

Toric flattening (sketch)

- (1) Let $(X, M \xrightarrow{\alpha} \mathcal{O}_X)$ be a scheme with a log structure (for the Zariski topology). We consider the reduced monoid algebra $\mathcal{O}[M]'$, it is a ~~sheaf~~ sheaf of $\mathcal{O}$ -algebras which is the quotient of $\mathcal{O}[M]$ by the ideal generated by $m-1$ where m is a local section of M with $\alpha(m)=1$. The stalk of $\mathcal{O}[M]'$ at x is $\overline{M}_x$ graded. The log structures considered will always be integral. An ideal in M gives an ideal in $\mathcal{O}[M]'$ with graded stalks.
- If $\eta: \pi \rightarrow \text{find object in the Zariski topology then } \mathcal{O}[M]'$ is η_1 of Lorenzon's algebra at x (Zariski topology version)
- Let F be an $\mathcal{O}_x$ -module. We say that F is t torsion free at $x \in X$ iff $\forall m \in M_x$ $\alpha(m)$ acts injectively on F_x . We say that F is t -flat at x iff for every ideal $I \subset M_x$ (these correspond bijectively to ideals in $\overline{M}_x$) we have
- $$\text{Tor}_1^{\mathcal{O}[M]'}(\mathcal{O}[M]'/I, F_x) = 0.$$
- t flatness implies t torsion freeness.
- One can extend the discussion of t -flatness to a situation where one works relative to an ideal sheaf $I \subset M$ that annihilates the modules we consider. If $I_x F_x \neq 0$ we say that F is t -flat rel to I at x if
- $$\text{Tor}_1^{\mathcal{O}[M]'/I}(\mathcal{O}[M]'/J, F_x) = 0 \text{ for every ideal } J \supset I.$$

- (2) Next we consider a situation without X : if P is an integral monoid $I \subset P$ an ideal and M a $\mathbb{Z}[P]/I$ -module we say that M is t -flat rel to I iff
- $$\text{Tor}_1^{\mathbb{Z}[P]/I}(\mathbb{Z}[P]/J, M) = 0 \quad \forall J \supset I.$$

- (3) Lemma Suppose there is a global chart $P \rightarrow \mathcal{O}_X$ for the log structure on X . Then ~~for~~ for I an ideal of P F is t -flat rel to IM_x iff F_x is t -flat over $\mathbb{Z}[P]/I$.

Proof We may assume $X = \text{Spec}(\mathcal{O}_{X,x})$. Let P' be

obtained from P by inverting $\alpha^{-1}(O^*)$. Then $\mathbb{Z}[P] \rightarrow \mathbb{Z}[P']$ is flat and one can reduce to the case $P = P'$.

Also

$$\begin{array}{ccc} \mathbb{Z}[P]/I & \longrightarrow & \mathbb{Z}[P]/J \\ \downarrow & & \downarrow \\ O[P]/I & \longrightarrow & O[P]/J \end{array} \quad \text{is cocartesian tor-independent square}$$

So one can write the definition in (2) as

$$\text{Tor}_1^{O[P]/I}(O[P]/J, F_x) = 0.$$

Now there is an equivalence of categories

$$\begin{array}{ccc} \text{pgp graded } O[P] \text{ - modules} & \longrightarrow & \bar{\mu}_x^{\text{pgp}} \text{ graded} \\ & & O[M]_x' \text{ - modules} \end{array}$$

Use free graded resolutions to compare Tor's, Note that F_x is not graded; what is needed here is that for N a pgp graded $O[P]$ - module

$$\text{Tor}_i^{O[P]}(N, O[M]_x') = 0 \text{ for } i > 0.$$

(4) We say that a combinatorial $\mathbb{Z}[P]$ - module is a pgp - graded $\mathbb{Z}[P]$ - module of the type $\mathbb{Z}A/\mathbb{Z}B$

where $B \subset A \subset \text{pgp}$ are subsets s.t. $PA \subset A$ $PB \subset B$,

The monoid law is denoted multiplicatively.

