefficients in the Cohen ring of the residue field of $\mathcal{R}$ (see Subsection 3.1). Section 3 proves Proposition 1 in three steps: the first one is the particular case mentioned, the second step appeals to Artin's approximation theorem in a context modeled on the first step, and the third step introduces the universal rings whose spectra have local rings of residue characteristic p inducing henselizations to which the second step applies. Section 4 proves Theorem 1.

We recall that the complete local ring $\widehat{R}$ is henselian. Thus directly from Theorem 1 (c) applied also to $\widehat{R}$ we get:

Corollary 1 *We assume that R is henselian of dimension 2. Then R is p-quasi-healthy if and only if $\widehat{R}$ is p-quasi-healthy.*

Subsection 4.4 proves the following consequence of Theorem 1 (c).

Corollary 2 *Let k be a field of characteristic 2 (thus $p = 2$). Then, up to isomorphisms, $C(k)[[x]]$ is the only regular complete local ring of dimension 2 and residue field k which is 2-quasi-healthy.*

In Sections 6 we prove the following theorem:

Theorem 2 *There exists a smallest $n_p \in \mathbb{N}^*$ with the property that for each regular local ring R of dimension 2 whose henselization R^{h} is not p-quasi-healthy, there exists a local étale homomorphism $R \to R'$ such that the following three properties hold:*

(i) *The residue field of R' is k.*

(ii) *We have $[\mathrm{Frac}(R') : \mathrm{Frac}(R)] \leq n_p$.*

(iii) *The regular local ring R' is not p-quasi-healthy and in fact there exists a local finite étale homomorphism $R' \to R'_+$ with the property that there exists a homomorphism $\gamma' : G_{X'_+} \to \boldsymbol{\mu}_{\mathbf{p},X'_+}$ over $X'_+ := \mathrm{Spec}(R'_+)$ whose pullback to the spectrum of the henselization of R'_+ is a homomorphism as in Theorem 1 (b).*

Five general lemmas required to prove Theorem 2, the below Theorem 3, and Theorem 4 of Subsection 7.5 are gathered in Section 5. In Section 7 we study the universal constant n_p and in particular we prove the following theorem (see Subsection 7.3):

Theorem 3 *We assume that R is of dimension 2 and that there exists a regular system of parameters x, y of R and constants $a, b \in R$ and $c \in R^\times$ such that $p + ax^p + by^p + cx^{p-1}y^{p-1} = 0$. If the pair (R, pR) is henselian (in the sense of [Ga2], Sect. 0 or [E], Subsect. 0.1; e.g., this holds if R is p-adically complete) or if $p \in \{2, 3\}$, then R is not p-quasi-healthy and in fact there exists a homomorphism $\gamma : G \to \boldsymbol{\mu}_{\mathbf{p},X}$ of finite flat commutative group schemes annihilated by p over X such that the following three properties:*

(i) *The order of G is p^2 and the closed fiber G_k is connected.*

(ii) *It is not an epimorphism.*

(iii) *Its restriction to U is an epimorphism with a kernel that extends to a form of $(\mathbb{Z}/p\mathbb{Z})_X$; moreover, this form of $(\mathbb{Z}/p\mathbb{Z})_X$ is trivial if and only if the image of c in the residue field k of R has a $p-1$-th root in k.*

For a weaker version of Theorem 3 for $p \geq 5$ see Theorem 4 of Subsection 7.5. The proofs of Theorems 3 and 4 appeal to cohomological properties which are gathered in Subsections 4.3, 7.1, 7.2, and 7.4 and which rely heavily on a concrete case of [Ga2], Thm. 1.

Lemma 4 of Section 5 uses a theorem of Elkik (see [E], Thm. 5) to prove that if the pair (R, pR) is henselian, then R is p-quasi-healthy if and only if its p-adic completion is p-quasi-healthy. This result, Theorem 1 (c), Corollary 1, and Lemma 2 are, so far, the only 'if and only if' statements on p-quasi-healthy regular local rings.

Section 8 presents two variants of Proposition 1 which are modeled on [VZ], Thm. 28 (i) and (ii) and which provide examples in arbitrary dimension $d \geq 2$ of regular local rings which are not p-quasi-healthy.

We recall that [Ga1], [VZ], Thm. 28, the 'only if' part of Theorem 1 (c), Theorems 2 and 3, and Propositions 1 to 3 represent also counterexamples to [FC], Ch. V, Thms. 6.4 and 6.4′.

2 Some matrices

Let J be an arbitrary commutative ring. Let $u, v, \alpha_1, \alpha_2, \alpha_3 \in J$. Let

$$A := \begin{pmatrix} u^{p-1} - \alpha_1 v & 0 & \alpha_1 \\ 0 & v^{p-1} - \alpha_2 u & \alpha_2 \\ u^{p-1}v^p + \alpha_3 v & -\alpha_3 u & 0 \end{pmatrix} \in M_3(J).$$

We have

$$A(u\ v\ uv)^T = (u^p\ v^p\ u^p v^p)^T. \tag{1}$$

We compute

$$\det(A) = \alpha_1(u^{p-1}v^p + \alpha_3 v)(\alpha_2 u - v^{p-1}) + \alpha_2\alpha_3 u(u^{p-1} - \alpha_1 v)$$

$$= \alpha_2 u^p(\alpha_3 + \alpha_1 v^p) - \alpha_1 v^p(\alpha_3 + u^{p-1}v^{p-1}).$$

Let $a, b \in J$. We assume that v belongs to the Jacobson radical of J. Thus $1 + u^{p-1}v^{p-1}$ and $1 + \alpha_1 v^p$ are units of J. From now on we take

$$\alpha_1 := -b(1 + u^{p-1}v^{p-1})^{-1}, \ \ \alpha_2 := a(1 + \alpha_1 v^p)^{-1} \ \text{ and } \ \alpha_3 := 1,$$

and therefore we have

$$\det(A) = au^p + bv^p.$$

Let B be the adjugate matrix of A. Let $A_0 := B^T$ and $B_0 := A^T$. We have

$$AB = BA = (au^p + bv^p)I_3 = A_0 B_0 = B_0 A_0 \tag{2}$$

and

$$\det(A_0) = \det(B) = (au^p + bv^p)^2. \tag{3}$$

As the reduction of A modulo the ideal (u, v) of J has the first two columns zero, the reductions of B and A_0 modulo (u, v) are the zero 3×3 matrix. Transposing Equation (1), we get that $(u^p\ v^p\ u^pv^p) = (u\ v\ uv)B_0$ and by multiplying this identity with A_0 from the right we get

$$(u^p\ v^p\ u^pv^p)A_0 = (au^p + bv^p)(u\ v\ uv). \tag{4}$$

We consider four matrices

$$C := \begin{pmatrix} -v & 0 & 1 \\ 0 & -u & 1 \end{pmatrix} \in M_{2\times 3}(J),$$

$$C^{[p]} := \begin{pmatrix} -v^p & 0 & 1 \\ 0 & -u^p & 1 \end{pmatrix} \in M_{2\times 3}(J),$$

$$A' := \begin{pmatrix} -\alpha_1 v^p - 1 & 1 \\ -u^{p-1}v^{p-1} - 1 & u^{p-1}v^{p-1} - \alpha_2 u^p + 1 \end{pmatrix} \in M_2(J),$$

and

$$\mathfrak{A}' := \begin{pmatrix} -\alpha_1 v^p - 1 & 1 \\ -u^{p-1}v^{p-1} - 1 - p & u^{p-1}v^{p-1} - \alpha_2 u^p + 1 \end{pmatrix} \in M_2(J).$$

If $p > 2$ or if $p = 2$ and J is of characteristic 2, then $C^{[p]}$ is the matrix obtained from C by raising each entry to its p-th power.

Simple calculations show that the following three identities hold

$$A'C = C^{[p]}A = \begin{pmatrix} \alpha_1 v^{p+1} + v & -u & -\alpha_1 v^p \\ u^{p-1}v^p + v & -u^pv^{p-1} + \alpha_2 u^{p+1} - u & -\alpha_2 u^p \end{pmatrix}, \tag{5}$$

$$\begin{aligned}\det(A') &= -\alpha_1 u^{p-1}v^{2p-1} + \alpha_1\alpha_2 u^pv^p - \alpha_1 v^p - u^{p-1}v^{p-1} + \alpha_2 u^p - 1 + u^{p-1}v^{p-1} + 1 \\ &= \alpha_2 u^p(1 + \alpha_1 v^p) - \alpha_1 v^p(1 + u^{p-1}v^{p-1}) = au^p + bv^p, \end{aligned} \tag{6}$$

and

$$\det(\mathfrak{A}') = p + \det(A') = p + au^p + bv^p. \tag{7}$$

Thus, if B' and $\mathfrak{B}'$ are the adjugates of A' and $\mathfrak{A}'$ (respectively) and if A'_0, B'_0, $\mathfrak{A}'_0$, C_0, and $C_0^{[p]}$ are the transposes of B', A', $\mathfrak{B}'$, C, and $C^{[p]}$ (respectively), then from Equations (5) to (7) we get that the following three identities hold

$$(au^p + bv^p)A_0C_0 = (au^p + bv^p)C_0^{[p]}A'_0, \tag{8}$$

$$A_0'B_0' = B_0'A_0' = (au^p + bv^p)I_2, \tag{9}$$

and

$$\det(A_0') = au^p + bv^p \quad \text{and} \quad \det(\mathfrak{A}_0') = p + au^p + bv^p. \tag{10}$$

If $au^p + bv^p$ is a non-zero-divisor of J, then from Equation (8) we get that

$$A_0C_0 = C_0^{[p]}A_0'. \tag{11}$$

3 Proof of Proposition 1

We will prove Proposition 1 in three steps. The first two steps will prove Proposition 1 in two particular cases (see Subsections 3.1 and 3.2). The third step will complete the proof of Proposition 1 based on the first two steps and standard (universal) pullback arguments (see Subsection 3.3).

In Subsections 3.1 and 3.2 we will assume that the hypotheses of Proposition 1 hold and moreover $\mathcal{R}$ is regular and (x, y) is part of a regular system of parameters of $\mathcal{R}$. Let $\mathcal{V} := \mathcal{X} \setminus \mathrm{Spec}(\mathcal{R}/(x, y)) = \mathrm{Spec}(\mathcal{R}) \setminus \mathrm{Spec}(\mathcal{R}/(x, y))$. To ease the notation, the residue field of $\mathcal{R}$ will also be denoted by k.

3.1 The case when $\mathcal{R}$ is complete

We first consider the case when $\mathcal{R}$ is complete. Let $d \geq 2$ be the dimension of $\mathcal{R}$. Let $x, y, z_1, \ldots, z_{d-2}$ be a regular system of parameters of $\mathcal{R}$. Let

$$\mathfrak{S} := C(k)[[x, y, z_1, \ldots, z_{d-2}]]$$

and

$$S := k[[x, y, z_1, \ldots, z_{d-2}]].$$

For $\star \in \mathfrak{S}$, let $\bar{\star} \in S$ be its reduction modulo p.

We lift the epimorphism $C(k) \to k$ to a homomorphism $C(k) \to \mathcal{R}$. We extend $C(k) \to \mathcal{R}$ to an epimorphism

$$\mathfrak{S} \to \mathcal{R}$$

which maps each $t \in \{x, y, z_1, \ldots, z_{d-2}\}$ (viewed as an indeterminate) to the element t of $\mathcal{R}$. We consider elements of $\mathfrak{S}$ which map to the elements a, b, c of $\mathcal{R}$ (respectively) and to ease the notation we denote them also by a, b, c

(respectively). If $c \in R$ is 0, then we choose $c \in \mathfrak{S}$ to be 0 also. The kernel I of the epimorphism $\mathfrak{S} \to \mathcal{R}$ contains the regular parameter

$$f = p + ax^p + by^p + cx^{p-1}y^{p-1} \in \mathfrak{S}.$$

The epimorphism $\mathfrak{S}/(f) \to \mathcal{R}$ of regular local rings of dimension d is an isomorphism and therefore we have $I = (f)$.

Let σ_k be a Frobenius endomorphism of $C(k)$. Let σ be the Frobenius endomorphism of $\mathfrak{S}$ which is compatible with σ_k and which maps each element $t \in \{x, y, z_1, \ldots, z_{d-2}\}$ to t^p. For such an element t we often denote simply by t its reduction $\bar{t}$ modulo p. We denote also by σ its reduction modulo p, i.e., the Frobenius endomorphisms of S. For a $\mathfrak{S}$-module N let $N^{(\sigma)} := \mathfrak{S} \otimes_{\sigma,\mathfrak{S}} N$. In what follows by a nilpotent Breuil module or window we mean a (covariant) nilpotent Breuil module or window for the frame

$$(\mathfrak{S}, I, \mathcal{R}, \sigma, \dot{\sigma}, \sigma(f))$$

as used in [L1], Subsects. 10.4 and 10.5 and Def. 11.1; here

$$\dot{\sigma} : I \to \mathfrak{S}$$

is the σ-linear map defined by the rule $\dot{\sigma}(fx) := \sigma(x)$. If k is perfect, we will also use (covariant) Breuil modules for this frame which are not nilpotent.

For $r \in \mathbb{N}$ let $M_r := S^r$. We identify naturally $M_r^{(\sigma)} = S \otimes_{\sigma,S} M_r$ with M_r. Let $\varphi_1 : M_1 \to M_1^{(\sigma)}$ be the S-linear map defined by the multiplication by $\bar{f}$.

We consider two subcases on the possible values of c.

Subcase 1: $c = 0$. We assume that $c = 0$; thus $\bar{f} = \bar{a}x^p + \bar{b}y^p \in S$. Let $A_0 \in M_{3\times 3}(S)$ be such that we have (see Equations (2) to (4) of Section 2 applied with $J = S$ and $u = x$ and with $v = y$ belonging to $\mathfrak{m}_J = \mathfrak{m}_S$)

$$(x^p \; y^p \; x^p y^p)A_0 = \bar{f}(x \; y \; xy), \tag{12}$$

$\det(A_0) = (\bar{a}x^p + \bar{b}y^p)^2 = \bar{f}^2$, the reduction of A_0 modulo (x, y) is the zero 3×3 matrix, and there exists $B_0 \in M_{3\times 3}(S)$ such that we have

$$B_0 A_0 = A_0 B_0 = \bar{f} I_3. \tag{13}$$

Let $\varphi_3 : M_3 \to M_3^{(\sigma)}$ be the S-linear map whose matrix representation with respect to the standard bases of M_3 and $M_3^{(\sigma)} = M_3$ is A_0. From (13) we get that the cokernel of φ_3 is annihilated by $\bar{f}$. Thus the pair (M_3, φ_3) is the nilpotent Breuil module of a connected finite flat commutative group scheme $\mathcal{G}$ over $\mathcal{X}$ annihilated by p and of order p^3, see [L1], Thm. 10.7. Similarly, the pair (M_1, φ_1) is the nilpotent Breuil module of $\boldsymbol{\mu}_{\mathbf{p},\mathcal{X}}$.

Equation (12) shows that the S-linear map $\beta_0 : M_3 \to M_1$ defined by the matrix $(x\ y\ xy)$ has cokernel $M_1/(x,y)M_1 = S/(x,y)$ and is a morphism of nilpotent Breuil modules

$$\beta_0 : (M_3, \varphi_3) \to (M_1, \varphi_1).$$

We consider the homomorphism $\delta_0 : \mathcal{G} \to \boldsymbol{\mu}_{\mathbf{p},\mathcal{X}}$ associated to β_0, see [L1], Thm. 10.7. We recall that (see [L1], Lem. 8.2 and Subsect. 10, [Z], Thm. 6 and Cor. 97, and [BM], Cor. 3.2.11):

(♯) *The morphism of (covariant) Dieudonné crystals*

$$\mathbb{D}(\delta_0) : \mathbb{D}(\mathcal{G}) \to \mathbb{D}(\boldsymbol{\mu}_{\mathbf{p},\mathcal{X}})$$

of the nilpotent crystalline site $\mathrm{NCris}(\mathrm{Spec}(\mathcal{R}/p\mathcal{R})/\mathrm{Spec}(\mathcal{R}/p\mathcal{R}))$ *(see [Be], Ch. 3, Subsect. 1.3.1 and [BBM], Ch. 3) is defined by the S-linear map*

$$\beta_0^{(\sigma)} : M_3^{(\sigma)} \to M_1^{(\sigma)}.$$

If k is perfect, then property (♯) also follows from [L2], Prop. 7.1.

Let $\mathfrak{p}$ be a prime ideal of $\mathcal{R}$ that contains p and let κ be the perfection of its residue field. We consider the composite homomorphism $\mathfrak{S} \to \mathcal{R} \to \kappa$. As $1_\kappa \otimes \beta_0$ is not surjective if and only if $\mathfrak{p} \supset (x,y)$, from the property (♯) we get that $\mathbb{D}(\delta_{0,\kappa})$ is not surjective if and only if $\mathfrak{p} \supset (x,y)$. From this and the classical Dieudonné theory (see [BBM], Thm. 4.2.14) we get that δ_0 is not an epimorphism at $\mathfrak{p}$ if and only if $\mathfrak{p} \supset (x,y)$. This implies that $\delta_{0,\mathcal{V}} : \mathcal{G}_{\mathcal{V}} \to \boldsymbol{\mu}_{\mathbf{p},\mathcal{V}}$ is an epimorphism while at every point of $\mathrm{Spec}(\mathcal{R}/(x,y))$ the homomorphism δ_0 is not an epimorphism.

Let $\mathcal{F}_{\mathcal{V}} := \mathrm{Ker}(\delta_{0,\mathcal{V}})$. Let $\mathcal{F}$ be the affine $\mathcal{X}$-scheme of global sections of $\mathcal{F}_{\mathcal{V}}$. We check that $\mathcal{F}$ is naturally a connected truncated Barsotti–Tate group of level 1 over $\mathcal{X}$ of height 2 and dimension 1 that extends $\mathcal{F}_{\mathcal{V}}$. We consider the nilpotent Breuil module (M_2, ϕ_2), where ϕ_2 is the S-linear matrix whose matrix representation with respect to the standard bases of M_2 and $M_2^{(\sigma)} = M_2$ is the matrix

$$A_0' = \begin{pmatrix} 1 - \overline{\alpha_2}x^p + x^{p-1}y^{p-1} & x^{p-1}y^{p-1} + 1 \\ -1 & -1 - \overline{\alpha_1}y^{p-1} \end{pmatrix}.$$

Let $\alpha_1, \alpha_2 \in \mathfrak{S}$ be as in Section 2 applied with $(J, u, v, a, b) = (\mathfrak{S}, x, y, a, b)$. Let $\overline{\alpha_1}$ and $\overline{\alpha_2} \in S$ be the reductions modulo p of α_1 and α_2 (respectively); thus $\overline{\alpha_1}, \overline{\alpha_2}$ are as in Section 2 applied with $(J, u, v, a, b) = (S, x, y, \bar{a}, \bar{b})$. From Equation (10) we get that $\det(A'_0) = \bar{a}x^p + \bar{b}y^p = \bar{f}$. Either from this or from Equation (9) we get that the cokernel of ϕ_2 is annihilated by $\bar{f}$. If $A_0'^{[p]} \in M_{2\times 2}(S)$ is the matrix obtained from A'_0 by raising each entry to its p-th power, i.e., is the matrix representation of $\phi_2^{(\sigma)}$, then the reductions of $A_0'^{[p]}A'_0$ and $(A'_0)^2$ modulo (x, y) are the zero 2×2 matrix.

