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On the Ergodicity of Frame Flows

M. Brin and M. Gromov

Institut des Hautes Etudes Scientifiques, 35, route de Chartres, F-91440 Bures-sur-Yvette, France

1. Statement of Results

Let V be a closed (i.e. compact without boundary) connected n-dimensional Riemannian manifold of class C^3. Denote by $St_k(V)$ *the space of the orthonormal tangent k-frames of V*. There is a natural fibration $St_k(V) \to V$ associated to the tangent bundle $T(V)$; its fiber is the Stiefel manifold St_k^n of the orthonormal k-frames in $\mathbb{R}^n$. According to our notations $St_1(V)$ is the bundle of the unit tangent vectors. The Riemannian structure on V induces an R-action in $St_1(V)$ called the *geodesic flow*. Consider the natural projection $St_k(V) \to St_1(V)$ (its fiber is St_{k-1}^{n-1}) and lift the geodesic flow to a flow in $St_k(V)$ as follows:

Take a frame $(e_1, e_2, \ldots, e_k)$ at a point $v \in V$, $e_i \in T_v(V)$. The geodesic flow sends e_1 to vectors tangent to the geodesic determined by e_1. The lifted flow sends our original frame to the frames parallel to it along the geodesic. This flow in $St_k(V)$ is called the *k-frame flow*.

1.1. **Main Theorem.** *If the sectional curvature of V is negative, and if the dimension of V is odd and different from 7, then the k-frame flow is ergodic for each* $k = 1, 2, \ldots, n-1 = \dim V - 1$. *Moreover, it is Bernoullian.*

1.2. *Remarks.* (a) The ergodicity of the 1-frame flows (i.e. of the geodesic flows) on the negatively curved manifolds is a well known fact (see [1]).

(b) Each k-frame flow is a natural factor of the $(k+1)$-frame flow, hence, the ergodicity of the $(n-1)$-frame flow implies the ergodicity of all k-frame flows with $k \leqq n-1$.

(c) We will show in Sect. 3 that the *ergodicity of the 2-frame flow for the even dimensional manifolds* of negative curvature *implies the ergodicity of the* $(n-1)$-*frame flow* with one possible exception of $n=8$.

(d) For the exceptional dimension 7 we shall establish the ergodicity of the 2-frame flow.

(e) The n-frame flow can be ergodic only when V is not orientable. In this case its ergodicity is equivalent to the ergodicity of the $(n-1)$-frame flow (see [5]).

Notice also that for $n=3$ the ergodicity was proved in [6] and for $n=5$ this is an unpublished result of D. Anosov.

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Examples and Counterexamples

(i) When V has constant negative curvature, all the frame flows are ergodic, and this property is stable under small perturbations of the metric (see [7]). One can show for an arbitrary V that the set of the negatively curved metrics with the ergodic n-frame flow is open and dense in the space of all C^3-metrics on V of negative curvature (the density is established in [5], and the openness requires a simple additional argument).

(ii) When V is a Kähler manifold (of real dimension greater than 2), none of the k-frame flows with $k \geqq 2$ is ergodic because the complex structure on a Kähler manifold is invariant under the parallel translation. The simplest examples of such manifolds are obtained by dividing the complex hyperbolic spaces by cocompact lattices (see [2]). This gives for each even dimension (>2) an example of a negatively curved manifold with the nonergodic 2-frame flow. Observe that the other locally symmetric spaces of rank 1 (the quaternion hyperbolic spaces and the hyperbolic Cayley plane) provide similar examples.

Let us describe the ergodic components for these actions. We know that V is covered by a symmetric space $\bar{V}$ of rank 1 (of the non-compact type), i.e. $V = \bar{V}/\Gamma$, $\Gamma = \pi_1(V)$. The group of isometries of $\bar{V}$ acts on each manifold $St_k(\bar{V})$. This action is *not transitive* ($k \geqq 2$) unless V has constant sectional curvature. Take an orbit $\bar{S} \subset St_k(\bar{V})$ and project it into $St_k(V)$. We obtain a smooth compact manifold $S \subset St_k(V)$ which is fibered over $St_1(V)$ and is invariant under the k-frame flow. One can easily show that *this action is ergodic in* S and, hence, by varying the orbit $\bar{S}$ we get the ergodic decomposition of $St_k(V)$.