Any f.g. combinatorial $\mathbb{Z}[P]$ - module has a finite filtration with factors $\cong \mathbb{Z}[P]/J$ up to ~~some~~ gradation shift

Hence in the t-flat case $\text{Tor}_1^{\mathbb{Z}[P]/I}(N, M) = 0$

for every combinatorial N annihilated by I .

(5) Lemma In the t-flat case $\text{Tor}_i^{\mathbb{Z}[P]/I}(N, M) = 0$ for every $i > 0$ and every P -graded torsion free $\mathbb{Z}[P]/I$ -module.

Proof. (1) Consider N combinatorial. Reduce to $N = \mathbb{Z}[P]/J$, then if $i > 1$ reduce to the $i-1$ statement with $N = J/I$.

(2) If M is a $\mathbb{Z}[P]/I$ module F a field and N is a graded $\mathbb{Z}[P]/I$ -module then every f.g. graded submodule of N has a finite filtration with factors $\cong \mathbb{Z}[P]/J$ up to gradation shift

so we reduce to the case treated in (1).

(3) Now let R be a Dedekind ring and M a t-flat $R[P]/I$ -module; then we want to show that $\text{Tor}_i^{R[P]/I}(N, M) = 0$ for every graded t.f. $R[P]/I$ -module N .

Reduce to R a DVR with uniformizer π $k = R/\pi$, $K = \text{Frac}(R)$.

We may assume N_K is f.g., filter it and take the induced filtration on N so w.m.a. $N_K \cong K[P]/J$

Then w.m.a. N f.g. so

$$\pi^i R[P]/J \subset N \subset \pi^j R[P]/J \text{ for some } N \text{ is } i \leq j.$$

Say $j=0$. Let $Q = (R[P]/J)/N$. Consider the π -adic filtration on Q . By Tor-sequences reduce to show

$$\text{Tor}_i^{R[P]/I}(Q, M) = 0 \quad \forall i \geq 2$$

where Q is a graded $k[P]/I$ -module.

By derissage w.m.a. $Q = k[P]/J$ and the statement

follows from $0 \rightarrow R[P]/J \xrightarrow{\pi} R[P]/J \rightarrow k[P]/J \rightarrow 0$.

(6) If P is finitely generated then by a combinatorial analogue of the prime filtration theorem in commutative algebra it suffices in the definition of t -flatness to consider $J \supset I$ that are prime.

There are finitely many of these.

Hence one gets openness of flatness in the noetherian case:

If (X, M) is a locally noetherian scheme with a ~~coherent~~ integral log structure and a coherent ideal sheaf $I \subset M$ and F is a coherent sheaf on X killed by I then $\{x \in X \mid F_x \text{ is } t\text{-flat rel } I\}$ is open.

(7) Let $M_x^+ = \{m \in M_x \text{ s.t. } d(m) \text{ is not invertible}\}$. This is an ideal.
(if $x \in V(I)$)

Lemma In the case of (6), F_x is t -flat rel I iff

$$\text{Tor}_1^{O[M]_x/I} (O[M]_x/M_x^+, F_x) = 0.$$

Proof Use a chart $P \xrightarrow{\alpha} M_x$ with $\alpha^*(\alpha^*)$ invertible.

Prove $\text{Tor}_1^{\mathbb{Z}[P]/I} (\mathbb{Z}[P]/J, F_x) = 0$ by noetherian induction on J , wma J prime. Then either $J = P^+$ or $J \subsetneq P^+$; in the latter case take some $p \in P^+ - J$ and consider

$$0 \rightarrow \mathbb{Z}[P]/J \xrightarrow{x} \mathbb{Z}[P]/J \rightarrow \mathbb{Z}[P]/J\mathbb{Z}[P]_+ \times \mathbb{Z}[P] \rightarrow 0$$

and the Tor sequences and Nakayama's Lemma.