If $\zeta_0 : M_2 \to M_3$ is the S-linear map whose matrix representation with respect to the standard bases of M_2 and M_3 is

$$C_0 := \begin{pmatrix} -y & 0 \\ 0 & -x \\ 1 & 1 \end{pmatrix} \in M_{3\times 2}(S),$$

then from Equation (11) we get a morphism of nilpotent Breuil modules $\zeta_0 : (M_2, \phi_2) \to (M_3, \varphi_3)$ associated to a morphism $\mathcal{F}' \to \mathcal{G}$ of connected finite flat group schemes over $\mathcal{X}$.

Let $\mathfrak{M}_2 := \mathfrak{S}^2$. We identify naturally $\mathfrak{M}_2^{(\sigma)} = \mathfrak{S} \otimes_{\sigma,\mathfrak{S}} \mathfrak{M}_2$ with $\mathfrak{M}_2$. We consider the $\mathfrak{S}$-linear map $\Phi_2 : \mathfrak{M}_2 \to \mathfrak{M}_2^{(\sigma)}$ whose matrix representation with respect to the standard bases of $\mathfrak{M}_2$ and $\mathfrak{M}_2^{(\sigma)} = \mathfrak{M}_2$ is

$$\mathfrak{A}'_0 := \begin{pmatrix} 1 - \alpha_2 x^p + x^{p-1}y^{p-1} & x^{p-1}y^{p-1} + p + 1 \\ -1 & -1 - \alpha_1 y^{p-1} \end{pmatrix}.$$

From Equation (10) applied with $(J, u, v) = (\mathfrak{S}, x, y)$ we get the identity $\det(\mathfrak{A}'_0) = p + ax^p + by^p = f \in \mathfrak{S}$ and thus the cokernel of Φ_2 is annihilated by f. From this and the fact that the reduction of $(\mathfrak{M}_2, \Phi_2)$ modulo p is (M_2, ϕ_2), we get that $(\mathfrak{M}_2, \Phi_2)$ is a nilpotent Breuil window associated to a connected Barsotti–Tate group $\mathcal{D}$ over $\mathcal{X}$ whose truncated Barsotti–Tate group of level 1 over $\mathcal{X}$ is $\mathcal{F}' = \mathcal{D}[p]$.

As $(x\ y\ xy)C_0 = 0$ we have $\beta_0 \circ \zeta_0 = 0$; in fact we have $\mathrm{Im}(\zeta_0) = \mathrm{Ker}(\beta_0)$. Therefore $\mathcal{F}' \to \mathcal{G} \to \boldsymbol{\mu}_{\mathbf{p},\mathcal{X}}$ is a complex of connected finite flat group schemes over $\mathcal{X}$. An argument similar to the one above which checked that $\delta_{0,\mathcal{V}}$ is an epimorphism, shows that the complex $\mathcal{F}'_{\mathcal{V}} \to \mathcal{G}_{\mathcal{V}} \to \boldsymbol{\mu}_{\mathbf{p},\mathcal{V}}$ is exact. Thus $\mathcal{F}'_{\mathcal{V}} = \mathrm{Ker}(\delta_{0,\mathcal{V}})$. From this and the fact that the codimension of $\mathcal{X} \setminus \mathcal{V}$ in $\mathcal{X}$ is 2 we get that $\mathcal{F} = \mathcal{F}' = \mathcal{D}[p]$.

As $\det(\mathfrak{A}'_0) = f$, $\mathrm{Coker}(\Phi_2)$ is isomorphic to the $\mathfrak{S}$-module $\mathcal{R}$. Thus $\mathcal{D}$ has height 2 and dimension 1. This implies that $\mathcal{D}$ has a principal quasi-polarization Ψ.

We consider the $C(k)$-module $N := C(k)^2$; we naturally identify it with $N^{(\sigma_k)} = C(k)\otimes_{\sigma_k, C(k)} N$. The connected Barsotti–Tate group $\mathcal{D}_k$ over $\mathrm{Spec}(k)$ is associated to the nilpotent Breuil window for the frame

$$(C(k), pC(k), k, \sigma_k, \dot{\sigma}_k, p)$$

with $\dot{\sigma}_k(wp) := \sigma_k(w)$ for each $w \in C(k)$, defined by the $C(k)$-linear map $N \to N^{(\sigma_k)}$ whose matrix representation with respect to the standard bases of N and $N^{(\sigma_k)} = N$ is the reduction

$$E := \begin{pmatrix} 1 & p+1 \\ -1 & -1 \end{pmatrix} \in M_{2\times 2}(C(k))$$

modulo $(x, y, z_1, \ldots, z_{d-2})$ of $\mathfrak{A}'_0$. This implies that $\mathcal{D}_k$ is defined over $\mathrm{Spec}(\mathbb{F}_p)$ and, as $E^2 = -pI_2$, in fact it is the pullback to $\mathrm{Spec}(k)$ of the Barsotti–Tate group $\mathcal{D}_{\mathbb{F}_p}$ over $\mathrm{Spec}(\mathbb{F}_p)$ whose covariant Dieudonné module is the pair $(\mathbb{Z}_p^2, \upsilon)$, where the $\mathbb{Z}_p$-linear endomorphism $\upsilon : \mathbb{Z}_p^2 \to \mathbb{Z}_p^2$ is defined by the rules $\upsilon(e_1) = pe_2$ and $\upsilon(e_2) = -e_1$, $\{e_1, e_2\}$ being the standard basis of the $\mathbb{Z}_p$-module $\mathbb{Z}_p^2$.

We note that $\mathcal{D}_{\mathbb{F}_{p^2}}$ is the Barsotti–Tate group of a supersingular elliptic curve over $\mathrm{Spec}(\mathbb{F}_{p^2})$ as one can easily check based on [BG-JGP], Lem. 3.21. Thus if k contains $\mathbb{F}_{p^2}$, $(\mathcal{D}, \Psi)$ is the principally quasi-polarized Barsotti–Tate group of an elliptic curve over $\mathcal{X}$ whose fiber over k is supersingular.

Subcase 2: $c \in \mathcal{R}^\times$. Assuming $\bar{f} = \bar{a}x^p + \bar{b}y^p + \bar{c}x^{p-1}y^{p-1} \in S$ with $c \in \mathcal{R}^\times$, [VZ], Thm. 2.8 (iii) and its proof can be easily adapted to give that Proposition 1 holds in this subcase as well. We recall and enlarge the computations of loc. cit. Let (M_2, φ_2) be the nilpotent Breuil module defined by the S-linear map $\varphi_2 : M_2 \to M_2^{(\sigma)}$ whose matrix representation with respect to the standard bases of M_2 and $M_2^{(\sigma)} = M_2$ is

$$A_1 := \begin{pmatrix} \bar{a}x + \bar{c}y^{p-1} & \bar{a}y \\ \bar{b}x & \bar{b}y + \bar{c}x^{p-1} \end{pmatrix}.$$

Note that A_1 modulo (x, y) is the zero matrix and we have

$$\det(A_1) = \bar{c}(\bar{a}x^p + \bar{b}y^p + \bar{c}x^{p-1}y^{p-1}) = \bar{c}\bar{f}.$$

As $\bar{c} \in S^\times$, we get that $\bar{f}$ annihilates the cokernel of φ_2.

Let $M_2 \to M_1$ be the S-linear map defined by the matrix $(x\ y)$. It defines a morphism of nilpotent Breuil modules

$$\beta_1 : (M_2, \varphi_2) \to (M_1, \varphi_1).$$

As in Subcase 1, β_1 is associated to a homomorphism $\delta_1 : \mathcal{G} \to \boldsymbol{\mu}_{\mathbf{p},\mathcal{X}}$ between finite flat commutative group schemes annihilated by p over $\mathcal{X}$ with $\mathcal{G}$ connected of order p^2. As in Subcase 1 one gets that $\delta_{1,\mathcal{V}} : \mathcal{G}_{\mathcal{V}} \to \boldsymbol{\mu}_{\mathbf{p},\mathcal{V}}$ is an epimorphism while at every point of $\mathrm{Spec}(\mathcal{R}/(x,y))$ the homomorphism δ_1 is not an epimorphism.

Let $\mathcal{F}_{\mathcal{V}} := \mathrm{Ker}(\delta_{1,\mathcal{V}})$. Let $\mathcal{F}$ be the affine $\mathcal{X}$-scheme of global sections of $\mathcal{F}_{\mathcal{V}}$. If $d = 2$ or if $\mathcal{F}$ is a flat $\mathcal{X}$-scheme, then $\mathcal{F}$ is a group scheme over $\mathcal{X}$ which extends $\mathcal{F}_{\mathcal{V}}$. We check that in fact $\mathcal{F}$ is always a group scheme that is a form of $(\mathbb{Z}/p\mathbb{Z})_{\mathcal{X}}$ which is trivial if and only if there exists a $(p-1)$-th root of c in $\mathcal{R}$. To check that this holds, by considering the pullback via the local flat morphism $\mathrm{Spec}(W(k_1)[[x, y, z_1, \ldots, z_{d-2}]]/(f)) \to \mathcal{X}$ with k_1 as the perfection of k (see [GV], Fact 3.1 for the flatness part) we can assume that the field k is perfect.

Let $\zeta_1 : M_1 \to M_2$ be the S-linear map that maps the element $1 \in M_1 = S$ to $(y, -x) \in M_2 = S^2$. We have $\mathrm{Im}(\zeta_1) = \mathrm{Ker}(\beta_1)$ and one computes that

$$\varphi_2(y, -x) = (\bar{c}y^p, -\bar{c}x^p) = \bar{c}(1 \otimes y, -1 \otimes x) \in M_2^{(\sigma)} = S \otimes_{\sigma,S} M_2 = M_2.$$

For $\nu \in S^\times$, if $\varphi_1^{\mathrm{t},\nu} : M_1 \to M_1^{(\sigma)}$ is the S-linear map defined by the multiplication by ν (thus $\varphi_1^{\mathrm{t},1}$ is the Breuil dual of φ_1), we get a morphism of Breuil modules $\zeta_1 : (M_1, \varphi_1^{\mathrm{t},\bar{c}}) \to (M_2, \varphi_2)$ which is associated to a homomorphism $\mathcal{F}' \to \mathcal{G}$ over $\mathcal{X}$, see [L2], Cor. 6.8 (the equivalent conditions of [L2], Prop. 6.2 hold in our context, see [L2], Rm. 6.3 and the fact that we have $\sigma(x) = x^p$ and $\sigma(y) = y^p$). As in Subcase 1 we argue that we have $\mathcal{F} = \mathcal{F}'$. Thus $\mathcal{F}$ and $(\mathbb{Z}/p\mathbb{Z})_{\mathcal{X}}$ are isomorphic if and only if $(M_1, \varphi_1^{\mathrm{t},\bar{c}})$ and $(M_1, \varphi_1^{\mathrm{t},1})$ are isomorphic which is equivalent to the existence of a $(p-1)$-th root of $\bar{c}$ in S and therefore to the existence of a $(p-1)$-th root of c in $\mathcal{R}$. Such $(p-1)$-th roots exist after a pullback via a local flat morphism $\mathrm{Spec}(W(k_1)[[x, y, z, \ldots, z_{d-2}]]/(f)) \to \mathcal{X}$ with k_1 a finite separable extension of k and hence indeed $\mathcal{F}$ is a form of $(\mathbb{Z}/p\mathbb{Z})_{\mathcal{X}}$.

From Subcases 1 and 2 we get that Proposition 1 hold in this case.

3.2 The case when $\mathcal{R}$ is the henselization of a regular local ring of a finitely generated $\mathbb{Z}_{(p)}$-algebra

We assume that $\mathcal{R}$ is the henselization of a regular local ring of a finitely generated $\mathbb{Z}_{(p)}$-algebra. Let $\widehat{\mathcal{X}} := \mathrm{Spec}(\widehat{\mathcal{R}})$. Let

$$\widehat{\mathcal{V}} := \widehat{\mathcal{X}} \setminus \mathrm{Spec}(\widehat{\mathcal{R}}/(x,y)) = \widehat{\mathcal{X}} \times_{\mathcal{X}} \mathcal{V}.$$

Let $\delta_{\widehat{\mathcal{X}}} : \widehat{\mathcal{G}} \to \boldsymbol{\mu}_{\mathbf{p},\widehat{\mathcal{X}}}$ be a homomorphism between connected finite flat group schemes over $\widehat{\mathcal{X}}$, with $\widehat{\mathcal{G}}$ of order p^2 or p^3, whose restriction to $\widehat{\mathcal{V}}$ is an epimorphism and which at every point of $\widehat{\mathcal{X}} \setminus \widehat{\mathcal{V}}$ is not an epimorphism (see Subsection 3.1 applied to the complete local ring $\widehat{\mathcal{R}}$).

We write

$$\widehat{\mathcal{R}} = \lim\mathrm{ind}_{\lambda\in\mathcal{F}} R_\lambda$$

as a filtered colimit of local rings R_λ of residue k which are localizations of finitely generated $\mathcal{R}$-algebras. Let $\lambda_0 \in \mathcal{F}$ be such that $\delta_{\widehat{\mathcal{X}}}$ is the pullback to $\widehat{\mathcal{X}}$ of a homomorphism $\delta_{\lambda_0} : G_{\lambda_0} \to \boldsymbol{\mu}_{\mathbf{p},X_{\lambda_0}}$ between finite flat commutative group schemes annihilated by p over $X_{\lambda_0} := \mathrm{Spec}(R_{\lambda_0})$. For $\lambda \in \mathcal{F}$ such that $\lambda \geq \lambda_0$, let $X_\lambda := \mathrm{Spec}(R_\lambda)$, let $\delta_\lambda : G_\lambda \to \boldsymbol{\mu}_{\mathbf{p},X_\lambda}$ be the pullback of δ_{λ_0} to X_λ, let V_λ be the largest non-empty open subscheme of X_λ with the property that the restriction of δ_λ to V_λ is an epimorphism, and let $Y_\lambda := X_\lambda \setminus V_\lambda$ be endowed with the reduced structure. Note that the Zariski closure $\widetilde{Y}_\lambda$ of the image of $\mathrm{Spec}(\widehat{R}/(x,y))$ in X_λ is a reduced closed subscheme of Y_λ. Moreover, $\widetilde{Y}_\lambda$ is also a closed subscheme of the reduced scheme Z_λ defined by $\mathrm{Spec}(R_\lambda/(x,y))$.

With $\dagger \in \{Y, Z\}$, the projective limit of the schemes $\dagger_\lambda$ for $\lambda \geq \lambda_0$ is $\mathrm{Spec}(\widehat{\mathcal{R}}/(x,y))$ and for each $\lambda' \geq \lambda \geq \lambda_0$, $\dagger_{\lambda'}$ is the reduced inverse image of $\dagger_\lambda$ via the morphism $X_{\lambda'} \to X_\lambda$. It is easy to see that these properties imply that there exists $\lambda_1 \in \mathcal{F}$ with $\lambda_1 \geq \lambda_0$ such that the image of the $\mathcal{X}$-morphism $\dagger_{\lambda_1} \to \dagger_{\lambda_0}$ is contained in $\widetilde{Y}_{\lambda_0}$. Thus we have the following property:

($\mho$) The closed subschemes Y_{λ_1} and Z_{λ_1} coincide with the reduced inverse image via the $\mathcal{X}$-morphism $X_{\lambda_1} \to X_{\lambda_0}$ of the closed subscheme $\widetilde{Y}_{\lambda_0}$ of X_{λ_0}.

Let $R_1 := R_{\lambda_1}$ and $X_1 := \mathrm{Spec}(R_1)$. We know that $\delta_{\widehat{\mathcal{X}}}$ is the pullback of δ_{λ_1}. As $\mathcal{R}$ is the henselization of a local ring of a finitely generated $\mathbb{Z}_{(p)}$-algebra, from Artin's approximation theorem (for instance, see [BLR], Sect. 3.6, Thm. 16) we get that the morphism $X_1 \to \mathcal{X}$ has a section. By pulling back δ_{λ_1} via this section we get a homomorphism $\delta : \mathcal{G} \to \boldsymbol{\mu}_{\mathbf{p},\mathcal{X}}$ of connected finite flat commutative group schemes annihilated by p over $\mathcal{X}$. Due to the property $(\mho)$, the reduced inverse image of $\widetilde{Y}_{\lambda_0}$ via the composite $\mathcal{X}$-morphism $\mathcal{X} \to X_1 \to X_{\lambda_0}$ is the reduced closed subscheme $\mathcal{X} \setminus \mathcal{V}$ of $\mathcal{X}$. Thus $\delta_{\mathcal{V}}$ is an epimorphism while at every point of $\mathcal{X} \setminus \mathcal{V}$ the homomorphism δ is not an epimorphism. Moreover, the order of $\mathcal{G}$ is the same as the order of $\widehat{\mathcal{G}}$ and therefore it is p^2 if $c \in \mathcal{R}^\times$ and it is p^3 if $c = 0$, see Subsection 3.1.

If $c = 0$, then the kernel of the restriction of $\delta_{\widehat{\mathcal{X}}}$ to $\widehat{\mathcal{V}}$ is the restriction to $\widehat{\mathcal{V}}$ of a truncated Barsotti–Tate group of level 1 over $\widehat{\mathcal{X}}$ of height 2 and dimension 1 whose fiber over k is supersingular (see Subcase 1 of Subsection 3.1) and thus we can choose λ_0 such that the kernel of the restriction of δ_{λ_0} to V_{λ_0} extends to a similar type of a Barsotti–Tate group of level 1 over X_{λ_0}. Hence $\mathrm{Ker}(\delta_{\mathcal{V}})$ extends to a connected truncated Barsotti–Tate group of level 1 over $\mathcal{X}$ of height 2 and dimension 1 (its fiber over k is automatically supersingular). If k contains $\mathbb{F}_{p^2}$, then we can assume that this connected Barsotti–Tate group of level 1 over $\mathcal{X}$ is the one of an elliptic curve over $\mathcal{X}$.

If $c \in \mathcal{R}^\times$, then as in the previous paragraph we argue that we can assume that the kernels of the restrictions of $\delta_{\widehat{\mathcal{X}}}$ to $\widehat{\mathcal{V}}$ and of $\delta_{\mathcal{X}}$ to $\mathcal{V}$ extend to $\widehat{\mathcal{X}}$ and $\mathcal{X}$ (respectively). As these two extensions have the same fiber over $\mathrm{Spec}(k)$, we get that they are forms of $(\mathbb{Z}/p\mathbb{Z})_{\widehat{\mathcal{X}}}$ and $(\mathbb{Z}/p\mathbb{Z})_{\mathcal{X}}$ (respectively) which are trivial if and only their fibers over k are trivial and thus if and only if there exists a $(p-1)$-th root of c in $\mathcal{R}$ (see end of Subsection 3.1).

We conclude that Proposition 1 hold in this case.

3.3 End of the proof of Proposition 1

We are now ready to prove Proposition 1.

If $c = 0$, we consider the universal ring

$$R_{\mathrm{univ},0} := \mathbb{Z}_{(p)}[a, b, x, y]/(p + ax^p + by^p)$$

and the homomorphism $\psi_0 : R_{\mathrm{univ},0} \to \mathcal{R}$ with the property that for each $t \in \{a, b, c, x, y\}$ we have $\psi_0(t + (p + ax^p + by^p)) = t \in \mathcal{R}$.