(4) One has an analogue of the local criterion of flatness.

(2.1) If $I \subset J$ M a $\mathbb{Z}[P]/I$ -module

then M is t -flat over $\mathbb{Z}[P]/I \Rightarrow M/JM$ is t -flat over $\mathbb{Z}[P]/J$.

if some power of J is contained in I then

$$M \text{ is } t\text{-flat over } \mathbb{Z}[P]/I \Leftrightarrow \text{Tor}_1^{\mathbb{Z}[P]/I} (\mathbb{Z}[P]/J, M) = 0$$

and M/JM is t -flat over $\mathbb{Z}[P]/J$.

(8.2) If P is f.g. and M a f.g. $\Phi[P]$ -module, Φ a comm. noeth. ring then by an Artin-Rees argument

$$\text{Tor}_1^{\mathbb{Z}[P]/I}(\mathbb{Z}[P]/J, M) \longrightarrow \text{Tor}_1^{\mathbb{Z}[P]/I+J^n}(\mathbb{Z}[P]/I, M/I^n M)$$

is an isomorphism for large n

(8.3) if $I \supset P_+^n$ and M a $\mathbb{Z}[P]/I$ -module, the following are equivalent

(a) M is t -flat rel I

(b) $\text{Tor}_1^{\mathbb{Z}[P]}(\mathbb{Z}[P]/P_+, M) = 0$ $R = \mathbb{Z}[P]/I$

(c) $P_+^m R / P_+^{m+1} R \otimes_R M \longrightarrow P_+^m M / P_+^{m+1} M$ is an isom. $\forall m$.

(8.4) Same hypotheses without $I \supset P_+^n$, M a f.g. $\Phi[P]/I$ -module, Φ noetherian. Then the conditions are equivalent with (a) replaced by t -flatness of the localization of M at primes of $\Phi[P]/I$ containing P_+ .

If (c) holds ($I \supset P_+^n$ not assumed) we say that M is weakly t -flat over R .

This is equivalent to $M/P_+^n M$ t -flat over $R/P_+^n R$ for every n .

(a) Lemma. Let M be a $\mathbb{Z}[P]$ -module. Then there is an ideal I s.t.

~~AND~~ $M/I M$ is weakly t -flat over $\mathbb{Z}[P]/I$

iff $J \supset I$.

Proof. it suffices to prove stability under intersection of ideals. This is reduced to the case of ideals containing some fixed power of P_+ .

the point is that if $A_\alpha \subset B \subset \mathbb{Z}[p^{\text{gp}}]$ are combinatorial submodules then

$$B / \bigcap_{\alpha \in \Sigma} A_\alpha \otimes_{\mathbb{Z}[p]} M \longrightarrow \prod_{\alpha \in \Sigma} (B / A_\alpha) \otimes M$$

is injective.

Seen by inspection.

One way to describe the canonical ideal I is as follows: Let $\Sigma \subset M$ be a set of coset representatives of $M \bmod p_+ M$ with $0 \in M$.

In the p_+ -adic completion of M one may have

$$\text{relations of the type } \sum p_\alpha \sigma_\alpha = 0 \quad \sigma_\alpha \in \Sigma - \{0\}$$

$$p_\alpha \in p_+^{n_\alpha}, \quad n_\alpha \rightarrow \infty, \quad p_\alpha \text{ distinct,} \\ (\alpha \text{'s in a finite or countable set, } p_\alpha \notin p_+^{n_\alpha+1}.$$

~~THEOREM~~ I is generated by all the p_α 's that appear in such relations

(10) Let $P \subset Q$ be an injective homomorphism of integral ~~monoids~~ monoids.