If $c \in \mathcal{R}^\times$, we consider the universal ring

$$R_{\mathrm{univ},1} := \mathbb{Z}_{(p)}[a, b, c, c^{-1}, x, y]/(p + ax^p + by^p + cx^{p-1}y^{p-1}),$$

its finite étale extension

$$R'_{\mathrm{univ},1} := R_{\mathrm{univ},1}[e]/(e^{p-1} - c) = \mathbb{Z}_{(p)}[a, b, e, e^{-1}, x, y]/\big(p + ax^p + by^p + (exy)^{p-1}\big),$$

and the homomorphism $\psi_1 : R_{\mathrm{univ},1} \to \mathcal{R}$ with the property that for each $t \in \{a, b, c, x, y\}$ we have $\psi_1(t + (p + ax^p + by^p + cx^{p-1}y^{p-1})) = t \in \mathcal{R}$.

Let $\epsilon \in \{0, 1\}$ be such that we have a homomorphism $\psi_\epsilon : R_{\mathrm{univ},\epsilon} \to \mathcal{R}$ and let R_ϵ be the henselization of the localization of $R_{\mathrm{univ},\epsilon}$ at the prime ideal $\psi_\epsilon^{-1}(\mathfrak{m}_{\mathcal{R}})$.

The four rings $R_{\mathrm{univ},\epsilon}$, $R_{\mathrm{univ},\epsilon}/(x)$, $R_{\mathrm{univ},\epsilon}/(y)$, and $R_{\mathrm{univ},\epsilon}/(x, y)$ are regular and x, y is a regular sequence in $R_{\mathrm{univ},\epsilon}$ and therefore the same properties hold for R_ϵ. Let $X_\epsilon := \mathrm{Spec}(R_\epsilon)$ and $V_\epsilon := X_\epsilon \setminus \mathrm{Spec}(R_\epsilon/(x, y))$. As $\mathcal{R}$ is henselian, we have a natural local homomorphism $R_\epsilon \to \mathcal{R}$ which defines a local morphism $\tau_\epsilon : \mathcal{X} \to X_\epsilon$ with the property that $\tau_\epsilon^{-1}(V_\epsilon) = \mathcal{V}$.

From Subsection 3.2 applied to R_ϵ we get that there exists a homomorphism of finite flat commutative group schemes annihilated by p over X_ϵ whose domain is connected of order $p^{3-\epsilon}$, whose codomain is $\boldsymbol{\mu}_{\mathbf{p},X_\epsilon}$, whose restriction to V_ϵ is an epimorphism, and which at every point of $X_\epsilon \setminus V_\epsilon$ is not an epimorphism. By pulling back this homomorphism via $\tau_\epsilon : \mathcal{X} \to X_\epsilon$ we get a homomorphism $\delta : \mathcal{G} \to \boldsymbol{\mu}_{\mathbf{p},\mathcal{X}}$ for which properties (i) and (ii) of Proposition 1 hold.

If $\epsilon = 0$, then from the end of Subsection 3.2 applied with $c = 0$ to R_0 we get that the property (iii) of Proposition 1 holds if $c = 0$.

If $\epsilon = 1$, then there exists a $(p-1)$-th root of c in $\mathcal{R}$ if and only if $\psi_1 : R_{\mathrm{univ},1} \to \mathcal{R}$ factors through a homomorphism $\psi_1' : R'_{\mathrm{univ},1} \to \mathcal{R}$. Based on this and the last paragraph of Subsection 3.2 applied to the henselizations R'_1 of each localization of $R'_{\mathrm{univ},1}$ at a prime ideal that dominates the prime ideal $\psi_1^{-1}(\mathfrak{m}_{\mathcal{R}})$ of $R_{\mathrm{univ},1}$, we get that we have $\mathrm{Ker}(\delta_{\mathcal{V}}) = (\mathbb{Z}/p\mathbb{Z})_{\mathcal{V}}$ if and only if there exists a $(p-1)$-th root of c in $\mathcal{R}$. Thus the property (iii) of Proposition 1 holds if $c \in \mathcal{R}^\times$. $\square$

3.4 Variants of Proposition 1

The finite flat group scheme $\mathcal{G}$ of Subcase 1 (resp. Subcase 2) of Subsection 3.1 is the quotient of a connected Barsotti–Tate group of level 1 over $\mathcal{X}$ of order at most p^6 (resp. at most p^4), see the proofs of [L1], Lem. 10.8 and Thm. 8.5. Thus we have a variant of the first part of Proposition 1 in which $\mathcal{G}$ is a connected Barsotti–Tate group of level 1 over $\mathcal{X}$ of order p^4 if $c \in \mathcal{R}^\times$ and of order p^6 if $c = 0$, without being able to say anything about either $\mathrm{Ker}(\delta_{\mathcal{V}})$ or its extensions to $\mathcal{X}$.

We refer to Subcase 1 of Subsection 3.1 in one of the following three situations which relate to [VZ], Thm. 28:

(i) The element $\bar{f} = \bar{a}x^p + \bar{b}y^p \in S$ is such that there exists an element $\bar{c}_1$ of the ideal (x, y) of S which divides both $\bar{a}$ and $\bar{b}$ in S.

(ii) We have a product factorization $\bar{a}x^p + \bar{b}y^p = x^{p-1}y^{p-1}(\bar{a}_1 x + \bar{b}_1 y)$, with $\bar{a}_1, \bar{b}_1 \in S$.

(iii) We have a product factorization $\bar{f} = \bar{a}x^p + \bar{b}y^p = wu^{p-1}v^{p-1}$ with $w \in S^\times$ and with $u, v \in S$ such that $g.c.d.(u, v) = 1$ (e.g., for $p = 2$ this holds with $w = 1$ if the principal ideal $(\bar{f})$ of S is not a power of a principal prime ideal of S) and the radical of the ideal (u, v) of S is (x, y).

In this paragraph we assume that (i) holds. Writing $\bar{c}_1 = sx+ty$, $\bar{a} = \bar{c}_1\bar{a}'$, $\bar{b} = \bar{c}_1\bar{b}'$, with $s, t, \bar{a}', \bar{b}' \in S$, we have

$$\bar{f} = \bar{c}_1(\bar{a}_1 x^p + \bar{b}_1 y^p + \bar{c}_1 x^{p-1}y^{p-1}),$$

where $\bar{a}_1 := \bar{a}' - sy^{p-1}$ and $\bar{b}_1 := \bar{b}' - tx^{p-1}$. In this situation we have a variant of Subcase 1 of Subsection 4.1 which is similar to [VZ], Thm. 28 (iii) and which is modeled on Subcase 2 of Subsection 4.1 but working with the triple $(\bar{a}_1, \bar{b}_1, \bar{c}_1) \in S^3$ instead of the triple $(\bar{a}, \bar{b}, \bar{c}) \in S^3$. We end up with a homomorphism $\mathcal{G} \to \mathcal{H}$ between finite flat group schemes over $\mathcal{X}$ such that:

• $\mathcal{H}$ is associated to the nilpotent Breuil module defined by the S-linear map $M_1 \to M_1^{(\sigma)}$ which is the multiplication by the factor $\bar{a}_1 x^p + \bar{b}_1 y^p + \bar{c}_1 x^{p-1}y^{p-1}$ of $\bar{f}$ and thus it is connected of order p but non-isomorphic to $\boldsymbol{\mu}_{\mathbf{p},\mathcal{X}}$;

• $\mathcal{G}$ is connected of order p^2;

• $\mathrm{Ker}(\delta_{\mathcal{V}})$ extends to a connected finite flat group scheme over $\mathcal{X}$ associated to the nilpotent Breuil module defined by the S-linear map $M_1 \to M_1^{(\sigma)}$ which is the multiplication by $\bar{c}_1$.

In this paragraph we assume that (ii) holds. In this situation we have a variant of Subcase 1 of Subsection 4.1 which is similar to [VZ], Thm. 28 (ii): we get a morphism of nilpotent Breuil modules $\beta_2 : (M_2, \varphi_{2,1}) \to (M_1, \varphi_1)$, where $\varphi_{2,1} : M_2 \to M_2^{(\sigma)}$ is the S-linear map whose matrix representation with respect to standard bases is the matrix

$$(\bar{a}_1 x + \bar{b}_1 y) \begin{pmatrix} y^{p-1} & 0 \\ 0 & x^{p-1} \end{pmatrix}.$$

We end up with a homomorphism $\mathcal{G} \to \boldsymbol{\mu}_{\mathbf{p},\mathcal{X}}$ between connected finite flat group schemes over $\mathcal{X}$ with $\mathcal{G}$ of order p^2 and with $\mathrm{Ker}(\delta_{\mathcal{V}})$ extending to a connected finite flat group scheme over $\mathcal{X}$ associated to the nilpotent Breuil module defined by the S-linear map $M_1 \to M_1^{(\sigma)}$ which is the multiplication by $\bar{a}_1 x + \bar{b}_1 y$.

In this paragraph we assume that (iii) holds. In this situation we have a variant of Subcase 1 of Subsection 4.1 which is similar to [VZ], Thm. 28 (ii) as in the previous paragraph: we get a morphism of nilpotent Breuil modules $\beta_3 : (M_2, \varphi_{2,2}) \to (M_1, \varphi_1)$, where β_3 and $\varphi_{2,2} : M_2 \to M_2^{(\sigma)}$ are the S-linear maps whose matrix representations with respect to standard bases are respectively $(u\ v)$ and

$$w \begin{pmatrix} v^{p-1} & 0 \\ 0 & u^{p-1} \end{pmatrix}.$$

We end up with a homomorphism $\mathcal{G} \to \boldsymbol{\mu}_{\mathbf{p},\mathcal{X}}$ between connected finite flat group schemes over $\mathcal{X}$ with $\mathcal{G}$ of order p^2 and with $\mathrm{Ker}(\delta_{\mathcal{V}})$ extending to an étale finite flat group scheme over $\mathcal{X}$ which is $(\mathbb{Z}/p\mathbb{Z})_{\mathcal{X}}$ if $w = 1$.

4 Proof of Theorem 1

We first prove Theorem 1 (a) and thus also the ‘if’ part of Theorem 1 (c). By replacing f with a multiple of it by a unit of $C(k)$ or (in the case when p is a regular parameter of R) with its image under an automorphism of $C(k)[[x, y]]$, we can assume that f is normalized as in [VZ], i.e., $h := p - f$ belongs to the ideal (x, y) of $C(k)[[x, y]]$.

If l is an algebraic closure of k, then the composite homomorphism

$$R \to \widehat{R} \simeq C(k)[[x, y]]/(f) \to W(l)[[x, y]]/(f)$$

is faithfully flat (see [GV], Fact 3.1). Thus the fact that R is p-quasi-healthy follows from [VZ], Thm. 3.

4.1 The proof of Theorem 1 (b)

We now show that Theorem 1 (b) follows from Proposition 1. Let

$$\rho : C(k)[[x, y]] \to \widehat{R}$$

be an epimorphism of rings whose kernel $\mathrm{Ker}(\rho)$ is generated by an element $f \in (p, x^p, y^p, x^{p-1}y^{p-1})$. Let $u, v \in R$ be such that $\mathfrak{m}_R = (u, v)$. We consider a $C(k)$-epimorphism $\rho' : C(k)[[u,v]] \to \widehat{R}$ which maps the indeterminates u and v to the elements u and v of $\widehat{R}$. Let $g \in C(k)[[u,v]]$ be such that it generates the kernel of ρ'. Let

$$\omega : C(k)[[u,v]] \to C(k)[[x,y]]$$

be a $C(k)$-homomorphism such that $\rho' = \rho \circ \omega$. The cotangent spaces of the local rings R/pR and $\widehat{R}/p\widehat{R} \simeq k[[x,y]]/(\bar{f})$ are k-vector space of dimension 2 having as bases the images of u and v and the images of x and y (respectively). This implies that the cotangent map of ω is an isomorphism. Therefore ω is an isomorphism such that we have $\omega((g)) = (f)$.

The ideal $(x^p, y^p, x^{p-1}y^{p-1})$ of $k[[x,y]]$ does not depend on the regular system x, y of parameters of $\mathfrak{m}_{k[[x,y]]}$ as it is equal to $\mathfrak{m}_{k[[x,y]]}^{[p]} + \mathfrak{m}_{k[[x,y]]}^{2p-2}$, where $\mathfrak{m}_{k[[x,y]]}^{[p]}$ is the ideal generated by all p-th powers of elements of $\mathfrak{m}_{k[[x,y]]}$. Hence, as ω is an isomorphism, the two ideals $(p, \omega(u)^p, \omega(v)^p, \omega(u)^{p-1}\omega(v)^{p-1})$ and $(p, x^p, y^p, x^{p-1}y^{p-1})$ of $C(k)[[x,y]]$ coincide.

Thus $(f) \subset (p, x^p, y^p, x^{p-1}y^{p-1}) = (p, \omega(u)^p, \omega(v)^p, \omega(u)^{p-1}\omega(v)^{p-1})$ if and only if $(g) = \omega^{-1}((f)) \subset (p, u^p, v^p, u^{p-1}v^{p-1})$. As the 1-dimensional k-vector spaces generated by the images of g and p in the cotangent space of the local ring $C(k)[[u,v]]$ are equal, the inclusion $(g) \subset (p, u^p, v^p, u^{p-1}v^{p-1})$ is equivalent to inclusions $pC(k)[[x,y]] \subset (u^p, v^p, u^{p-1}v^{p-1}, g) \subset C(k)[[x,y]]$ and thus to inclusions $p\widehat{R} \subset (u^p, v^p, u^{p-1}v^{p-1}) \subset \widehat{R}$ and hence also to inclusions $pR \subset (u^p, v^p, u^{p-1}v^{p-1}) \subset R$. A similar argument shows that the inclusions $(f) \subset (p, x^p, y^p) \subset \widehat{R}$ are equivalent to inclusions $pR \subset (u^p, v^p) \subset R$.

As we are assuming that $(f) \subset (p, x^p, y^p, x^{p-1}y^{p-1})$, based on the previous paragraph we conclude that there exists an identity

$$p + au^p + bv^p + cu^{p-1}v^{p-1} = 0 \tag{14}$$

with $a, b, c \in R$. Note that we have $c \notin R^\times$, i.e., we can write

$$c = c_1 u + c_2 v \in \mathfrak{m}_R$$

with $c_1, c_2 \in R$, if and only if we have

$$p + \tilde{a}u^p + \tilde{b}v^p = 0$$

with $(\tilde{a}, \tilde{b}) := (a + c_1 v^{p-1}, b + c_2 u^{p-1})$.

The last two paragraphs imply that either $f \in (p, x^p, y^p)$ and Equation (14) holds with $c = 0$ or $f \notin (p, x^p, y^p)$ and Equation (14) holds with $c \in R^\times$.

As hypotheses of Proposition 1 hold in the context of R and of elements $u, v \in \mathfrak{m}_R$ and $a, b, c \in R$, Theorem 1 (b) follows from the particular case of Proposition 1 proved in Subsection 3.1.

Moreover, if $c = 0$ and k contains $\mathbb{F}_{p^2}$, then from the Subcase 1 of Subsection 3.1 we get that $\mathrm{Ker}(\gamma_U)$ extends to a Barsotti–Tate group of level 1 over X which is the one associated to an elliptic curve over X. Similarly, if $c \in R^\times$, from the the Subcase 1 of Subsection 3.1 we get that $\mathrm{Ker}(\gamma_U)$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})_U$ if and only if there exists a $(p-1)$-th root of c in R.

4.2 End of the proof of Theorem 1

We are left to show that the 'only if' part of Theorem 1 (c) follows from Theorem 1 (b). Using the contrapositive it suffices to show that the assumption that $f \in (p, x^p, y^p, x^{p-1}y^{p-1})$ implies that R is not p-quasi-healthy. We consider a homomorphism $\gamma : G \to \boldsymbol{\mu}_{\mathbf{p},X}$ of connected finite flat group schemes annihilated by p over X which is not an epimorphism and which extends an epimorphism $\gamma_U : G_U \to \boldsymbol{\mu}_{\mathbf{p},U}$ over U, see Theorem 1 (b). Using it, from [VZ], Lem. 27 we get that R is not p-quasi-healthy. Thus Theorem 1 holds.$\square$

4.3 Cohomology classes

The homomorphism $\mathrm{Ext}^1_{\mathrm{fppf}}(\boldsymbol{\mu}_{\mathbf{p},X}, (\mathbb{Z}/p\mathbb{Z})_X) \to \mathrm{Ext}^1_{\mathrm{fppf}}(\boldsymbol{\mu}_{\mathbf{p},U}, (\mathbb{Z}/p\mathbb{Z})_U)$ is injective regardless of what R is and thus we can define the quotient group

$$\mathcal{A}_U := \mathrm{Ext}^1_{fppf}(\boldsymbol{\mu}_{\mathbf{p},U}, (\mathbb{Z}/p\mathbb{Z})_U)/\mathrm{Ext}^1_{fppf}(\boldsymbol{\mu}_{\mathbf{p},X}, (\mathbb{Z}/p\mathbb{Z})_X)$$

which depends only on U.

We refer to Theorem 1 (b) and assume that $f = p + ax^p + by^p + cx^{p-1}y^{p-1}$ with $c \in C(k)[[x, y]]^\times$. The short exact sequence

$$0 \to \mathrm{Ker}(\gamma_U) \to G_U \to \boldsymbol{\mu}_{\mathbf{p},U} \to 0 \tag{15}$$

defined by the epimorphism γ_U does not extend to a short exact sequence over X (as otherwise γ would be an epimorphism).

If there exists no $(p-1)$-th root in k of the image of c in k (equivalently, $\mathrm{Ker}(\gamma_U)$ is not isomorphic to $(\mathbb{Z}/p\mathbb{Z})_U$) and if moreover we have an identity $ax^p + by^p + cx^{p-1}y^{p-1} = ch^{p-1}$ with $h \in C(k)[[x, y]$ such that the image of (h)

in $\widehat{R}$ is a product of distinct prime ideals of $\widehat{R}$ (e.g., this holds if $a = b = 0$), then it can be checked that $\mathrm{Ext}^1_{\mathrm{fppf}}(\boldsymbol{\mu}_{\mathbf{p},U}, (\mathbb{Z}/p\mathbb{Z})_U) = 0$.

If there exists a $(p-1)$-th root of c in R, then (15) is a short exact sequence

$$0 \to (\mathbb{Z}/p\mathbb{Z})_U \to G_U \to \boldsymbol{\mu}_{\mathbf{p},U} \to 0 \tag{16}$$

(see end of Subsection 4.1). Thus (16) defines a non-zero class of $\mathcal{A}_U$; such short exact sequences (16) first show up in [Ga1].

4.4 Proof of Corollary 2

We assume that $p = 2$ and that R is complete of dimension 2 and is 2-quasi-healthy. Considering an isomorphism $R \simeq C(k)[[x, y]]/(f)$, from Theorem 1 (c) we get that the reduction $\bar{f}$ of f modulo 2 is an element of $\mathfrak{m}_{k[[x,y]]}$ which does not belong to the ideal (x^2, y^2, xy). Thus $\bar{f}$ is a regular parameter of $k[[x, y]]$ and by interchanging the roles of x and y if needed, we can assume that $k[[x, y]] = k[[x, \bar{f}]]$. Therefore $C(k)[[x, y]] = C(k)[[x, f]]$. We conclude that $R \simeq W[[x, f]]/(f) = C(k)[[x]]$. □

5 Five general lemmas

We first include three general lemmas required to prove Theorem 2 in Section 6 and then we include two extra general lemmas needed to prove Theorems 3 and 4 in Section 7. The first lemma recalls a particular case of [GV], Lem. 2.6.