10.1 If M is a t -flat $\mathbb{Z}[P]$ -module then $M \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ is t -flat over $\mathbb{Z}[Q]$ and

$$\text{Tor}_i^{\mathbb{Z}[P]}(M, \mathbb{Z}[Q]) = 0 \quad \forall i > 0.$$

10.2 ^(n > 0) If M has t -tor dimension $\leq n$ (i.e. the Tor_i are 0 for $i > n$ in the definition of t -flatness, equivalently an n -th syzygy of M is t -flat) then

$$\text{Tor}_i^{\mathbb{Z}[P]}(M, \mathbb{Z}[Q]) = 0 \quad \forall i \geq n$$

Proof. 10.1. $\mathbb{Z}[Q]$ is $(Q/p)^{\text{gp}}$ graded with ~~homogeneous~~ homogeneous components combinatorial $\mathbb{Z}[P]$ modules (up to choice of coset representatives to determine the gradation)

Hence the Tor - vanishing ; since a combinatorial $\mathbb{Z}[Q]$ -module restricted ~~to~~ to $\mathbb{Z}[P]$ is a $\oplus$ of combinatorial modules the result follows by $\hat{\sim}$ change of rings formula.

(10.2) ~~Taking syzygies reduce to $n=1$ i.e. we have an exact sequence~~

$$0 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0 \quad F_i \text{ t-flat.}$$

$$\mathbb{Z}[Q] \subset \oplus \mathbb{Z} A_\alpha \quad A_\alpha \subset \mathbb{Z}[P^{\text{op}}] \text{ combinatorial.}$$

$$\star \operatorname{Tor}_i^{\mathbb{Z}[Q]}(M, A_\alpha) \simeq \operatorname{Tor}_{i+1}^{\mathbb{Z}[Q]}(M, \mathbb{Z}[P^{\text{op}}]/A_\alpha).$$

(11) Let X be a fine noetherian log scheme $\mathcal{F}$ a coherent $\mathcal{O}_X$ -module, $\forall x \in X$ we have by (9) a canonical ideal $I_{\mathcal{F}}(x)$ in M_x s.t. $\mathcal{F}/\mathcal{F}$ is t flat at x rel. an ideal J iff $J \supset I_{\mathcal{F}}(x)$.

If there is a global chart $P \rightarrow M$ we get an ideal $\alpha^{-1}(I_{\mathcal{F}}(x)) \subset \mathbb{P}$

Lemma. ~~There~~ $x \mapsto \alpha^{-1}(I_{\mathcal{F}}(x))$ is a constructible function, and in particular takes finitely many values.

Proof. Openness of t-flatness implies that $x \mapsto \alpha^{-1}(I_{\mathcal{F}}(x))$ is upper semicontinuous in the sense that every point x has a neighborhood U_x s.t. $\forall y \in U_x \quad \alpha^{-1}(I_{\mathcal{F}}(y)) \subset \alpha^{-1}(I_{\mathcal{F}}(x))$. Quite generally an upper semicontinuous function from a sober noetherian space to a poset is constructible. One reduces to showing that if the space is irreducible the function is constant on some non empty open.

$I_{\mathcal{F}}(x)$ ~~corresponds~~ corresponds to an ideal $\bar{I}_{\mathcal{F}}(x)$ in $\bar{M}_x$.

$\bar{M}$ is a constructible sheaf of monoids, $x \mapsto \bar{I}_{\mathcal{F}}(x)$ is not a subsheaf in general but it depends constructibly on x on strata where $\bar{M}$ is constant.

(12) Let P be a fine monoid and $p^{\text{sat}} \subset P^{\text{gp}}$ its saturation which is also fine. If P^{gp} is torsion free $\mathbb{Z}[p^{\text{sat}}]$ is the normalization of $\mathbb{Z}[P]$; in all cases it is finite over $\mathbb{Z}[P]$.

Lemma. There is a non empty ideal $I \subset P$ s.t. the blowing up of $\text{Spec}(\mathbb{Z}[P])$ along $V(I\mathbb{Z}[P])$ is $\text{Spec} \mathbb{Z}[p^{\text{sat}}]$.

Proof. We show that for every $p \in p^{\text{sat}}$ one can get $\mathbb{Z}[P + \mathbb{N}p]$ as a blow up of $\mathbb{Z}[P]$; then one takes the product of the ideals corresponding to a finite generating set.

Let $p = a/b$ $a, b \in P$. For some $n \geq 1$ $p^n \in P$. We blow up $aP + bP$ i.e. the ideal (a, b) in $\text{Spec} \mathbb{Z}[P]$. One open in the blow up is what we want. It suffices to show its complement is empty i.e.

$$\mathbb{Z}[P][b/a] = \mathbb{Z}[P][b/a, a/b],$$

$$\text{Indeed } a/b = p^n (b/a)^{n-1},$$

Using the universal property of Kato's valuative log space X^{val} its construction and the above one sees that if X is a scheme with a fine log structure then

$$X^{\text{val}} \simeq (X^{\text{sat}})^{\text{val}}.$$

(13) There are several variants of the inverse system used in the construction of X^{val} .

For a fine monoid P Consider the set of non ~~empty~~ ^{empty} ideals with the partial order $I \leq J$ if $\exists K$ s.t. $J = IK$.

One can also consider the preorder by domination

$$I < J \text{ if } \text{bl}_J \mathbb{Z}[P](\mathbb{Z}[P]) \text{ dominates } \text{bl}_I \mathbb{Z}[P](\mathbb{Z}[P]).$$

$I < J$ iff $\exists p \in P \setminus \{0\}, n \in \mathbb{N}_{>0}$ and an ideal K s.t.

$$p J^n = IK.$$

$(\text{ideals}, \leq) \rightarrow (\text{ideals}, \prec)$ is cofinal so the proobjects
 $"\varprojlim" Bl_{I \in \mathbb{Z}[P]} (\text{Spec } \mathbb{Z}[P])$ defined using these indexing
 categories are canonically isomorphic.

We work below with toric varieties, Let T be a split torus
 over $\mathbb{Z}$ $N = X_*(T)$. For a fan Δ in N (a finite
 non empty set of rational strongly convex polyhedral cones in $N_{\mathbb{R}}$ closed
 under face and s.t. the intersection $\sigma \cap \tau$ is a face of σ) one
 has a standard toric variety $X(\Delta)$ over $\mathbb{Z}$ obtained
 by gluing affine toric varieties

$$U_{\sigma} = \text{Spec } \mathbb{Z}[\sigma^{\vee} \cap N^{\vee}]$$

U_{σ} has a closed stratum $Z_{\sigma} = \text{Spec } \mathbb{Z}[(R\sigma)^{\perp} \cap N^{\vee}]$
 $Z_{\sigma} \simeq$ a quotient of T , Z_{σ} 's form a partition of $X(\Delta)$.

$X(\Delta)$ is endowed with a log structure. By a t -ideal
 we mean a non empty coherent ideal in the sheaf of
 monoids. Such a t -ideal corresponds on each U_{σ} to a
 non empty ideal in $\sigma^{\vee} \cap N^{\vee}$ and these should patch.

For the purpose of stability under blow ups it would
 be convenient to have a more general notion of toric varieties
 constructed from not necessarily saturated monoids but this
 can be avoided.

Let P be an integral monoid. If $I \subset P$ is an ideal
 let the integral closure $\bar{I}$ be $\{p \in P \mid \exists n \geq 1 \text{ s.t. } p^n \in I^n\}$.
 When $P \otimes \mathbb{Q}$ is torsion free this corresponds to integral closure
 of ideals in $\mathbb{Z}[P]$. Let $Q \subset P \times N$ be the submonoid
 generated by $P \times \{0\}$ and $I \times \{1\}$. Then the saturation
 of Q in $P \times N$ is $\coprod_{n \in \mathbb{N}} \bar{I}^n \times \{n\}$. If P is finitely

generated then by a general fact this saturation is f.g.

so $\bigoplus_n \mathbb{Z} \bar{I}^n$ is finite over $\bigoplus_n \mathbb{Z} I^n$

so $\exists c > 0$ s.t. $\bar{I}^n = I^{n-c} \bar{I}^c$ for $n > c$.