Lemma 1 *Let R' be a regular local ring of the same dimension as R that is a faithfully flat R-algebra. If R' is p-quasi-healthy, then R is p-quasi-healthy.*

Lemma 2 *Let R' be a regular local ring which is an ind-finite ind-étale R-algebra. We assume that one of the following two conditions holds:*

(i) *The homomorphism $R \to R'$ is in fact finite.*

(ii) *We have* $\dim(R) = 2$.

Then R' is p-quasi-healthy if and only if R is p-quasi-healthy.

Proof: The 'only if' parts follow from Lemma 1. We are left to show that if R is p-quasi-healthy, then R' is p-quasi-healthy. We will follow the ideas of [V1], Rm. 3.2.2 4) and thus we will rely on properties of Weil restriction of scalars as in [BLR], Sect. 7.6 and [V2], Subsect. 2.3. Let $X' := \mathrm{Spec}(R')$; so $U' := X' \times_X U$ is the punctured spectrum of R'. Let R_1 be a regular local ring which is an ind-finite ind-étale R'-algebra such that the composite homomorphism $R \to R_1$ is Galois of Galois group Θ. If the condition (i) holds, then we can assume that Θ is finite. If R_1 is p-quasi-healthy, then R' is p-quasi-healthy (see Lemma 1). Thus by replacing R' with R_1 we can assume that the homomorphism $R \to R'$ is Galois of Galois group Θ.

To end the proof it suffices to show that each Barsotti–Tate group $D'_{U'}$ over U' extends to a Barsotti–Tate group D' over X'.

We first prove that D' exists when the condition (i) holds. Thus Θ is finite. Defining $D_U[p^n] := \mathrm{Res}_{U'/U} D'_{U'}[p^n]$, the inductive system $D_U[p^n]$ indexed by $n \in \mathbb{N}^*$ is a Barsotti–Tate group over U which extends to a Barsotti–Tate group D over X as R is p-quasi-healthy. We have

$$D_{U'} := U' \times_U D_U = \prod_{h \in \Theta} U' \times_{U',h} D'_{U'},$$

see [V2], Prop. 2.3.1 applied in the context of Barsotti–Tate groups of finite levels over affine open subschemes of U and their pullbacks to U'. Thus there exists a projector of $D_{U'}$ whose image is $D'_{U'}$. It extends to a projector of $D_{X'} := X' \times_X D_X$ whose image is a Barsotti–Tate group over X' that extends $D'_{U'}$.

We now prove that D' exists when the condition (ii) holds. Thus R has dimension 2. Let $m, n \in \mathbb{N}^*$. The short exact sequence

$$0 \to D'_{U'}[p^m] \to D'_{U'}[p^{n+m}] \to D'_{U'}[p^n] \to 0 \tag{17}$$

over U' is defined over $U_{m,n} := X_{m,n} \backslash \mathrm{Spec}(k_{m,n})$, where $R_{m,n} \subset R'$ is local and a finite étale R-algebra of residue field $k_{m,n}$ and where $X_{m,n} := \mathrm{Spec}(R_{m,n})$. Let $\mathcal{S}_{m,n}$ be a short exact sequence over $U_{m,n}$ whose pullback to U' is (17). As R has dimension 2, the Weil restriction $\mathrm{Res}_{U_{m,n}/U} \mathcal{S}_{m,n}$ extends uniquely (see [GV], Fact 2.1 (b)) to a complex $0 \to D_m \to D_{m+n} \to D_n \to 0$ of commutative finite flat group schemes of p-th power order over X whose restriction to U is a short exact sequence; this complex depends not only on m and n but also on the choice of $R_{m,n}$. If the homomorphism $D_{m+n} \to D_n$ is not an epimorphism, then from [VZ], Lem. 27 we get that R is not p-quasi-healthy and this is a contradiction. Thus the homomorphism $D_{m+n} \to D_n$ is an epimorphism and this implies that $0 \to D_m \to D_{m+n} \to D_n \to 0$ is a short exact sequence whose pullback to $X_{m,n}$ will be denoted as $\mathcal{D}_{m,n}$. As in the previous paragraph, based on [V2], Prop. 2.3.1 we get that $\mathcal{S}_{m,n}$ is the image of a projector of $\mathcal{D}_{m,n,U_{m,n}}$ which extends to a projector of $\mathcal{D}_{m,n}$ whose image is a short exact sequence over $X_{m,n}$ which is the complex that extends $\mathcal{S}_{m,n}$ and therefore its pullback to X' is a short exact sequence $0 \to D'_m \to D'_{m+n} \to D'_n \to 0$ which is the complex that extends (17) and which depends on m and n but not on the choice of $R_{m,n}$. Thus the inductive limit D'_n indexed by $n \in \mathbb{N}^*$ is the Barsotti–Tate group over X' which extends $D'_{U'}$. $\square$

Let Q be a local ring which is an integral domain. Let Q^{h} be the henselization of Q. We assume that its strict henselization Q^{sh} is also an integral domain (e.g., this holds if Q is normal). For a Q-algebra Q_1 which is an integral domain and generically étale, let $[Q_1 : Q] := [\mathrm{Frac}(Q_1) : \mathrm{Frac}(Q)]$.

Lemma 3 *Let $m \in \mathbb{N}^*$. Then there exits $\chi(m) \in \mathbb{N}^*$ such that for each Q as above and for every Q-subalgebra Q_1 of Q^{sh} which is local and an étale Q-algebra with $[Q_1 : Q] \le m$, there exists a commutative diagram of local étale Q-homomorphisms*

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & Q_1 \arrow[d] \\ Q_2 \arrow[r] & Q_3, \end{tikzcd}$$

where Q_2 is a Q-subalgebra of Q^{h} satisfying $[Q_2 : Q] \le \chi(m)$ and where the homomorphism $Q_2 \to Q_3$ is finite.

If Q is a regular ring of dimension at least 2 and mixed characteristic $(0,p)$ and if Q_1 is not p-quasi-healthy, then each such Q_2 is also not p-quasi-healthy.

Proof: The first part is a standard (direct) application of the fact that the ind-étale homomorphism $Q^{\mathrm{h}} \to Q^{\mathrm{sh}}$ is ind-Galois and thus its proof with $\chi(m) = m(m!)$ is left to the reader. We check the second part. As Q_1 is not p-quasi-healthy, from Lemma 1 we get that Q_3 is not p-quasi-healthy. From this and Lemma 2 we get that Q_2 is not p-quasi-healthy. $\square$

Lemma 4 *Let $\mathcal{R}$ be a flat $\mathbb{Z}_{(p)}$-algebra such that the pair $(\mathcal{R}, p\mathcal{R})$ is henselian (e.g., this holds if $\mathcal{R}$ is local henselian). Let $\mathcal{R}^\wedge$ be the p-adic completion of $\mathcal{R}$ and let $\mathcal{X}^\wedge := \mathrm{Spec}(\mathcal{R}^\wedge)$. For an arbitrary closed subscheme Y of $\mathrm{Spec}(\mathcal{R}/p\mathcal{R}) = \mathrm{Spec}(\mathcal{R}^\wedge/p\mathcal{R}^\wedge)$, let $Z := \mathcal{X} \setminus Y$ and $Z^\wedge := \mathcal{X}^\wedge \setminus Y = \mathcal{X}^\wedge \times_{\mathcal{X}} Z$. Then the following two properties hold:*

(a) *The pullback via the morphism $Z^\wedge \to Z$ defines an equivalence of categories between the category of finite flat commutative group schemes over Z and the category of finite flat commutative group schemes over $Z^\wedge$.*

(b) *We assume that the local regular ring R is such that the pair (R, pR) is henselian. Then R is p-quasi-healthy if and only if $R^\wedge$ is p-quasi-healthy.*

Proof: As the pair $(\mathcal{R}, p\mathcal{R})$ is henselian, the tensor product functor which takes a finite $\mathcal{R}$-algebra $\diamondsuit$ to $\mathcal{R}^\wedge \otimes_{\mathcal{R}} \diamondsuit$ induces an equivalence of categories between the category of finite $\mathcal{R}$-algebras S with the property that $S[\frac{1}{p}]$ is an étale $\mathcal{R}[\frac{1}{p}]$-algebra and the category of finite $\mathcal{R}^\wedge$-algebras $S^\wedge$ with the property that $S^\wedge[\frac{1}{p}]$ is an étale $\mathcal{R}^\wedge[\frac{1}{p}]$-algebra (see [E], Thm. 5).

Let $E^\wedge$ be a finite flat commutative group scheme over $Z^\wedge$. We consider a finite $\mathcal{R}^\wedge$-subalgebra $T^\wedge$ of the $\mathcal{R}^\wedge$-algebra of global sections of $E^\wedge$ such that we have $Z^\wedge \times_{\mathcal{X}^\wedge} \mathrm{Spec}(T^\wedge) = E^\wedge$. From the previous paragraph we get that there exists a finite $\mathcal{R}$-algebra T such that $T^\wedge = \mathcal{R}^\wedge \otimes_{\mathcal{R}} T$. Let $E := Z \times_{\mathcal{X}} \mathrm{Spec}(T)$; we have an identity $E^\wedge = Z^\wedge \times_Z E$ of $Z^\wedge$-schemes. A similar argument applied to homomorphisms coming from the multiplication, the inverse automorphism, and the identity section of $E^\wedge$ shows that E has a natural structure of a finite flat commutative group scheme over Z and that $E^\wedge = Z^\wedge \times_Z E$ is in fact an identity between group schemes over $Z^\wedge$. This proves that the faithful pullback functor of part (a) is surjective on objects. A similar argument for homomorphisms shows that this functor is also surjective on morphisms. Thus part (a) holds.

Taking $\mathcal{R} = R$ we have $X = \mathcal{X}$. Let $U^\wedge$ be the punctured spectra of $R^\wedge = \mathcal{R}^\wedge$; we have $U^\wedge = \mathcal{X}^\wedge \times_X U$. By taking $Y \in \{\mathrm{Spec}(k), \emptyset\}$ in part (a), we get that the pullback functors define equivalences of categories between the categories of truncated Barsotti–Tate groups over X and $\mathcal{X}^\wedge$ and between the categories of truncated Barsotti–Tate groups over U and $U^\wedge$. Part (b) follows from this and the fact that R (or $R^\wedge$) is p-quasi-healthy if and only if the pullback functor defines an equivalence of categories between the categories of Barsotti–Tate groups over X and U (or over $\mathcal{X}^\wedge$ and $U^\wedge$).$\square$

Let $\mathcal{O}_\nabla$ denote the structure ring sheaf of a scheme ∇.

Lemma 5 *Let $\mathcal{R}$ be a commutative unitary ring such that there exists an element $\theta \in \mathrm{Rad}(R)$ with the property that $\mathcal{R}[\frac{1}{\theta}]$ is a noetherian ring of dimension at most 1. For the henselization $(R^{\theta\mathrm{h}}, \theta R^{\theta\mathrm{h}})$ of the pair $(R, \theta R)$ let $\mathcal{Y}^{\theta\mathrm{h}} := \mathrm{Spec}(\mathcal{R}^{\theta\mathrm{h}}[\frac{1}{\theta}])$ and $\mathcal{Y} := \mathrm{Spec}(\mathcal{R}[\frac{1}{\theta}])$. We consider a finite étale* *morphism $\Pi^{\theta h}: \mathcal{Z}^{\theta h} \to \mathcal{Y}^{\theta h}$ of degree 2. Let $\mathcal{L}^{\theta h}$ be a line bundle over $\mathcal{Z}^{\theta h}$. We assume that both vector bundles $\Pi^{\theta h}_*(\mathcal{O}_{\mathcal{Z}^{\theta h}})$ and $\Pi^{\theta h}_*(\mathcal{L}^{\theta h})$ descent to vector bundles $\mathcal{M}$ and $\mathcal{L}$ (respectively) over $\mathcal{Y}$. Then the following two things hold:*

(a) *The pullback homomorphism $\mathrm{Pic}(\mathcal{Y}) \to \mathrm{Pic}(\mathcal{Y}^{\theta h})$ is a monomorphism. Moreover, $\mathcal{M}$ and $\mathcal{L}$ are up to isomorphisms the unique vector bundles over $\mathcal{Y}$ whose pullbacks to vector bundles over $\mathcal{Y}^{\theta h}$ are isomorphic to $\Pi^{\theta h}_*(\mathcal{O}_{\mathcal{Z}^{\theta h}})$ and $\Pi^{\theta h}_*(\mathcal{L}^{\theta h})$ (respectively).*

(b) *The pair $(\Pi^{\theta h}, \mathcal{L}^{\theta h})$ over $\mathcal{Y}^{\theta h}$ descends to $\mathcal{Y}$ in a way compatible with $\mathcal{M}$ and $\mathcal{L}$, i.e., there exists a finite étale morphism $\mathcal{Z} \to \mathcal{Y}$ of degree 2 for which the following two properties hold:*

(b.i) *The $\mathcal{O}_{\mathcal{Y}}$-module $\mathcal{O}_{\mathcal{Z}}$ is isomorphic to $\mathcal{M}$ and $\mathcal{L}$ gets the structure of a line bundle over $\mathcal{Z}$.*

(b.ii) *There exists an isomorphism $\mathcal{Z}^{\theta h} \to \mathcal{Y}^{\theta h} \times_{\mathcal{Y}} \mathcal{Z}$ of $\mathcal{Y}^{\theta h}$-schemes such that $\mathcal{L}^{\theta h}$ is isomorphic to the pullback of $\mathcal{L}$ to $\mathcal{Z}^{\theta h}$ under the composite morphism $\mathcal{Z}^{\theta h} \to \mathcal{Y}^{\theta h} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$.*

Proof: To prove the first sentence of part (a), let N be a finitely generated torsion free $\mathcal{R}$-module such that $\mathcal{R}^{\theta h}[\frac{1}{\theta}] \otimes_{\mathcal{R}} N$ is a free $\mathcal{R}^{\theta h}[\frac{1}{\theta}]$-module of rank 1. We deduce the existence of a section $s^{\theta h} \in \mathcal{R}^{\theta h} \otimes_{\mathcal{R}} N$ such that $(\mathcal{R}^{\theta h} \otimes_{\mathcal{R}} N)/(\mathcal{R}^{\theta h} s^{\theta h})$ is annihilated by θ^u for some $u \in \mathbb{N}$. The same property holds if $s^{\theta h}$ is replaced by a multiple of it by an element of $1 + \theta^{u+1}\mathcal{R}^{\theta h}$ and therefore, as $\mathcal{R}/\theta\mathcal{R} = \mathcal{R}^{\theta h}/\theta\mathcal{R}^{\theta h}$, we can assume that we have $s^{\theta h} = 1 \otimes s$ for some element $s \in N$. This implies that $N/\mathcal{R}s$ is annihilated by θ^u and thus the $\mathcal{R}[\frac{1}{\theta}]$-module $N[\frac{1}{\theta}]$ is free of rank 1. It follows that the pullback homomorphism $\mathrm{Pic}(\mathcal{Y}) \to \mathrm{Pic}(\mathcal{Y}^{\theta h})$ is a monomorphism.

We prove the second sentence of part (a) only for $\mathcal{M}$ as the argument for $\mathcal{L}$ is the same. From Serre's theorem we get the existence of a line bundle $\mathcal{F}$ over $\mathcal{Y}$ such that we have a $\mathcal{O}_{\mathcal{Y}}$-isomorphism $\mathcal{M} \simeq \mathcal{O}_{\mathcal{Y}} \oplus \mathcal{F}$ to be viewed as an identity (see [S], Thm. 1 or [Ba], Thm. 8.2). Moreover, $\mathcal{F}$ is uniquely determined up to isomorphisms as it is the determinant of $\mathcal{L}$. From the first sentence of part (a) we get that $\mathcal{F}$ is up to isomorphisms the unique line bundle over $\mathcal{Y}$ whose pullback to $\mathcal{Y}^{\theta h}$ is isomorphic to the determinant of $\Pi^{\theta h}_*(\mathcal{L}^{\theta h})$. This implies that the second sentence of part (a) holds for $\mathcal{L}$.

To prove part (b) we consider the Azumaya $\mathcal{O}_{\mathcal{Y}}$-algebra $\mathcal{A} := End(\mathcal{L})$.

To the isomorphism $\mathcal{L}^{\theta h} \to \mathcal{O}_{\mathcal{Y}^{\theta h}} \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{L}$, as $\mathcal{L}^{\theta h}$ is a $\mathcal{O}_{\mathcal{Z}^{\theta h}}$-module, corresponds a $\mathcal{O}_{\mathcal{Y}^{\theta h}}$-monomorphism

$$\mathcal{O}_{\mathcal{Z}^{\theta h}} = \mathcal{O}_{\mathcal{Y}^{\theta h}} \oplus \mathcal{O}_{\mathcal{Y}^{\theta h}} \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{F} \to \mathcal{O}_{\mathcal{Y}^{\theta h}} \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{A} \tag{18}$$

which is a maximal étale $\mathcal{O}_{\mathcal{Y}^{\theta\mathrm{h}}}$-subalgebra of $\mathcal{O}_{\mathcal{Y}^{\theta\mathrm{h}}} \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{A}$.

We consider the moduli vector group scheme

$$\mathbb{V} := \mathrm{Spec}(Sym(Hom(\mathcal{F}, \mathcal{A})^*))$$

over $\mathcal{Y}$ which parameterizes $\mathcal{O}_{\mathcal{Y}}$-linear homomorphisms from $\mathcal{F}$ to $\mathcal{A}$: for a $\mathcal{Y}$-scheme $\mathcal{W}$ we have

$$\mathbb{V}(\mathcal{W}) = \mathrm{Hom}_{\mathcal{O}_{\mathcal{W}}}(\mathcal{O}_{\mathcal{W}} \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{F}, \mathcal{O}_{\mathcal{W}} \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{A}) = H^0(\mathcal{W}, \mathcal{O}_{\mathcal{W}} \otimes_{\mathcal{O}_{\mathcal{Y}}} (\mathcal{F}^{-1} \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{A})),$$

equivalently, we have $\mathbb{V}(\mathcal{W}) = H^0(\mathcal{W}, \mathcal{O}_{\mathcal{W}} \otimes_{\mathcal{O}_{\mathcal{Y}}} Hom_{\mathcal{O}_{\mathcal{Y}}}(\mathcal{F}, \mathcal{A}))$. Let $\mathbb{W}$ be the open subscheme of $\mathbb{V}$ such that $\mathbb{W}(\mathcal{W})$ consists of all $\mathcal{O}_{\mathcal{W}}$-linear maps $l : \mathcal{O}_{\mathcal{W}} \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{F} \to \mathcal{O}_{\mathcal{W}} \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{A}$ with the property that:

(♮) The resulting sum $[\mathcal{O}_{\mathcal{W}} \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{Y}}] + \mathrm{Im}(l)$ is direct and a maximal étale $\mathcal{O}_{\mathcal{W}}$-subalgebra of $\mathcal{O}_{\mathcal{W}} \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{A}$ (in particular, l maps injectively into a direct summand of $\mathcal{O}_{\mathcal{W}} \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{A}$).