Then all powers of $\bar{I}^e$ are integrally closed. We call such ideals power integrally closed. (p.i.c.)

Similar discussion for t -ideals on $X(\Delta)$. If I is a t -ideal on $X(\Delta)$ the blow up w.r.t. the corresponding variety ideal in the structure sheaf is

$$\text{Proj}(\oplus I^n \mathcal{O}_{X(\Delta)})$$

the normalization of this is $\text{Proj}(\oplus \bar{I}^n \mathcal{O}_{X(\Delta)})$ which is

$$\text{Bl}_{\bar{I}^e}(X(\Delta)).$$

It is a toric variety $X(\Delta')$ for a subdivision of Δ .

Any proper modification $X(\Delta') \rightarrow X(\Delta)$ can be dominated

by the blow up of a p.i.c. t -ideal.

(14) Let R be a noetherian commutative ring, $X(\Sigma)$ a toric variety over $\mathbb{Z}$, U an open in $X(\Sigma)_R$. ~~TRANSFERRING~~ Consider a coherent sheaf F on U , a subdivision Σ' of Σ and the projection

$$p: X(\Sigma')_R \rightarrow X(\Sigma)_R.$$

If F has t -tor dimension ≤ 1 then by (10)

$$L_{p^*} F \simeq p^* F \quad \text{in } D(p^{-1}(U), \mathcal{O}_{p^{-1}(U)})$$

and one checks that $p^* F$ has t -tor dimension ≤ 1 .

If F is t -flat then $p^* F$ is t -flat and hence t -torsion free.

Theorem. (1) There is a subdivision Σ' s.t. $p^*(F)$ has t -tor dimension ≤ 1

(2) There is a subdivision Σ' s.t. $p^*(F)/(t\text{-torsion})$ is t -flat.

We remark that if a subdivision satisfies the assertion of part (i) of the theorem then every finer subdivision does. Also one can reduce (1) to (2); we may assume

$$F = \mathcal{O}_U^n / R. \quad \text{Then}$$

$p^*(R)/(t\text{-torsion}) = R \mathcal{O}_{p^{-1}(U)} \subset \mathcal{O}_{p^{-1}(U)}^n$ so (2) for R gives (1) for F .

(15) We give a Lemma that is the basis of the proof of the flattening theorem. ~~This proof is analogous to~~
 Let $x \in U$ be a point s.t. F is not t -flat at x ,
 $(Z_\sigma)_R$ the stratum containing x , $P = \sigma^\vee \cap N^\vee$ the monoid
 corresponding to U_σ , $J \subset P$ the non empty ideal
 of P defined by F_x as in (9).

Lemma. If Σ' is a subdivision of Σ s.t.
 $X(\Sigma')|_{U_\sigma} \rightarrow U_\sigma$ dominates $\text{Bl}_J(U_\sigma)$

$y \in X(\Sigma')_R$ a point over x , $(Z_\tau')_R$ the stratum containing y
~~then~~ and $F' = p^*(F) / J$ -torsion then the map

$$p^*(F) \otimes \mathcal{O}_{(Z_\tau')_R, y} \rightarrow F' \otimes \mathcal{O}_{(Z_\tau')_R, y} \quad \text{is not}$$

an isomorphism.