Note that (♮) is an open condition as it is equivalent to the $\mathcal{O}_{\mathcal{W}}$-linear map

$$\mathrm{Tr}_{\mathrm{red}} : (\mathcal{W} \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{F})^{\otimes 2} \to \mathcal{O}_{\mathcal{W}}$$

defined by the (reduced) discriminant rule (reduced trace)$^2 - 4\det$ being an isomorphism, (i.e., to the statement that, locally in the Zariski topology of $\mathcal{W}$, a generator of $\mathcal{O}_{\mathcal{W}} \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{F}$ maps to a $\mathcal{O}_{\mathcal{W}}$-linear endomorphism of $\mathcal{O}_{\mathcal{W}} \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{L}$ that has distinct eigenvalues at each point of $\mathcal{W}$).

Let $\mathcal{R}^{\theta\wedge}$ be the completion of $\mathcal{R}$ (or of $\mathcal{R}^{\theta\mathrm{h}}$) in the (θ)-adic topology. Let $\mathcal{Y}^{\theta\wedge} := \mathrm{Spec}(\mathcal{R}^{\theta\wedge}[\frac{1}{\theta}])$. We consider the (θ, θ)-adic topology of $\mathbb{W}(\mathcal{Y}^{\theta\mathrm{h}})$, $\mathbb{W}(\mathcal{Y}^{\theta\wedge})$, $\mathbb{V}(\mathcal{Y}^{\theta\mathrm{h}})$, and $\mathbb{V}(\mathcal{Y}^{\theta\wedge})$ defined in [GR], Subsects. 5.4.15 to 5.4.19; the inclusion $\mathbb{W}(\mathcal{Y}^{\theta\mathrm{h}}) \subset \mathbb{W}(\mathcal{Y}^{\theta\wedge})$ is continuous and has a dense image.

Let Γ be the finitely generated projective $\mathcal{R}[\frac{1}{\theta}]$-module of global sections of the $\mathcal{O}_{\mathcal{Y}}$-module $\mathcal{F}^{-1} \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{A}$. As Γ is a direct summand of $\mathcal{R}[\frac{1}{\theta}]^v$ for some $v \in \mathbb{N}$, $\mathbb{V}(\mathcal{Y}) = \Gamma$ is dense in $\mathbb{V}(\mathcal{Y}^{\theta\wedge}) = \mathcal{R}^{\theta\wedge}[\frac{1}{\theta}] \otimes_{\mathcal{R}[\frac{1}{\theta}]} \Gamma$ and by considering the open subscheme $\mathbb{W}$ of $\mathbb{V}$ we get that $\mathbb{W}(\mathcal{Y})$ is dense in $\mathbb{W}(\mathcal{Y}^{\theta\wedge})$.

To the $\mathcal{O}_{\mathcal{Y}^{\theta\mathrm{h}}}$-monomorphism of (18) corresponds a point $l^{\theta\mathrm{h}} \in \mathbb{W}(\mathcal{Y}^{\theta\mathrm{h}})$. From the last two paragraphs we get the existence of a point $l_0 \in \mathbb{W}(\mathcal{Y})$ closed to $l^{\theta\mathrm{h}}$ in the (θ, θ)-adic topology of $\mathbb{W}(\mathcal{Y}^{\theta\mathrm{h}})$.

Let $\mathcal{S} := \mathcal{O}_{\mathcal{Y}} + \mathrm{Im}(l_0)$ be the maximal étale subalgebra of $\mathcal{A}$ defined by l_0. The $\mathcal{O}_{\mathcal{Y}}$-module $\mathcal{S}$ is isomorphic to $\mathcal{O}_{\mathcal{Y}} \oplus \mathcal{F} \simeq \mathcal{M}$, see property (♮). Let $\mathcal{Z} \to \mathcal{Y}$ be the finite étale morphism of degree 2 defined by the coherent

$\mathcal{O}_{\mathcal{Y}}$-algebra $\mathcal{S}$. We have $\mathcal{O}_{\mathcal{Z}} = \mathcal{S} \subset \mathcal{A} = End(\mathcal{L})$, so we can view naturally $\mathcal{L}$ as a line bundle over either $\mathcal{S}$ or $\mathcal{Z}$. Thus the property (b.i) holds.

The affine smooth group scheme $Aut(\mathcal{A})$ over $\mathcal{Y}$ of $\mathcal{O}_{\mathcal{Y}}$-automorphisms of $\mathcal{A}$ acts via conjugation on $\mathbb{W}$ and the conjugation of $l^{\theta\mathrm{h}}$ defines an affine morphism $\iota_{\theta\mathrm{h}} : Aut(\mathcal{A})_{\mathcal{Y}^{\theta\mathrm{h}}} \to \mathbb{W}_{\mathcal{Y}^{\theta\mathrm{h}}}$ which is smooth. From this and [GR], Prop. 5.4.29 we get that the morphism

$$\iota_{\theta\mathrm{h}}(\mathcal{Y}^{\theta\mathrm{h}}) : Aut(\mathcal{A})_{\mathcal{Y}^{\theta\mathrm{h}}}(\mathcal{Y}^{\theta\mathrm{h}}) \to \mathbb{W}_{\mathcal{Y}^{\theta\mathrm{h}}}(\mathcal{Y}^{\theta\mathrm{h}})$$

is open and thus we can choose l_0 such that there exists an element $g \in Aut(\mathcal{A})_{\mathcal{Y}^{\theta\mathrm{h}}}(\mathcal{Y}^{\theta\mathrm{h}}) = Aut(\mathcal{A})(\mathcal{Y}^{\theta\mathrm{h}})$ such that $gl^{\theta\mathrm{h}}g^{-1} = l_0 \in \mathbb{W}_{\mathcal{Y}^{\theta\mathrm{h}}}(\mathcal{Y}^{\theta\mathrm{h}})$. This implies that we have a $\mathcal{Y}^{\theta\mathrm{h}}$-isomorphism $\mathcal{Z}^{\theta\mathrm{h}} \to \mathcal{Y}^{\theta\mathrm{h}} \times_{\mathcal{Y}} \mathcal{Z}$ induced by the conjugation by g isomorphism between maximal étale $\mathcal{Y}^{\theta\mathrm{h}}$-subalgebras of $\mathcal{Y}^{\theta\mathrm{h}} \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{A}$ to be viewed as an identification (the notation matches). Under this identification the pullback of $\mathcal{L}$ to $\mathcal{Z}^{\theta\mathrm{h}}$ gets identified with $\mathcal{L}^{\theta\mathrm{h}}$. Thus the property (b.ii) holds as well.[1] $\square$

Example 1 *We assume there exists a finite étale homomorphism* $\mathcal{R}[\frac{1}{\theta}] \to \mathcal{S}$ *such that* $\mathcal{Z}^{\theta\mathrm{h}} = \mathrm{Spec}(\mathcal{R}^{\theta\mathrm{h}}[\frac{1}{\theta}] \otimes_{\mathcal{R}[\frac{1}{\theta}]} \mathcal{S})$ *and that the norm* $N_{\mathcal{Z}^{\theta\mathrm{h}}/\mathcal{Y}^{\theta\mathrm{h}}}(\mathcal{L}^{\theta\mathrm{h}})$ *of* $\mathcal{L}^{\theta\mathrm{h}}$ *to a line bundle on* $\mathcal{Y}^{\theta\mathrm{h}}$ *is trivial. Let* $\Pi : \mathcal{Z} = \mathrm{Spec}(\mathcal{S}) \to \mathcal{Y}$*. Then as* $\mathcal{M}$ *we can take* $\Pi_*(\mathcal{O}_{\mathcal{Z}})$*. We include a simple argument for the formula*

$$\det(\Pi_*^{\theta\mathrm{h}}(\mathcal{L}^{\theta\mathrm{h}})) = \det(\Pi_*^{\theta\mathrm{h}}(\mathcal{O}_{\mathcal{Z}^{\theta\mathrm{h}}})) \otimes_{\mathcal{O}_{\mathcal{Y}^{\theta\mathrm{h}}}} N_{\mathcal{Z}^{\theta\mathrm{h}}/\mathcal{Y}^{\theta\mathrm{h}}}(\mathcal{L}^{\theta\mathrm{h}}) \tag{19}$$

which is a particular case of a general well-known identity on determinants. Let $\varrho^{\theta\mathrm{h}} : \mathcal{O}_{\mathcal{Z}^{\theta\mathrm{h}}} \to \mathcal{O}_{\mathcal{Z}^{\theta\mathrm{h}}}$ *be the involution of* $\mathcal{O}_{\mathcal{Y}^{\theta\mathrm{h}}}$*-algebras with the property that the coherent* $\mathcal{O}_{\mathcal{Y}^{\theta\mathrm{h}}}$*-subalgebra of* $\mathcal{O}_{\mathcal{Z}^{\theta\mathrm{h}}}$ *fixed by it is exactly* $\mathcal{O}_{\mathcal{Y}^{\theta\mathrm{h}}}$*. We view* $\mathcal{L}^{\theta\mathrm{h}}$ *as a fractional ideal of* $\mathcal{Z}^{\theta\mathrm{h}}$*. Thus both sides of Equation (19) are fractional ideals of* $\mathcal{Y}^{\theta\mathrm{h}}$*. Therefore to prove that Equation (19) holds it suffices to show that it holds after the pullback via* $(\Pi^{\theta\mathrm{h}})^*$*. Thus locally in the Zariski topology of* $\mathcal{Y}^{\theta\mathrm{h}}$ *we can assume that* $\mathcal{L}^{\theta\mathrm{h}} = j\mathcal{O}_{\mathcal{Z}^{\theta\mathrm{h}}}$ *for some element* j *of the ring of fractions of* $\mathcal{S}^{\theta\mathrm{h}}$ *and we have to prove that we have an identity*

$$\det(j\mathcal{O}_{\mathcal{Z}^{\theta\mathrm{h}}} \oplus \varrho^{\theta\mathrm{h}}(j)\mathcal{O}_{\mathcal{Z}^{\theta\mathrm{h}}}) = \det(\mathcal{O}_{\mathcal{Z}^{\theta\mathrm{h}}} \oplus \mathcal{O}_{\mathcal{Z}^{\theta\mathrm{h}}})j\varrho^{\theta\mathrm{h}}(j)$$

of fractional ideals of $\mathcal{Z}^{\theta\mathrm{h}}$*, which is obvious. As* $N_{\mathcal{Z}^{\theta\mathrm{h}}/\mathcal{Y}^{\theta\mathrm{h}}}(\mathcal{L}^{\theta\mathrm{h}})$ *is the trivial line bundle over* $\mathcal{Y}^{\theta\mathrm{h}}$*, we get that* $\det(\Pi_*^{\theta\mathrm{h}}(\mathcal{L}^{\theta\mathrm{h}})) = \det(\Pi_*^{\theta\mathrm{h}}(\mathcal{O}_{\mathcal{Z}^{\theta\mathrm{h}}}))$*. From this and Serre's theorem we get that both vector bundles* $\Pi_*^{\theta\mathrm{h}}(\mathcal{L}^{\theta\mathrm{h}})$ *and* $\Pi_*^{\theta\mathrm{h}}(\mathcal{O}_{\mathcal{Z}^{\theta\mathrm{h}}})$ *over* $\mathcal{Y}^{\theta\mathrm{h}}$ *are isomorphic to* $\mathcal{O}_{\mathcal{Z}^{\theta\mathrm{h}}} \oplus \det(\Pi_*^{\theta\mathrm{h}}(\mathcal{O}_{\mathcal{Z}^{\theta\mathrm{h}}}))$ *and therefore we can take* $\mathcal{L} = \mathcal{M}$*. Thus the hypotheses of Lemma 5 are satisfied.*

[1] The first part of the property (b.i) follows also from part (a) and the first part of the property (b.ii).

6 Proof of Theorem 2

To prove Theorem 2, we assume that R is of dimension 2 and its henselization is not p-quasi-healthy. Let $X^{\mathrm{h}} := R^{\mathrm{h}}$ and $U^{\mathrm{h}} := X^{\mathrm{h}} \times_X U = X^{\mathrm{h}} \setminus \mathrm{Spec}(k)$.

From Corollary 1 we get that $\widehat{R}$ is not p-quasi-healthy. From this and Theorem 1 (c) we get that the hypotheses of Theorem 1 (b) hold for R^{h}. Let $\gamma^{\mathrm{h}} : G^{\mathrm{h}} \to \boldsymbol{\mu}_{\mathbf{p},X^{\mathrm{h}}}$ be a homomorphism of connected finite flat group schemes over X^{h} which is not an epimorphism and whose restriction to U^{h} is an epimorphism. Such a homomorphism is defined over some $X' := \mathrm{Spec}(R')$, where R' is a local subring of R^{h} which is an étale R-algebra (automatically of residue field k). Let $\gamma' : G' \to \boldsymbol{\mu}_{\mathbf{p},X'}$ be a homomorphism over X' whose pullback to X^{h} is γ^{h}. Its restriction to $U' := X' \setminus \mathrm{Spec}(k)$ is an epimorphism but γ' is not an epimorphism. From this and [VZ], Lem. 27 we get that R' is not p-quasi-healthy.

To check that we can bound $[R' : R]$ independently of p, let u, v be a regular system of parameters of R. From Subsection 4.1 applied to R^{h} we get that $p \in (u^p, v^p, u^{p-1}v^{p-1}) \subset R^{\mathrm{h}}$ and thus also $p \in (u^p, v^p, u^{p-1}v^{p-1}) \subset R$. Thus Equation (14) holds for R, i.e., as in Subsection 4.1 we argue that we can write $p + au^p + bv^p + cu^{p-1}v^{p-1} = 0$, where $a, b, c \in R$ are such that either $c = 0$ or $c \in R^{\times}$. Therefore we get the existence of a homomorphism $\psi_\epsilon : R_{\mathrm{univ},\epsilon} \to R$, where $\epsilon \in \{0, 1\}$ and $R_{\mathrm{univ},\epsilon}$ are as in Subsection 3.3, which maps the images of x, y, a, and b in $R_{\mathrm{univ},\epsilon}$ to u, v, a, and b (respectively).

Let S_ϵ be the localization of $R_{\mathrm{univ},\epsilon}$ at its prime ideal $\mathfrak{p}_\epsilon := \psi_\epsilon^{-1}(\mathfrak{m}_R)$. The prime ideal $\mathfrak{p}_\epsilon$ contains p, x, and y and thus it can be viewed as a point of $\mathrm{Spec}(P_{\mathrm{univ},\epsilon})$, where $P_{\mathrm{univ},\epsilon} := R_{\mathrm{univ},\epsilon}/(p, x, y)$. We have

$$P_{\mathrm{univ},0} = \mathbb{F}_p[a, b] \quad \text{and} \quad P_{\mathrm{univ},1} = \mathbb{F}_p[a, b, c, c^{-1}].$$

As γ^{h} we can take the pullback of a similar homomorphism $\gamma_{\mathfrak{p}_\epsilon}$ over the spectrum of the henselization $R_\epsilon := S_\epsilon^{\mathrm{h}}$ of S_ϵ, see Subsection 3.3. As $\gamma_{\mathfrak{p}_\epsilon}$ can be defined over a local étale S_ϵ-algebra S'_ϵ which is an S_ϵ-subalgebra of R_ϵ, we can take R' to be a localization of $S'_\epsilon \otimes_{S_\epsilon} R$ and therefore we have an inequality $[R' : R] \leq [S'_\epsilon : S_\epsilon]$. Unfortunately, it is not easy to find directly an upper bound of $[S'_\epsilon : S_\epsilon]$ that works for all points of $\mathrm{Spec}(P_{\mathrm{univ},\epsilon})$ as we are working with henselizations and not strict henselizations and the requirement that S'_ϵ has the same residue field as S_ϵ is not preserved under localizations. To go around this difficulty we will use Lemma 3.

As $\mathrm{Spec}(P_{\mathrm{univ},\epsilon})$ is quasi-compact, there exist an étale morphism

$$\mathrm{Spec}(R^+_{\mathrm{univ},\epsilon}) \to \mathrm{Spec}(R_{\mathrm{univ},\epsilon})$$

whose image contains $\mathrm{Spec}(P_{\mathrm{univ},\epsilon})$ and a homomorphism

$$\gamma^{+}_{\mathrm{univ},\epsilon} : G^{+} \to \boldsymbol{\mu}_{\mathbf{p},\mathrm{Spec}(R^{+}_{\mathrm{univ},\epsilon})}$$

between finite flat group schemes over $\mathrm{Spec}(R^{+}_{\mathrm{univ},\epsilon})$ whose restriction to

$$\mathrm{Spec}(R^{+}_{\mathrm{univ},\epsilon}) \times_{\mathrm{Spec}(R_{\mathrm{univ},\epsilon})} [\mathrm{Spec}(R_{\mathrm{univ},\epsilon}) \setminus \mathrm{Spec}(P_{\mathrm{univ},\epsilon})]$$

is an epimorphism and which at every point of $\mathrm{Spec}(R^{+}_{\mathrm{univ},\epsilon})$ that maps to $\mathrm{Spec}(P_{\mathrm{univ},\epsilon})$ is not an epimorphism. The homomorphism $\gamma^{+}_{\mathrm{univ},\epsilon}$ is obtained by extending different homomorphisms $\gamma_{\mathfrak{p}_\epsilon}$ with $\mathfrak{p}_\epsilon$ a closed point of $\mathrm{Spec}(P_{\mathrm{univ},\epsilon})$. Let $m_p(\epsilon) \in \mathbb{N}^*$ be the smallest integer such that we can choose $\mathrm{Spec}(R^{+}_{\mathrm{univ},\epsilon})$ with the property that each connected component of it is generically a finite cover of $\mathrm{Spec}(R_{\mathrm{univ},\epsilon})$ of degree at most $m_p(\epsilon)$.

Let R_1 be a local ring of $R^{+}_{\mathrm{univ},\epsilon} \otimes_{R_{\mathrm{univ},\epsilon},\psi_\epsilon} R$ which dominates R. We have $[R_1 : R] \leq m_p(\epsilon)$ and the pullback of $\gamma^{+}_{\mathrm{univ},\epsilon}$ to $X_1 := \mathrm{Spec}(R_1)$ is not an epimorphism but its restriction to $U_1 := X_1 \times_X U$ is an epimorphism. From [VZ], Lem. 27 we get that R_1 is not p-quasi-healthy. From this and Lemma 3 we get the existence of a smallest constant

$$n_p(\epsilon) \in \mathbb{N}^*$$

such that for each regular local ring of dimension 2 equipped with a homomorphism $\psi_\epsilon : R_{\mathrm{univ},\epsilon} \to R$ as above, there exists a local étale homomorphism $R \to R_2$ with the properties that $[R_2 : R] \leq n_p(\epsilon)$ and that R_2 has residue field k and is not p-quasi-healthy and in fact there exists a local finite étale homomorphism $R_2 \to R_3$ for which there exists a homomorphism $\gamma' : G_{X_3} \to \boldsymbol{\mu}_{\mathbf{p},X_3}$ over $X_3 = \mathrm{Spec}(R_3)$ whose extension to the spectrum of the henselization of R_3 is a homomorphism which is a pullback of $\gamma^{+}_{\mathrm{univ},\epsilon}$ and is as in Theorem 1 (b). We have $n_p(\epsilon) \leq \chi(m_p(\epsilon))$, where $\chi(m_p(\epsilon))$ is as in Lemma 3.

Theorem 2 follows from this by taking

$$n_p := \max\{n_p(0), n_p(1)\}$$

and by choosing R' to be R_2. □

7 On $n_p(1)$

In this section we study $n_p(1)$ and thus extensions of $\boldsymbol{\mu}_{\mathbf{p}}$ by forms of $\mathbb{Z}/p\mathbb{Z}$.

Let R be of dimension 2. We consider semilocal flat noetherian R-algebras $R_\diamond$ of dimension 2 such that all maximal ideals of $R_\diamond$ intersect R in $\mathfrak{m}_R$. Let $X_\diamond := \mathrm{Spec}(R_\diamond)$. By the punctured spectrum $U_\diamond$ of $R_\diamond$ we mean $X_\diamond \times_X U$, i.e., the complement in $X_\diamond$ of the finite set of all closed points of $X_\diamond$. Let $R_\diamond^{p\mathrm{h}}$ be the $R_\diamond$-algebra such that the pair $(R_\diamond^{p\mathrm{h}}, pR_\diamond^{p\mathrm{h}})$ is the henselization of the pair $(R_\diamond, pR_\diamond)$. If $R_\diamond$ is local, then $R_\diamond^{p\mathrm{h}}$ is an $R_\diamond$-subalgebra of the henselization $R_\diamond^{\mathrm{h}}$ of $R_\diamond$. Let $Y_\diamond := \mathrm{Spec}(R_\diamond[\frac{1}{p}])$.

In Subsection 7.1 and 7.2 we recall how forms of $(\mathbb{Z}/p\mathbb{Z})_{X_\diamond}$ and suitable extensions of $\boldsymbol{\mu}_{\mathbf{p},U_\diamond}$ by forms of $(\mathbb{Z}/p\mathbb{Z})_{U_\diamond}$ (respectively) descend to spectra and punctured spectra (respectively) of suitable semilocal R-subalgebras of $R_\diamond$. In Subsection 7.3 we prove Theorem 3. In Theorem 4 of Subsection 7.5 we prove a weaker form of Theorem 3 for $p \geq 5$ that does not assume that the pair (R, pR) is henselian and which relies on Lemma 5 (b) and on Lemma 7 of Subsection 7.4 which pertains to descent of line bundles.

Let $X_\diamond^\wedge$ be the spectrum of the p-adic completion $R_\diamond^\wedge$ of $R_\diamond$. Let $U_\diamond^\wedge$ be the punctured spectrum of $R_\diamond^\wedge$. Let $\Pi(2) := \{1\}$, $q(2) := 1$, and $m_1 := 1$. If $p > 2$, let $\Pi(p)$ be the set of prime divisors of $p-1$, let $q(p) := \frac{p-1}{\prod_{l\in\Pi(p)} l}$, and we write $p-1 = \prod_{l\in\Pi(p)} l^{m_l}$ with each $m_l \in \mathbb{N}^*$. We have $q(p) = \prod_{l\in\Pi(p)} l^{m_l-1}$ for all primes $p \geq 2$.

7.1 Descending forms of $\mathbb{Z}/p\mathbb{Z}$

For $l \in \Pi(p)$ let R_l be the finite étale R-subalgebra of R^{h} generated by R and by all roots of unity of order l^{m_l}. Let R'_l be the largest finite étale R-subalgebra of R_l such that $[R'_l : R]$ divides l^{m_l-1}. Let R'_0 be the finite R-subalgebra of R^{h} generated by all R'_l with $l \in \Pi(p)$. We have identities $R_l/pR_l = (R/pR)^{[R_l:R]}$ and $R'_0/pR'_0 = (R/pR)^{[R'_0:R]}$.

We consider the localizations $R'_{l,+}$ and R'_+ of R'_l and R'_0 (respectively) at the maximal ideal of R'_l and R'_0 (respectively) contained in $\mathfrak{m}_{R^{\mathrm{h}}}$. We have $R'_+/pR'_+ = R/pR$ and therefore R'_+ is an R-subalgebra of $R^{p\mathrm{h}}$. Moreover, $[R'_+ : R]$ divides $q(p)$ and the $R[\frac{1}{p}]$-algebra $R'_+[\frac{1}{p}]$ is a localization of the finite étale $R[\frac{1}{p}]$-algebra $R'_0[\frac{1}{p}]$. Let $X'_+ := \mathrm{Spec}(R'_+)$, $X'_{l,+} := \mathrm{Spec}(R'_{l,+})$ and $X_{l,+} := \mathrm{Spec}(R_{l,+})$, where $R_{l,+} := R'_{l,+} \otimes_{R'_l} R_l$.

Fact 1 *Let $\mathcal{E}_p^{\mathrm{h}}$ be a form of $(\mathbb{Z}/p\mathbb{Z})_{X^{\mathrm{h}}}$. Then there exists a form $\mathcal{E}'_{p,+}$ of $(\mathbb{Z}/p\mathbb{Z})_{X'_+}$ whose pullback to X^{h} is $\mathcal{E}_p^{\mathrm{h}}$.*

Proof: The form $\mathcal{E}_p^{\mathrm{h}}$ is defined by a class

$$\eta^{\mathrm{h}} \in H^1_{\text{ét}}(X^{\mathrm{h}}, \mathbb{F}_p^*) = H^1_{\text{ét}}(X^{\mathrm{h}}, \boldsymbol{\mu}_{\mathbf{p-1},X^{\mathrm{h}}}) = k^*/(k^*)^{p-1} = \prod_{l\in\Pi(p)} k^*/(k^*)^{l^{m_l}},$$

and for $l \in \Pi(p)$, let η_l^{h} be the component of η^{h} in the factor $k^*/(k^*)^{l^{m_l}}$. It suffices to show that for each $l \in \Pi(p)$ there exists a class $\eta'_{l,+} \in H^1_{\text{ét}}(X'_{l,+}, \mathbb{F}_p^*)$ which maps to η_l^{h}. Let $s_l \in \{0, \ldots, m_l\}$ be such that the order of η_l^{h} is l^{s_l}. As R_l contains all roots of unity of order l^{m_l}, the natural homomorphism $H^1_{\text{ét}}(X_{l,+}, \boldsymbol{\mu}_{\mathbf{l^{m_l}},X_{l,+}}) \to H^1_{\text{ét}}(X^{\mathrm{h}}, \boldsymbol{\mu}_{\mathbf{l^{m_l}},X^{\mathrm{h}}}) = k^*/(k^*)^{l^{m_l}}$ is surjective and there exists a class $\eta_{l,+} \in H^1_{\text{ét}}(X_{l,+}, \boldsymbol{\mu}_{\mathbf{l^{m_l}},X_{l,+}})$ of order l^{s_l} which maps to η_l^{h} and which is defined by $\mathcal{E}_{l,+} = \mathrm{Spec}(R_{l,+}[z]/(z^{l^{m_l}} - a_l^{l^{m_l-s_l}}))$ for some unit a_l of $R'_{l,+}$. The composite Galois homomorphism $R'_{l,+} \to R_{l,+} \to R_{l,+}[z]/(z^{l^{s_l}} - a_l)$ is abelian of order $l^{s_l} d_l$ with d_l dividing $l-1$ (one can easily check this modulo p). Thus $E'_{l,+} := \mathrm{Spec}(R'_{l,+}[z]/(z^{l^{s_l}} - a_l)) \to X'_{l,+}$ is the only Galois morphism dominated by $\mathrm{Spec}(R_{l,+}[z]/(z_l^{l^{s_l}} - a_l))$ which has a Galois group of order l^{s_l}; it defines a class $\eta'_{l,+} \in H^1_{\text{ét}}(X'_{l,+}, \mathbb{F}_p^*)$ of order l^{s_l} which maps to $\eta_{l,+}$ and thus also to η_l^{h}. $\square$

7.2 Descending cohomology classes

Let $\mathcal{E}_p$ be a form of $(\mathbb{Z}/p\mathbb{Z})_X$. Let $\mathcal{J} := \underline{Hom}_{\text{ét}}(\boldsymbol{\mu}_{\mathbf{p},X}, \mathcal{E}_p)$ and we denote by $\mathcal{J}_\triangle$ its pullback to the étale site of the X-scheme $\triangle$. Over a separably closed field K, each extension of $\boldsymbol{\mu}_{\mathbf{p},\mathrm{Spec}(K)}$ by $\mathcal{E}_{p,\mathrm{Spec}(K)}$ splits. Thus extensions of the sheaf of the fppf site of the X-scheme $\flat$ defined by $\boldsymbol{\mu}_{\mathbf{p},X}$ and the sheaf of the fppf site of $\flat$ defined by $\mathcal{E}_p$ split locally in the étale topology of $\flat$.

Each localization of $R_\diamond$ at a maximal ideal of it is faithfully flat over R and thus its depth is at least 2. From this we easily get that the pullback homomorphism $H^1_{\text{ét}}(X_\diamond, \mathcal{J}_{X_\diamond}) \to H^1_{\text{ét}}(U_\diamond, \mathcal{J}_{U_\diamond})$ is injective and therefore we can define the group

$$\mathcal{B}_{U_\diamond} := H^1_{\text{ét}}(U_\diamond, \mathcal{J}_{U_\diamond})/H^1_{\text{ét}}(X_\diamond, \mathcal{J}_{X_\diamond}) = \mathrm{Ext}^1_{fppf}(\boldsymbol{\mu}_{\mathbf{p},U_\diamond}, \mathcal{E}_{p,U_\diamond})/\mathrm{Ext}^1_{fppf}(\boldsymbol{\mu}_{\mathbf{p},X_\diamond}, \mathcal{E}_{p,X_\diamond})$$

which depends only on $U_\diamond$ and $J_{X_\diamond}$. If $\mathcal{E}_p = (\mathbb{Z}/p\mathbb{Z})_X$, then $\mathcal{B}_{U_\diamond}$ is $\mathcal{A}_{U_\diamond}$ of Subsection 4.3 and we will use the notation $\mathcal{A}_{U_\diamond}$.

If $R_\diamond$ is local of residue field $k_\diamond$ and if $R_{\diamond,+}$ is a local ind-étale $R_\diamond$-algebra of residue field $k_\diamond$, then we have a commutative diagram of étale cohomology groups

$$\begin{array}{ccc} \mathcal{B}_{U_\diamond} & \longrightarrow & H^2_{\mathrm{Spec}(k_\diamond)}(X_\diamond, \mathcal{J}_{X_\diamond}) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{nat}} \\ \mathcal{B}_{U_{\diamond,+}} & \longrightarrow & H^2_{\mathrm{Spec}(k_\diamond)}(X_{\diamond,+}, \mathcal{J}_{X_{\diamond,+}}) \end{array} \tag{20}$$

with injective horizontal arrows and an excision isomorphism nat.

Lemma 6 *The following four properties hold:*

(a) *If $R'_\diamond$ is a semilocal flat noetherian R-subalgebra of $R_\diamond$ of dimension 2 such that the homomorphism $R'_\diamond \to R_\diamond$ is faithfully flat, then the pullback homomorphism $b_{U'_\diamond/U_\diamond} : \mathcal{B}_{U'_\diamond} \to \mathcal{B}_{U_\diamond}$ is injective (and thus in what follows it will be viewed as an inclusion).*

(b) *We assume that $R_\diamond$ is local of residue field $k_\diamond$. If the monomorphism $\mathcal{B}_{U_\diamond} \to H^2_{\mathrm{Spec}(k_\diamond)}(X_\diamond, \mathcal{J}_{X_\diamond})$ is an isomorphism (i.e., if the homomorphism $H^2_{\acute{e}t}(X_\diamond, \mathcal{J}_{X_\diamond}) \to H^2_{\acute{e}t}(U_\diamond, \mathcal{J}_{U_\diamond})$ is injective), then all arrows of Diagram (20) are isomorphisms and in particular we have $\mathcal{B}_{U_\diamond} = \mathcal{B}_{U_{\diamond,+}} = \mathcal{B}_{U_\diamond^{\mathrm{ph}}} = \mathcal{B}_{U_\diamond^{\mathrm{h}}}$.*

(c) *We consider a semilocal $R_\diamond$-algebra $R_{\diamond,+}$ which is a Galois extension of $R_\diamond$ of Galois group Λ of order prime to p. Then inside $\mathcal{B}_{U^{\mathrm{ph}}_{\diamond,+}}$ we have identities $\mathcal{B}_{U_\diamond} = \mathcal{B}^\Lambda_{U_{\diamond,+}} = \mathcal{B}_{U_{\diamond,+}} \cap \mathcal{B}_{U_\diamond^{\mathrm{ph}}}$.*

(d) *We assume that $R_\diamond$ is semilocal and we consider a finite semilocal $R_\diamond$-algebra $R_{\diamond,+}$ which has the same number of maximal ideals as $R_\diamond$ and for which the homomorphism $R_\diamond[\frac{1}{p}] \to R_{\diamond,+}[\frac{1}{p}]$ is Galois with finite Galois group Λ that leaves $R_{\diamond,+}$ invariant. Then we have $\mathcal{B}_{U_\diamond} = b^{-1}_{U_\diamond^{\mathrm{ph}}/U^{\mathrm{ph}}_{\diamond,+}}(\mathcal{B}_{U_{\diamond,+}}) \cap \mathcal{B}_{U_\diamond^{\mathrm{ph}}}$.*

Proof: Part (a) follows from faithfully flat descent: a short exact sequence of finite flat group schemes over $U_\diamond$ extends to a complex of finite flat group schemes over $X_\diamond$ if and only its pullback to a short exact sequence of finite flat group schemes over $U'_\diamond$ extends to a complex over $X'_\diamond$ and a complex of finite flat group schemes over $X_\diamond$ is exact if and only if its pullback to $X'_\diamond$ is exact. Part (b) follows from Diagram (20).

To prove part (c) we consider the short exact sequence of Λ-modules

$$0 \to H^1_{\acute{e}t}(X_{\diamond,+}, \mathcal{J}_{X_{\diamond,+}}) \to H^1_{\acute{e}t}(U_{\diamond,+}, \mathcal{J}_{U_{\diamond,+}}) \to \mathcal{B}_{U_{\diamond,+}} \to 0.$$

As the order of Λ is relatively prime to the exponent p of $\mathcal{J}$ we have $H^1(\Lambda, H^1_{\acute{e}t}(X_{\diamond,+}, \mathcal{J}_{X_{\diamond,+}})) = 0$. This implies that we have a short exact

$$0 \to H^1_{\acute{e}t}(X_{\diamond,+}, \mathcal{J}_{X_{\diamond,+}})^\Lambda \to H^1_{\acute{e}t}(U_{\diamond,+}, \mathcal{J}_{U_{\diamond,+}})^\Lambda \to \mathcal{B}^\Lambda_{U_{\diamond,+}} \to 0$$

which (by faithfully flat descent) is identified with the short exact sequence

$$0 \to H^1_{\text{ét}}(X_\diamond, \mathcal{J}_{X_\diamond}) \to H^1_{\text{ét}}(U_\diamond, \mathcal{J}_{U_\diamond}) \to \mathcal{B}_{U_\diamond} \to 0.$$

From this part (c) follows.

To prove part (d), we note that the inclusion $\mathcal{B}_{U_\diamond} \subset b^{-1}_{U^{p\mathrm{h}}_\diamond/U^{p\mathrm{h}}_{\diamond,+}}(\mathcal{B}_{U_{\diamond,+}}) \cap \mathcal{B}_{U^{p\mathrm{h}}_\diamond}$ follows from part (a). To prove the other inclusion, we first consider the case when $R_\diamond$ is local; thus $R_{\diamond,+}$ is also local. We remark that the group $H^1_{\text{ét}}(X^{\mathrm{h}}_\diamond, \mathcal{J}_{X^{\mathrm{h}}_\diamond})$ is trivial and thus $\mathcal{B}_{U^{\mathrm{h}}_\diamond} = H^1_{\text{ét}}(U^{\mathrm{h}}_\diamond, \mathcal{J}_{U^{\mathrm{h}}_\diamond})$. Therefore to prove that $\mathcal{B}_{U_\diamond} \supset b^{-1}_{U^{p\mathrm{h}}_\diamond/U^{p\mathrm{h}}_{\diamond,+}}(\mathcal{B}_{U_{\diamond,+}}) \cap \mathcal{B}_{U^{p\mathrm{h}}_\diamond} = b^{-1}_{U^{\mathrm{h}}_\diamond/U^{\mathrm{h}}_{\diamond,+}}(\mathcal{B}_{U_{\diamond,+}}) \cap \mathcal{B}_{U^{\mathrm{h}}_\diamond}$ it suffices to show that the commutative diagram

$$\begin{tikzcd}
H^1_{\text{ét}}(U_\diamond, \mathcal{J}_{U_\diamond}) \arrow[r] \arrow[d] & H^1_{\text{ét}}(U^{\mathrm{h}}_\diamond, \mathcal{J}_{U^{\mathrm{h}}_\diamond}) \arrow[d] \\
H^1_{\text{ét}}(U_{\diamond,+}, \mathcal{J}_{U_{\diamond,+}})^\Lambda \arrow[r] & H^1_{\text{ét}}(U^{\mathrm{h}}_{\diamond,+}, \mathcal{J}_{U^{\mathrm{h}}_{\diamond,+}})^\Lambda
\end{tikzcd}$$

is cartesian. But this is a direct consequence of the facts that we have the relations $H^1_{\text{ét}}(Y_{\diamond,+}, \mathcal{J}_{Y_{\diamond,+}})^\Lambda = H^1_{\text{ét}}(Y_\diamond, \mathcal{J}_{Y_\diamond})$ and $R_\diamond \subset R^\Lambda_{\diamond,+} \cap R^{\mathrm{h}}_\diamond \subset R_\diamond[\frac{1}{p}] \cap R^{\mathrm{h}}_\diamond$ (by Galois descent) and $R_\diamond[\frac{1}{p}] \cap R^{\mathrm{h}}_\diamond = R_\diamond$ (as the $R_\diamond$-algebra $R^{\mathrm{h}}_\diamond$ is faithfully flat) and thus $R_\diamond = R^\Lambda_{\diamond,+} \cap R^{\mathrm{h}}_\diamond = R_\diamond[\frac{1}{p}] \cap R^{\mathrm{h}}_\diamond$.

The general case when $R_\diamond$ is just semilocal follows from the local case by standard gluing arguments of short exact sequences. □

Corollary 3 *We assume that $R_\diamond$ is local of residue field $k_\diamond$. If the pair $(R_\diamond, pR_\diamond)$ is henselian, i.e., we have $R_\diamond = R^{p\mathrm{h}}_\diamond$ (e.g., this holds if $R_\diamond$ is p-adically complete), then the monomorphism $\mathcal{B}_{U_\diamond} \to H^2_{\mathrm{Spec}(k_\diamond)}(X_\diamond, \mathcal{J}_{X_\diamond})$ is an isomorphism and we have $\mathcal{B}_{U_\diamond} = \mathcal{B}_{U_{\diamond,+}} = \mathcal{B}_{U^{\mathrm{h}}_\diamond}$, with $R_{\diamond,+}$ as in Diagram (20).*

Proof: As $\mathcal{J}_{\mathrm{Spec}(R_\diamond/pR_\diamond)} = 0$ and as the pair $(R_\diamond, pR_\diamond)$ is henselian, from [Ga2], Thm. 1 we get that $H^2_{\text{ét}}(X_\diamond, \mathcal{J}_{X_\diamond}) = 0$. From this and Lemma 6 (b) we get that the monomorphism $\mathcal{B}_{U_\diamond} \to H^2_{\mathrm{Spec}(k_\diamond)}(X_\diamond, \mathcal{J}_{X_\diamond})$ is an isomorphism and moreover we have $\mathcal{B}_{U_\diamond} = \mathcal{B}_{U_{\diamond,+}} = \mathcal{B}_{U^{\mathrm{h}}_\diamond}$. □

7.3 Proof of Theorem 3

We assume that R is of dimension 2 and that there exists a regular system of parameters x, y of R and elements $a, b \in R$ and $c \in R^\times$ such that we have

$p + ax^p + by^p + cx^{p-1}y^{p-1} = 0$. We know that R^{h} is not p-quasi-healthy and in fact there exists a form $\mathcal{E}_p^{\mathrm{h}}$ of $(\mathbb{Z}/p\mathbb{Z})_{X^{\mathrm{h}}}$ with the property that there exists a complex $0 \to \mathcal{E}_p^{\mathrm{h}} \to G^{\mathrm{h}} \to \boldsymbol{\mu}_{\mathbf{p},X^{\mathrm{h}}} \to 0$, with G^{h} connected of order p^2 and annihilated by p, which is not a short exact sequence but whose restriction to U^{h} is a short exact sequence, see Theorem 1 (b). Moreover, we have $\mathcal{E}_p^{\mathrm{h}} = (\mathbb{Z}/p\mathbb{Z})_{X^{\mathrm{h}}}$ if and only if there exists a $(p-1)$-th root of c in R^{h}.