Proof. Let $p_1, \dots, p_k$ be a minimal basis of J . We can
 assume y lies ~~over~~ the affine open $U = \text{Spec } \mathbb{Z}[P][\frac{p_2}{p_1}, \dots, \frac{p_k}{p_1}]$

of $\text{Bl}_J \text{Spec}(\mathbb{Z}[P])$. If $\mathcal{O}$ is the local ring of U at x ~~then~~
 Here $M_x = F_x$

and $\hat{M}_x$ the J -adic completion of M_x it follows from
 the description of J in terms of formal series relations
 in (9) that there are $m_i \in M_x$ ($1 \leq i \leq k$), $m_i \notin P^+ M_x$ s.t. in

$\hat{M}_x$ we have $\sum p_i m_i \in J$ -adic closure of $P^+ J M_x$ in $\hat{M}_x$.
 But all submodules of M_x are J -adically closed so

$\sum p_i m_i \in P^+ J M_x$. Modifying the m_i 's by elements of
 $P^+ M_x$ we may assume
 $\sum p_i m_i = 0$.

Hence in F'_x ~~we have~~ $m_1 + \sum_{i>1} \left(\frac{p_i}{p_1}\right) m_i = 0$.

But $Z'_\tau \rightarrow Z_\sigma$ is a split epimorphism of tori (quotients of T).

We have maps $U_\tau \rightarrow U_1 \rightarrow U_\sigma$ corresponding to monoid homomorphisms

$$P \rightarrow P \cdot \prod_i (p_i/p_1)^{\mathbb{N}} \rightarrow Q$$

Z_σ is defined by P^+ and Z'_τ is defined by Q^+ ,

Since for $i \geq 1$ $p_i/p_1 \notin P$ we have either $p_i/p_1 \in Q^+$ or $p_i/p_1 \in Q^* - P^*$, i.e. either p_i/p_1 vanishes on Z'_τ

or $p_i/p_1|_{Z'_\tau}$ is a character of Z'_τ that does not come from Z_σ .

Now $\sum m_i \otimes (p_i/p_1)$ is an element of the kernel

$$\text{of the map } F_x \otimes_{\mathcal{O}} \mathcal{O}_{(Z'_\tau)_{R,y}} \rightarrow F'_y \otimes_{\mathcal{O}_{X(\Sigma)_R,y}} \mathcal{O}_{(Z'_\tau)_{R,y}}.$$

To show it is non zero one observes that

Lemma If $\mathcal{O}$ is a local ring $T \rightarrow \text{Spec}(\mathcal{O})$ affine smooth with connected fibers then for every point y of the closed fiber and an $\mathcal{O}$ -module M

$$M \otimes_{\mathcal{O}} \mathcal{O}(T) \rightarrow (p^*M)_y \text{ is injective.}$$

We need this only for T a power of $\mathbb{G}_m$ a case that can be handled e.g. by reducing to the noetherian case where one can determine the associated primes of p^*M .

We now

(16) ~~To~~ prove the flattening theorem part (2).

If $x \in (Z_\sigma)_R \cap U$ is a point where F is not t -flat we considered an ideal $J = J_x$ in the monoid defining U_σ . This can be viewed as a t -ideal on U_σ and we extend it to a t -ideal on $X(\Sigma)$ by the operation that corresponds to taking the schematic closure of the subscheme defined by the ideal. Only finitely many t -ideals ~~are~~ on $X(\Sigma)$ are obtained in this way. Let I be their product.

Let $X(\Sigma^{(1)})$ be the normalization of $\text{bl}_I X(\Sigma)$.

$$\gamma: X(\Sigma^{(1)})_R \rightarrow X(\Sigma^{(0)})_R = X(\Sigma)_R.$$

Let $F^{(1)} = p^*F / t\text{-torsion} = p^*(F) / I\text{-torsion}.$

If $F^{(1)}$ is not t -flat on $U^{(1)} = p^{-1}(U)$ we blow up $X(\Sigma^{(1)})$ following the same recipe.
If the theorem does not hold we get an infinite tower

$$\rightarrow X(\Sigma^{(2)}) \rightarrow X(\Sigma^{(1)}) \rightarrow X(\Sigma^{(0)})$$

with $F^{(n)}$ on $U^{(n)} = \text{inverse image of } U \text{ in } X(\Sigma^{(n)})_R$.