We have $q(2) = q(3) = 1$ and thus we have $R'_+ = R$, see Subsection 7.1. Thus, as either $X = X^{p\mathrm{h}}$ or $p \in \{2, 3\}$, from Fact 1 we get that $\mathcal{E}_p^{\mathrm{h}}$ is the pullback of a form $\mathcal{E}_p$ of $(\mathbb{Z}/p\mathbb{Z})_X$ and therefore we can speak about the $\mathcal{B}_{U_\diamond}$ groups. From the previous paragraph we know that $\mathcal{B}(U^{\mathrm{h}}) \neq 0$. If there exists a $(p-1)$-th root of c in R^{h} (e.g., this holds if $p = 2$), then we choose $\mathcal{E}_p = (\mathbb{Z}/p\mathbb{Z})_X$. If $p = 3$ and $\mathcal{E}_p \neq (\mathbb{Z}/p\mathbb{Z})_X$, let k_1 be the quadratic extension of k obtained by adjoining a square root of the image of c in k and let R_1 be a local finite étale R-algebra of residue field k_1.

If $\mathcal{B}_U = \mathcal{B}_{U^{\mathrm{h}}}$, then as $\mathcal{B}_{U^{\mathrm{h}}} \neq 0$ we get that over X there exists a complex $0 \to \mathcal{E}_p \to G \to \boldsymbol{\mu}_{\mathbf{p},X} \to 0$ which is not a short exact sequence and whose restriction to U is a short exact sequence and from [VZ], Lem. 27 we get that R is not p-quasi-healthy. Thus if $\mathcal{B}_U = \mathcal{B}_{U^{\mathrm{h}}}$, then Theorem 3 holds.

If (R, pR) is a henselian pair, from Corollary 3 we get that $\mathcal{B}_U = \mathcal{B}_{U^{\mathrm{h}}}$.

We assume that $p \in \{2, 3\}$. If $p = 3$ and $\mathcal{E}_3 \neq (\mathbb{Z}/p\mathbb{Z})_X$, the equality $\mathcal{B}_U = \mathcal{B}_{U^{\mathrm{h}}}$ is equivalent to the equality $\mathcal{B}_{U_1} = \mathcal{B}_{U_1^{\mathrm{h}}}$ (see Lemma 6 (c)); thus by replacing R with R_1 we can assume that $\mathcal{E}_3 = (\mathbb{Z}/p\mathbb{Z})_X$. As $\mathcal{E}_p = (\mathbb{Z}/p\mathbb{Z})_X$ regardless of $p \in \{2, 3\}$, below we will use $\mathcal{A}_{U_\diamond}$ instead of $\mathcal{B}_{U_\diamond}$.

We have $\mathcal{A}_{U^{\mathrm{h}}} = \mathcal{A}_{U^{p\mathrm{h}}}$, see Corollary 3. Based on Lemma 4 (a) we can identify $H^1_{\text{ét}}(U^{p\mathrm{h}}, \mathcal{J}_{U^{p\mathrm{h}}}) = H^1_{\text{ét}}(U^\wedge, \mathcal{J}_{U^\wedge})$ and $H^1_{\text{ét}}(X^{p\mathrm{h}}, \mathcal{J}_{X^{p\mathrm{h}}}) = H^1_{\text{ét}}(X^\wedge, \mathcal{J}_{X^\wedge})$ and thus also $\mathcal{A}_{U^{p\mathrm{h}}} = \mathcal{A}_{U^\wedge}$. Therefore, to prove that the subgroup $\mathcal{A}_U$ of $\mathcal{A}_{U^{\mathrm{h}}} = \mathcal{A}_{U^{p\mathrm{h}}} = \mathcal{A}_{U^\wedge}$ is $\mathcal{A}_{U^{\mathrm{h}}}$ itself, it suffices to show that the pullback homomorphism $\eta_U : H^1_{\text{ét}}(U, \mathcal{J}_U) \to H^1_{\text{ét}}(U^\wedge, \mathcal{J}_{U^\wedge})$ is surjective.

Let $j : Y = \mathrm{Spec}(R[\frac{1}{p}]) \to U$ be the open embedding. As $p \in \{2, 3\}$ we have $\mathcal{J} = j_!(\boldsymbol{\mu}_{\mathbf{p},Y})$. We consider the short exact sequence

$$0 \to \mathcal{J}_U \to \boldsymbol{\mu}_{\mathbf{p},U} \to \boldsymbol{\mu}_{\mathbf{p},\Upsilon} \to 0,$$

where $\Upsilon := U \cap \mathrm{Spec}(R/pR)$ is the finite affine scheme of generic points of the closed subscheme $\mathrm{Spec}(R/pR)$ of X. Associated to it and its analog over

$U^\wedge$ we get a commutative digram with exact rows

$$\begin{tikzcd}
H^0_{\text{ét}}(\Upsilon, \boldsymbol{\mu}_{\mathbf{p},\Upsilon}) \arrow[r] \arrow[d, equal] & H^1_{\text{ét}}(U, \mathcal{J}_U) \arrow[r] \arrow[d, "\eta_U"] & H^1_{\text{ét}}(U, \boldsymbol{\mu}_{\mathbf{p},U}) \arrow[r] \arrow[d, "\xi_U"] & H^1_{\text{ét}}(\Upsilon, \boldsymbol{\mu}_{\mathbf{p},\Upsilon}) \arrow[d, equal] \\
H^0_{\text{ét}}(\Upsilon, \boldsymbol{\mu}_{\mathbf{p},\Upsilon}) \arrow[r] & H^1_{\text{ét}}(U^\wedge, \mathcal{J}_{U^\wedge}) \arrow[r] & H^1_{\text{ét}}(U^\wedge, \boldsymbol{\mu}_{\mathbf{p},U^\wedge}) \arrow[r] & H^1_{\text{ét}}(\Upsilon, \boldsymbol{\mu}_{\mathbf{p},\Upsilon}).
\end{tikzcd}$$

Thus to prove that η_U is surjective it suffices to show that the pullback homomorphism $\xi_U : H^1_{\text{ét}}(U, \boldsymbol{\mu}_{\mathbf{p},U}) \to H^1_{\text{ét}}(U^\wedge, \boldsymbol{\mu}_{\mathbf{p},U^\wedge})$ is surjective. By considering the standard short exact sequence $0 \to \boldsymbol{\mu}_{\mathbf{p}} \to \mathbb{G}_m \to \mathbb{G}_m \to 0$ over U and $U^\wedge$ and using the fact that the Picard groups of U, X, $U^{p\mathrm{h}}$ and $X^\wedge$ are trivial, the fact that ξ_U is an epimorphism follows from the well-known fact that the functorial homomorphism $\xi_U : R^*/(R^*)^p \to (R^\wedge)^*/((R^\wedge)^*)^p$ is surjective.

Thus, if $p \in \{2, 3\}$ we have $\mathcal{A}_U = \mathcal{A}_{U^{\mathrm{h}}}$, and therefore in all cases we have $\mathcal{B}_U = \mathcal{B}_{U^{\mathrm{h}}}$. Thus Theorem 3 holds. $\square$

Corollary 4 *We have $n_2(1) = n_3(1) = 1$.*

7.4 Descending line bundles

Lemma 7 *For each semilocal flat noetherian R-algebras $R_\diamond$ of dimension 2 such that all maximal ideals of $R_\diamond$ intersect R in $\mathfrak{m}_R$ we have a functorial commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}(U_\diamond) \arrow[r, "r_\diamond"] \arrow[d] & \mathrm{Pic}(U_\diamond^{p\mathrm{h}}) \arrow[r] \arrow[d] & \mathrm{Pic}(U_\diamond^{p\mathrm{h}})/\mathrm{Pic}(U_\diamond) \arrow[r] \arrow[d, hook, "q_\diamond"] & 0 \\
0 \arrow[r] & \mathrm{Pic}(Y_\diamond) \arrow[r, "t_\diamond"] & \mathrm{Pic}(Y_\diamond^{p\mathrm{h}}) \arrow[r] & \mathrm{Pic}(Y_\diamond^{p\mathrm{h}})/\mathrm{Pic}(Y_\diamond) \arrow[r] & 0.
\end{tikzcd}$$

whose arrows are pullbacks or passages to quotients, whose rows are short exact sequences and whose homomorphism $q_\diamond$ is injective.

Proof: The fact that $t_\diamond$ is injective is a particular case of Lemma 5 (a) applied to $(R_\diamond, p)$ instead of $(\mathcal{R}, \theta)$. The closed subschemes of $U_\diamond$ and $U_\diamond^{p\mathrm{h}}$ defined by the equation $p = 0$ coincide. Based on this, the fact that $q_\diamond$ and $r^{q_\diamond}$ are injective follows from [FR], Prop. 4.2 applied to affine localizations of the faithfully flat morphism $U_\diamond \to U_\diamond^{p\mathrm{h}}$ and from the injectivity of $t_\diamond$. $\square$

7.5 A variant of Theorem 3 for $p \geq 5$

Theorem 4 *We assume that $p \geq 5$. Then the inequality $n_p(1) \leq q(p)2^{\frac{p-1}{2}}$ holds. More precisely, for any regular local ring R of dimension 2 for which there exists a regular system of parameters x, y of R and elements $a, b \in R$ and $c \in R^\times$ such that $p + ax^p + by^p + cx^{p-1}y^{p-1} = 0$, there exists a local étale homomorphism $R \to R'$ such that the following three properties hold:*

(i) *We have $R'/pR' = R/pR$ (thus R' has residue field k).*

(ii) *The $R[\frac{1}{p}]$-algebra $R'[\frac{1}{p}]$ is a localization of a finite étale $R[\frac{1}{p}]$-algebra and we have $[R' : R] \leq q(p)2^{\frac{p-1}{2}}$.*

(iii) *The local regular ring R' is not p-quasi-healthy and in fact there exists a homomorphism $\gamma' : G_{X'} \to \boldsymbol{\mu}_{\mathbf{p},X'}$ over $X' = \mathrm{Spec}(R')$ whose extension to the spectrum of the henselization of R (or R') is a homomorphism as in Theorem 1 (b).*

Proof: Let R and $0 \to \mathcal{E}_p^{\mathrm{h}} \to G^{\mathrm{h}} \to \boldsymbol{\mu}_{\mathbf{p},X^{\mathrm{h}}} \to 0$ be as in the first paragraph of Subsection 7.3 with $p \geq 5$. From Subsection 7.1 we get that there exists a local étale morphism $X'_+ = \mathrm{Spec}(R'_+) \to X$ such that $R'_+/pR'_+ = R/pR$, the étale $R[\frac{1}{p}]$-algebra $R'_+[\frac{1}{p}]$ is the localization of a finite étale $R[\frac{1}{p}]$-algebra, we have $[R'_+ : R] \leq q(p)$ and $\mathcal{E}_p^{\mathrm{h}}$ is the pullback of a form of $(\mathbb{Z}/p\mathbb{Z})_{X'_+}$. Thus by replacing R with R'_+ we can assume that $\mathcal{E}_p^{\mathrm{h}}$ is the pullback of a form $\mathcal{E}_p$ of $(\mathbb{Z}/p\mathbb{Z})_X$ and we have to show that there exists a local étale homomorphism $R \to R'$ which has all the required properties and moreover we have $[R' : R] \leq 2^{\frac{p-1}{2}}$.

We recall that $Y = \mathrm{Spec}(R[\frac{1}{p}])$. We consider the form

$$\mathcal{C} := (\boldsymbol{\mu}_{\mathbf{p},X}^{-2} \otimes_{\mathbb{Z}/p\mathbb{Z}} \otimes \mathcal{E}_p)_Y$$

of the étale sheaf $(\mathbb{Z}/p\mathbb{Z})_Y$. Let Y_1 be a connected component of the affine Y-scheme $Isom((\mathbb{Z}/p\mathbb{Z})_Y, \mathcal{C})$ which is a torsor under the étale finite group scheme $(\mathbb{F}_p)_Y^\times$ of automorphisms of $(\mathbb{Z}/p\mathbb{Z})_Y$. Let Y_2 be the connected component of the quotient of $Isom((\mathbb{Z}/p\mathbb{Z})_Y, \mathcal{C})$ by the subgroup $\{-1, 1\}_Y$ of $(\mathbb{F}_p)_Y^\times$ which is dominated by Y_1. Let R_1 and R_2 be the normalizations of R in Y_1 and Y_2 (respectively); we have $Y_1 = \mathrm{Spec}(R_1[\frac{1}{p}])$ and $Y_2 = \mathrm{Spec}(R_2[\frac{1}{p}])$ and two isomorphisms $(\mathbb{Z}/p\mathbb{Z})_{Y_1} \simeq \mathcal{C}_{Y_1}$ and $\mathcal{J}_{Y_1} \simeq \boldsymbol{\mu}_{\mathbf{p},Y_1}$, to be viewed as identifications. As $\mathcal{E}_p$ splits over the strict henselization of X and as $\{-1, 1\}$ acts trivially on $\boldsymbol{\mu}_{\mathbf{p},X}^{-2}$, the finite homomorphism $R_2 \to R_1$ is étale.

Let $\varepsilon \in \mathcal{B}_{U^{p\mathrm{h}}} = \mathcal{B}_{U^{\mathrm{h}}}$ be the non-zero class defined by the above extension $0 \to \mathcal{E}_p^{\mathrm{h}} \to G^{\mathrm{h}} \to \boldsymbol{\mu}_{\mathbf{p},X^{\mathrm{h}}} \to 0$, see Corollary 3 for the equality part. We have $\mathcal{B}_{U^{p\mathrm{h}}} \leqslant \mathcal{B}_{U_2^{p\mathrm{h}}} \leqslant \mathcal{B}_{U_1^{p\mathrm{h}}}$, see Lemma 6 (a). The class $\varepsilon \in \mathcal{B}_{U_2^{\mathrm{h}}}$ is the image of a class $\eta_2^{p\mathrm{h}} \in H^1_{\text{ét}}(U_2^{p\mathrm{h}}, \mathcal{J}_{U_2^{p\mathrm{h}}})$. Let $\eta_1^{p\mathrm{h}} \in H^1_{\text{ét}}(U_1^{p\mathrm{h}}, \mathcal{J}_{U_1^{p\mathrm{h}}})$ be the image of $\eta_2^{p\mathrm{h}}$. By considering the standard short exact sequence $0 \to \boldsymbol{\mu}_{\mathbf{p}} \to \mathbb{G}_m \to \mathbb{G}_m \to 0$ over U_1 and $U_1^{\wedge}$ we get a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R_1^\times/(R_1^\times)^p & \longrightarrow & H^1_{\text{ét}}(U_1, \boldsymbol{\mu}_{\mathbf{p},U_1}) & \longrightarrow & \mathrm{Pic}(U_1)[p] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (R_1^{p\mathrm{h}})^\times/((R_1^{p\mathrm{h}})^\times)^p & \longrightarrow & H^1_{\text{ét}}(U_1^{p\mathrm{h}}, \boldsymbol{\mu}_{\mathbf{p},U_1^{p\mathrm{h}}}) & \longrightarrow & \mathrm{Pic}(U_1^{p\mathrm{h}})[p] & \longrightarrow & 0. \end{array} \tag{21}$$

Let $\mathcal{L}_{U_1^{p\mathrm{h}}}$ be a line bundle over $U_1^{p\mathrm{h}}$ of order p such that $[\mathcal{L}_{U_1^{p\mathrm{h}}}] \in \mathrm{Pic}(U_1^{p\mathrm{h}})[p]$ is the image of $\eta_1^{p\mathrm{h}}$. Let $\mathcal{L}_{Y_1^{p\mathrm{h}}}$ be its restriction to $Y_1^{p\mathrm{h}}$.

We consider three cases as follows:

Case 1: $R_1 = R_2$.

Case 2: $[R_1 : R_2] = 2$ and $\mathrm{Spec}(R_1^{p\mathrm{h}})$ has twice as many connected components as $\mathrm{Spec}(R_2^{p\mathrm{h}})$, i.e., $R_1/pR_1 = (R_2/pR_2)^2$ (or $R_1^{p\mathrm{h}} = (R_2^{p\mathrm{h}})^2$).

Case 3: $[R_1 : R_2] = 2$ and $\mathrm{Spec}(R_1^{p\mathrm{h}})$ and $\mathrm{Spec}(R_2^{p\mathrm{h}})$ have the same number of connected components.

In Case 1 (so $R_1 = R_2$), let $R_5 := R_2 \oplus R_2$ and we consider the line bundle $\mathcal{L}_{U_5^{p\mathrm{h}}} := \mathcal{L}_{U_1^{p\mathrm{h}}} \oplus (\mathcal{L}_{U_1^{p\mathrm{h}}})^{-1}$ over $U_5^{p\mathrm{h}} = U_2^{p\mathrm{h}} \times U_2^{p\mathrm{h}} = U_1^{p\mathrm{h}} \times U_1^{p\mathrm{h}}$ and its restriction $\mathcal{L}_{Y_5^{p\mathrm{h}}}$ to $Y_5^{p\mathrm{h}}$. From Lemma 5 (b) and Example 1 applied with $(\mathcal{R}, \theta, \mathcal{S}, \mathcal{L}^{\theta\mathrm{h}}) = (R_2, p, R_5[\frac{1}{p}], \mathcal{L}_{Y_5^{p\mathrm{h}}})$ we get that there exists a finite étale R_2-subalgebra R_4 of $R_5^{p\mathrm{h}}$ such that $R_4^{p\mathrm{h}} = (R_2^{p\mathrm{h}})^2 = R_5^{p\mathrm{h}}$, so $R_4/pR_4 = (R_2/pR_2)^2$, and there exists a line bundle $\mathcal{L}_{Y_4}$ over Y_4 whose pullback to $Y_5^{p\mathrm{h}}$ is $\mathcal{L}_{Y_5^{p\mathrm{h}}}$. Let $X_3 = \mathrm{Spec}(R_3)$ be the affine open subscheme of $X_4 = \mathrm{Spec}(R_4)$ such that $Y_3 = Y_4$ (i.e., we have $R_3[\frac{1}{p}] = R_4[\frac{1}{p}]$) and $R_3/pR_3 = R_2/pR_2$ is the first factor of $R_4/pR_4 = (R_2/pR_2) \oplus (R_2/pR_2)$. Let $\mathcal{L}_{Y_3} := \mathcal{L}_{Y_4}$. We have $R_3^{p\mathrm{h}} = R_2^{p\mathrm{h}}$.