Let $Z_n \subset U^{(n)}$ be the closed set where $F^{(n)}$ is not t -flat.

Z_{n+1} maps to Z_n .

$$\text{Let } \Sigma_{\text{nf}}^{(n)} = \{ \sigma \in \Sigma^{(n)} \mid (Z_\sigma)_R \cap Z_n \neq \emptyset \}.$$

$\Sigma_{\text{nf}}^{(n)}$ is a finite ^{non empty} subset of $\Sigma^{(n)}$ with $\Sigma_{\text{nf}}^{(n+1)}$ maps to $\Sigma_{\text{nf}}^{(n)}$.

Hence the inverse limit of the $\Sigma_{\text{nf}}^{(n)}$ is not empty. Fix

$n \mapsto \sigma^{(n)}$ in the inverse limit. $n \mapsto \dim \sigma^{(n)}$ is non increasing

Hence for $n \geq n_0$ it is constant.

$Z_{\sigma^{(n)}} \rightarrow Z_{\sigma^{(n_0)}}$ is an isomorphism for $n \geq n_0$.

We have coherent sheaves $F^{(n)} \otimes \mathcal{O}_{Z_{\sigma^{(n)}}/R}$

on $Z_{\sigma^{(n_0)}/R} \cap U$ for $n \geq n_0$.

By (15) the $n+1$ st sheaf is a quotient of the n -th sheaf by a non-zero subsheaf.

This contradicts the noetherian assumption.

(17) Let X be a ^{locally} noetherian scheme with a coherent log structure for the Zariski topology. Kato defined a certain locally ringed space X^{val} with a valuative log structure over X .

Theorem. (i) The structure sheaf of X^{val} is universally coherent i.e. $\mathcal{O}[t_1, \dots, t_n]$ is coherent $\forall n$.

(ii) The t -torsion subsheaf of a coherent $\mathcal{O}[t_1, \dots, t_n]$ -module on an open in X^{val} is coherent.

We prove (i) for $n=0$.

Case a. The structure sheaf of $X(\Sigma)_R^{\text{val}}$ is coherent (notation as in (14)).

If $f_1, \dots, f_m$ are sections of $\mathcal{O}$ on an open $U \subset X(\Sigma)_R^{\text{val}}$ we have to show that $R(f_1, \dots, f_m)$ is of finite type.
(the module of relations)

We may assume U comes from an open $U_{\Sigma'} \subset X(\Sigma')_R$ for

some subdivision Σ' and the f_i 's descend to $U_{\Sigma'}$.

By the flattening theorem we have a subdivision Σ'' of Σ' s.t. $I \otimes_{\Sigma'} f_i \mathcal{O}_{U_{\Sigma''}}$ is t -flat.

Hence for every subdivision Σ''' of Σ'' and $p: X(\Sigma''')_R \rightarrow X(\Sigma'')_R$ the resulting projection

$p^*(I) \simeq I \mathcal{O}_{U_{\Sigma'''}}$. Hence the module of

relations on $U_{\Sigma''}$ generates the module of relations on $U_{\Sigma'''}$ and hence the module of relations in the limit.

Case b. General case. We may assume X is affine with a global chart $P \rightarrow \mathcal{O}(X)$. We can make P integral and saturated. Then X embeds in an affine toric variety V . We have a Cartesian square of locally ringed spaces

$$\begin{array}{ccc} X^{\text{val}} & \xrightarrow{i} & V^{\text{val}} \\ \downarrow & & \downarrow \\ X & \longrightarrow & V \end{array}$$

$i_* \mathcal{O}_{X^{\text{val}}}$ is an $\mathcal{O}_{V^{\text{val}}}$ -module of finite presentation, hence coherent by a as an $\mathcal{O}_{V^{\text{val}}}$ -module, hence coherent as a sheaf of rings, but i is a closed immersion so X^{val} is coherent.