In Case 2, let $R_5 := R_1$ and let $\mathcal{L}_{U_2^{p\mathrm{h}}}$ be the line bundle over $U_2^{p\mathrm{h}}$ such that over $U_5^{p\mathrm{h}} = U_2^{p\mathrm{h}} \times U_2^{p\mathrm{h}} = U_1^{p\mathrm{h}}$ we can identify $\mathcal{L}_{U_1^{p\mathrm{h}}} := \mathcal{L}_{U_1^{p\mathrm{h}}} = \mathcal{L}_{U_2^{p\mathrm{h}}} \oplus (\mathcal{L}_{U_2^{p\mathrm{h}}})^{-1}$. With R_4, R_3 and $\mathcal{L}_{Y_3}$ as in Case 1 we have $R_3^{p\mathrm{h}} = R_2^{p\mathrm{h}}$.

In Case 3, each connected component of $X_1^{p\mathrm{h}}$ is an étale cover of degree 2 of a connected component of $X_2^{p\mathrm{h}}$ and the norm of $\mathcal{L}_{Y_1^{p\mathrm{h}}}$ to a line bundle over $Y_2^{p\mathrm{h}}$ is trivial (as $\eta_1^{p\mathrm{h}}$ comes from $\eta_2^{p\mathrm{h}}$). Thus from Lemma 5 (b) and Example

1 applied with $(\mathcal{R},\theta,\mathcal{S},\mathcal{L}^{\theta\mathrm{h}})=(R_2,p,R_1[\frac{1}{p}],\mathcal{L}_{Y_1^{p\mathrm{h}}})$ we get that there exists a finite étale R_2-subalgebra $R_3=R_4$ of $R_1^{p\mathrm{h}}$ such that we have an identification $R_3^{p\mathrm{h}}=R_1^{p\mathrm{h}}$, hence $R_3/pR_3=R_1/pR_1$, and there exists a line bundle $\mathcal{L}_{Y_3}$ over Y_3 whose pullback to $Y_3^{p\mathrm{h}}=Y_1^{p\mathrm{h}}$ is $\mathcal{L}_{Y_1^{p\mathrm{h}}}$.

In all three cases, the norm of $\mathcal{L}_{Y_3}$ to a line bundle over Y_2 is trivial. From Lemma 7 applied with $R_\diamond=R_3$, we get that there exists a unique line bundle $\mathcal{L}_{U_3}$ over U_3 which extends $\mathcal{L}_{Y_3}$ and whose pullback to $U_3^{p\mathrm{h}}$ is $\mathcal{L}_{U_1^{p\mathrm{h}}}$ in Cases 1 and 3 and is $\mathcal{L}_{U_2^{p\mathrm{h}}}$ in Case 2. As the order of $\mathcal{L}_{U_1^{p\mathrm{h}}}$ is p, based on Lemma 7 we easily get that the order of $\mathcal{L}_{U_3}$ is also p.

As the homomorphism $R_3^\times/(R_3^\times)^p\to(R_3^{p\mathrm{h}})^\times/((R_3^{p\mathrm{h}})^\times)^p$ is surjective, from the analog of Diagram (21) for U_3 and $U_3^{p\mathrm{h}}$ and from the existence of $\mathcal{L}_{U_3}$ we get that there exists a class $\xi_3\in H^1_{\text{ét}}(U_3,\boldsymbol{\mu}_{\mathbf{p},U_3})$ which maps to the image $\xi_1^{p\mathrm{h}}\in H^1_{\text{ét}}(U_1^{p\mathrm{h}},\boldsymbol{\mu}_{\mathbf{p},U_1^{p\mathrm{h}}})$ of $\eta_1^{p\mathrm{h}}$ in Cases 1 and 3 and to the image $\xi_2^{p\mathrm{h}}\in H^1_{\text{ét}}(U_2^{p\mathrm{h}},\boldsymbol{\mu}_{\mathbf{p},U_2^{p\mathrm{h}}})$ of $\eta_2^{p\mathrm{h}}$ in Case 2.

If R_4 contains R_1, i.e., if in Cases 2 and 3 we have by chance $R_4=R_1$, then $\mathcal{J}_{Y_4}=\boldsymbol{\mu}_{\mathbf{p},Y_4}$ and by using a commutative digram as in Subsection 7.3 but for U_3 and $U_3^{p\mathrm{h}}=U_3^{p\mathrm{h}}$ instead of U and $U^{p\mathrm{h}}$ we get that ξ_3 is the image of a class $\eta_3\in H^1_{\text{ét}}(U_3,\mathcal{J}_{U_3})$ that defines a class in $\mathcal{B}_{U_3}$ which maps to $\varepsilon\in\mathcal{B}_{U_1^{p\mathrm{h}}}$. As $\mathcal{B}_{U_3}\leqslant\mathcal{B}_{U_3^{p\mathrm{h}}}$ (see Lemma 6 (a)), we conclude that $\varepsilon\in\mathcal{B}_{U_3}\cap\mathcal{B}_{U^{p\mathrm{h}}}$.

If we are in Case 2 with $R_4\neq R_1$, then let $X_3^+:=X_1\times_{X_2}X_3$ and an argument similar to the one of the previous paragraph shows that we have $\varepsilon\in\mathcal{B}_{U_3^+}\cap\mathcal{B}_{U^{p\mathrm{h}}}$.

The finite morphism $\pi:X_2\to X$ is flat. Let $\pi_*(X_3)$ be the relative Weil restriction X-scheme such that for a X-scheme $\dagger$ we have

$$\pi_*(X_3)(\dagger)=\mathrm{Hom}_X(\dagger,\pi_*(X_3))=\mathrm{Hom}_{X_2}(X_2\times_X\dagger,X_3).$$

The X-scheme $\pi_*(X_3)$ is affine (see [CGP], Prop. A.5.2 (2)) and quasi-affine. For $x\in X$, if the fiber of $X_2\to X$ at x has s geometric points, then the fiber of $\pi_*(X_3)\to X$ at x has 2^s geometric points if either $R_1\neq R_2$ or $x\in Y$ and it has s geometric points otherwise. In particular, the morphism $\pi_*(Y_3)=Y\times_X\pi_*(X_3)\to Y$ is finite étale of degree $2^{[R_2:R]}$ and thus of degree at most $2^{\frac{p-1}{2}}$. Let $\mathrm{Tr}:X_2\times_X\pi_*(X_3)\to X_3$ be the trace morphism that corresponds to the identity morphism $1_{\pi_*(X_3)}\in\mathrm{Hom}_X(\pi_*(X_3),\pi_*(X_3))$.

In this paragraph we assume that we are in Cases 1 or 2. We consider the morphism $\zeta:X^{p\mathrm{h}}\to\pi_*(X_3)$ that corresponds to the composite morphism $X_2\times_X X^{p\mathrm{h}}=X_2^{p\mathrm{h}}=X_3^{p\mathrm{h}}\to X_3$. Let $X'=\mathrm{Spec}(R')$ be the affine open subscheme of $\pi_*(X_3)$ such that $R'/pR' = R/pR$, we have $Y' = \pi_*(Y_3)$ (so $[R' : R] \le 2^{\frac{p-1}{2}}$), and moreover ζ factors as a morphism $\zeta' : X^{p\mathrm{h}} \to X'$. If we are in Case 1 or in Case 2 with $R_4 = R_1$, then the class $\varepsilon \in \mathcal{B}_{U_3}$ pulls back via the morphism $U_2 \times_U U' \to U_3$ induced by Tr to a class in $\mathcal{B}_{U_2 \times_U U'}$ whose correstriction is a class in $\mathcal{B}_{U'}$ whose pullback via $\zeta_{U'} : U^{p\mathrm{h}} \to U'$ is the class $[R_2 : R]\varepsilon \in \mathcal{B}_{U^{p\mathrm{h}}}$. Thus by replacing ε with its multiple by the natural number $[R_2 : R]$ prime to p, we can assume that $\varepsilon \in \mathcal{B}_{U^{p\mathrm{h}}}$ is the pullback via $\zeta_{U'}$ of a class in $\mathcal{B}_{U'}$ and therefore R' has all the desired properties. If we are in Case 2 with $R_4 \neq R_1$, then by pulling back the trace morphism Tr to a morphism of X_1-schemes we get a morphism $\mathrm{Tr}_{X_1} : X_1 \times_X \pi_*(X_3) \to X_1 \times_{X_2} X_3 = X_3^+$. The pullback of the class $\varepsilon \in \mathcal{B}_{U_3^+}$ via the morphism $U_1 \times_U U' \to U_3^+$ induced by Tr_{X_1} is a class in $\mathcal{B}_{U_1 \times_U U'}$ whose correstriction is a class in $\mathcal{B}_{U'}$ and as above we conclude that R' has all the desired properties.

In this paragraph we assume that we are in Case 3. Thus the morphism $X_3 \to X$ is finite, and based on this, we will show that in fact we can take $R' = R$. It suffices to show that inside $\mathcal{B}_U^{p\mathrm{h}}$ we have $\varepsilon \in \mathcal{B}_U$. Let $R \to R_0$ be a finite Galois extension of order a divisor of $p-1$ such that $\mathcal{E}_{p,X_0}$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})_{X_0}$ and the residue field k_0 of R_0 is a Galois extension of k of degree $[R_0 : R]$, see Subsection 7.1. Based on Lemma 6 (c) it suffices to show that inside $\mathcal{B}_{U_0^{p\mathrm{h}}}$ we have $\varepsilon \in \mathcal{B}_{\mathcal{U}_0}$. Let R_{20} be the Galois extension of R_2 generated by R_0; the R_2-algebra R_1 is a subalgebra of R_{20} and thus we have $\mathcal{J}_{Y_{20}} = \boldsymbol{\mu}_{\mathbf{p},Y_{20}}$. Let R_{30} be the Galois extension of R_{20} generated by R_{20} and R_3; we have $[R_{30} : R_{20}] \le 2$. As above (for the paragraph and context in which R_4 contains R_1) we argue that the image $\xi_{30} \in H^1_{\text{ét}}(U_3, \boldsymbol{\mu}_{\mathbf{p},U_{30}})$ of the class $\xi_3 \in H^1_{\text{ét}}(U_3, \boldsymbol{\mu}_{\mathbf{p},U_3})$ is the image of a class $\eta_{30} \in H^1_{\text{ét}}(U_3, \mathcal{J}_{U_{30}})$ that defines a class in $\mathcal{B}_{U_{30}}$ whose image in $\mathcal{B}_{U_{30}^{p\mathrm{h}}}$ is the same as the image of $\varepsilon \in \mathcal{B}_{U^{p\mathrm{h}}}$. From Lemma 6 (c) applied in the context of the finite Galois extension $R_{20} \to R_{30}$ we get that inside $\mathcal{B}_{U_{30}^{p\mathrm{h}}}$ we have $\varepsilon \in \mathcal{B}_{\mathcal{U}_{20}}$. Let R_{00} be the maximal finite R_0-subalgebra of R_{20} which is étale over R_0. The finite homomorphism $R_{00} \to R_{20}$ induces a bijection at the level of sets of maximal ideals and by inverting p we get an abelian extension $R_{00}[\frac{1}{p}] \to R_{20}[\frac{1}{p}]$ whose Galois group leaves R_{20} invariant. By applying Lemma 6 (d) to it we get that inside $\mathcal{B}_{U_{20}^{p\mathrm{h}}}$ we have $\varepsilon \in \mathcal{B}_{\mathcal{U}_{00}}$. From Lemma 6 (c) applied in the context of the finite Galois extension $R_0 \to R_{00}$ we get that inside $\mathcal{B}_{\mathcal{U}_{00}}$ we have $\varepsilon \in \mathcal{B}_{\mathcal{U}_0}$. $\square$

8 A complement to [VZ]

In this section we prove two variants of Proposition 1 which are modeled on [VZ], Thm. 28 (i) and (ii).

Proposition 2 *Let $\mathcal{R}_1$ be a henselian local ring whose residue field has characteristic p. We assume that there exist elements $x \in \mathfrak{m}_{\mathcal{R}_1}$ and $a \in \mathcal{R}_1$ such that we have an identity*

$$p + ax^p = 0.$$

Let $s \in \mathbb{N}^*$. Let $t_1, \ldots, t_s \in \mathfrak{m}_{\mathcal{R}_1}$. Then there exists a homomorphism $\delta_1 : \mathcal{G}_1 \to \mathcal{I}_1$ of connected finite flat commutative group schemes annihilated by p over $\operatorname{Spec}(\mathcal{R}_1)$, with $\mathcal{G}_1$ of order p^{3s} and with $\mathcal{I}_1$ of order p, whose restriction to $\operatorname{Spec}(\mathcal{R}_1) \setminus \operatorname{Spec}(\mathcal{R}_1/(x, t_1, \ldots, t_s))$ is an epimorphism and which at every point of $\operatorname{Spec}(\mathcal{R}_1/(x, t_1, \ldots, t_s))$ is not an epimorphism. In particular, if $\mathcal{R}_1$ is regular of mixed characteristic $(0, p)$ and dimension $d \geq 2$, then $\mathcal{R}_1$ is not p-quasi-healthy (thus formal power series rings in $d-1 \geq 1$ indeterminates over a complete discrete valuation ring of absolute ramification index at least p are not p-quasi-healthy).

Proof: The proof of the first part of the proposition is similar to the one of Proposition 1 with the new universal ring being

$$\mathbb{Z}_{(p)}[a, x, t_1, \ldots, t_s]/(p + ax^p).$$

The only difference is in the case when $\mathcal{R}_1$ is regular complete and moreover $x, t_1, \ldots, t_s, z_1, \ldots, z_{d-s-1}$ is a regular system of parameters of $\mathcal{R}_1$. We take

$$\mathfrak{S} := C(k)[[x, t_1, \ldots, t_s, z_1, \ldots, z_{d-s-1}]],$$

$$S := k[[x, t_1, \ldots, t_s, z_1, \ldots, z_{d-s-1}]],$$

$\mathcal{R}_1 := \mathfrak{S}/(p + ax^p)$ for some $a \in \mathfrak{S}$, and a Frobenius lift σ of $\mathfrak{S}$ which is compatible with σ_k and which maps each $t \in \{x, t_1, \ldots, t_s, z_1, \ldots, z_{d-s-1}\}$ to t^p. Let $M_{3s} := S^{3s}$ and let $\varphi_4 : M_{3s} \to M_{3s}^{(\sigma)}$ be the S-linear map whose matrix representation with the standard S-bases of M_{3s} and $M_{3s}^{(\sigma)}$ is a diagonal block matrix $\Gamma = (\Gamma_i)_{1 \leq i \leq s} \in M_{3s \times 3s}(S)$, where for $i \in \{1, \ldots, s\}$ we have

$$\Gamma_i = \Gamma(t_i) := \begin{pmatrix} 0 & 0 & x^p \\ t_i - t_i^p x^{p-1} & x & (x - t_i^{p-1})(t_i - t_i^p x^{p-1}) \\ x^{p-1} & 0 & x^{p-1}(x - t_i^{p-1}) \end{pmatrix}.$$

Let $M_1 := S$ and $\varphi_{1,1} : M_1 \to M_1^{(\sigma)}$ be the S-linear map defined by the multiplication with x^p. Let $\beta_4 : (M_{3s}, \varphi_4) \to (M_1, \varphi_{1,1})$ be the S-linear map defined by the matrix

$$(t_1, x, t_1x, t_2, x, t_2u, \ldots, t_s, x, t_sx).$$

If

$$\Delta_i = \Delta(t_i) := \begin{pmatrix} t_i^{p-1} - x & 0 & x \\ 0 & x^{p-1} & -t_i + t_i^p x^{p-1} \\ 1 & 0 & 0 \end{pmatrix} \in M_{3\times 3}(S),$$

then we have

$$\Gamma_i \Delta_i = \Delta_i \Gamma_i = x^p I_3$$

(see [VZ], proof of Thm. 28 (i)). Note that the adjugate of Γ_i is $-x^p \Delta_i$. If $\Delta = (\Delta_i)_{1\leq i\leq s} \in M_{3s\times 3s}(S)$ is the diagonal block matrix, then we have

$$\Gamma\Delta = \Delta\Gamma = x^p I_{3s}.$$

The identity

$$(t_1^p\ x^p\ t_1^p x^p\ t_2^p\ x^p\ t_2^p x^p\ \ldots\ t_s^p\ x^p\ t_s^p x^p)\Gamma = x^p(t_1\ x\ t_1x\ t_2\ x\ t_2u\ \ldots\ t_s\ x\ t_sx)$$

implies that β_4 is a morphism of nilpotent Breuil modules. The search for homomorphism $\delta_1 : \mathcal{G}_1 \to \mathcal{I}_1$ is the one associated to β_4.

The last part of the proposition follows from [VZ], Lem. 27 applied to δ_1 with $t_1, \ldots, t_s$ chosen such that the quotient ring $\mathcal{R}_1/(x, t_1, \ldots, t_s)$ is artinian. $\square$

Proposition 3 *Let $\mathcal{R}_2$ be a henselian local ring whose residue field has characteristic p. We assume that there exist $s \in \mathbb{N}^*$ and elements $x_1, \ldots, x_s \in \mathfrak{m}_{\mathcal{R}_2}$ and $a \in \mathcal{R}_2$ such that we have an identity*

$$p + a\prod_{i=1}^{s} x_i^{p-1} = 0.$$

Then there exists a homomorphism $\delta_2 : \mathcal{G}_2 \to \mathcal{I}_2$ of connected finite flat commutative group schemes annihilated by p over $\mathrm{Spec}(\mathcal{R}_2)$, with $\mathcal{G}_2$ of order p^s and with $\mathcal{I}_2$ of order p, whose restriction to $\mathrm{Spec}(\mathcal{R}_2) \setminus \mathrm{Spec}(\mathcal{R}_2/(x_1, \ldots, x_s))$ is an epimorphism and which at every point of $\mathrm{Spec}(\mathcal{R}_2/(x_1, \ldots, x_s))$ is not an epimorphism. In particular, if $\mathcal{R}_2$ is regular of mixed characteristic $(0, p)$ and dimension $d \geq 2$, then $\mathcal{R}_2$ is not p-quasi-healthy.

Proof: The proof of this proposition is entirely similar to the one of Proposition 2 with the new universal ring being

$$\mathbb{Z}_{(p)}[a, x_1, \ldots, x_s]/(p + a \prod_{i=1}^{s} x_i^{p-1})$$

once a suitable morphism $\beta_5 : (M_s, \varphi_5) \to (M_1, \varphi_{1,2})$ of nilpotent Breuil modules is constructed. We take $\varphi_5 : M_s \to M_s^{(\sigma)}$ to be the S-linear map whose matrix representation with respect to the standard S-bases of $M_s = S^s$ and $M_s^{(\sigma)}$ is the diagonal $s \times s$ matrix whose entry in the jj position with $j \in \{1, \ldots, s\}$ is $\prod_{1 \le i \le s, i \ne j} x_i^{p-1}$. We take $\varphi_{1,2} : M_1 \to M_1^{(\sigma)}$ to be the multiplication by $\prod_{i=1}^{s} x_i^{p-1}$. We take $\beta_5 : M_s \to M_1$ such that its matrix representation is $(x_1 \ \ldots \ x_s)$. $\square$

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