In the general case the ergodic decomposition is a refinement of the "holonomy decomposition" of $St_k(V)$. The following theorem provides examples when the reverse is also true.

Kähler Manifolds

Suppose that V is a compact Kähler manifold of complex dimension $m = n/2$. Denote by $Sc_k(V) \subset St_k(V)$, $k = 1, 2, \ldots, m$, the manifold of the unitary k-frames. The manifold $Sc_1(V)$ coincides with $St_1(V)$, and each $Sc_k(V)$ is fibered over $Sc_1(V)$; the fiber is the complex Stiefel manifold of the $(k-1)$-frames in $\mathbb{C}^{m-1}$. Each $Sc_k(V)$ is invariant under the k-frame flow.

1.3. **Theorem.** *If V has negative sectional curvature, then the frame flow in $Sc_k(V)$ is ergodic in the following two cases:*

(a) *m is odd, $k = 1, 2, \ldots, m$;*
(b) *$m = 2$, $k = 1, 2$.*

1.4. *Remark.* We will see in Sect. 6 that, if m is even, then the ergodicity of the 2-frame flow in $Sc_2(V)$ implies the ergodicity of the m-frame flow in $Sc_m(V)$.

About the Proofs. In Sect. 2 we reduce the ergodicity problem to some standard questions in algebraic topology which are resolved in Sects. 5 and 6. In Sect. 3 we present some necessary facts on Lie groups. Section 4 is devoted to an elementary proof of the ergodicity of the 2-frame flows.

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2. Reduction of the Problem to the Topology of Sphere Bundles

The fibration $St_{n-1}(V) \to St_1(V)$ is a principal fiber bundle with the structure group $SO(n-1)$.

2.1. **Ergodicity Lemma.** *If V has negative curvature and if the $(n-1)$-frame flow is not ergodic, then the structure group of the fibration $St_{n-1}(V) \to St_1(V)$ can be reduced to a proper closed subgroup $G \subset SO(n-1)$. Furthermore, if the 2-frame flow is not ergodic, then the action of G on the sphere S^{n-2} is not transitive* (*see* [5], [6]).

2.2. **The Complex Version of the Lemma.** *When V is Kähler and when the m-frame flow in $Sc_m(V)$ is not ergodic, then the structure group of the fibration $Sc_m(V) \to Sc_1(V)$ can be reduced to a proper closed subgroup $G \subset U(m-1)$. Furthermore, if the 2-frame flow in $Sc_2(V)$ is not ergodic, then the action of G on the sphere S^{2m-3} is not transitive. This also follows from* [5], [6].

2.3. *Remark.* The conclusions of the lemmas hold when "not ergodic" is replaced by "not Bernoullian". This follows from [6], [13], [14].

3. Groups Acting on Spheres

The table below contains the list of all subgroups of the special orthogonal group $SO(p+1)$ which act transitively on the sphere S^p (see e.g. [12]).

$p=2q$	$SO(p+1)$
$p=4q+1$	$SO(p+1)$, $U(2q+1)$, $SU(2q+1)$
$p=4q-1$	$SO(p+1)$, $U(2q)$, $SU(2q)$ $Sp(q)$, $Sp(q)\times U(1)$, $Sp(q)\times Sp(1)$
There are additional transitive actions on the following spheres	
$p=7$	Spin(7)
$p=9$	Spin(9)
$p=6$	$\mathbb{G}_2$

Using the first and the last lines of the table and the Ergodicity Lemma we immediately obtain Remark 1.2(c).

4. Proof of the Ergodicity of the 2-frame Flows

Start with a simple general fact.

4.1. **Lemma** (compare with [8]). *Let $X \to S^q$ be a fibration with the fiber S^p. If this fibration admits a structure group G which acts non-transitively on the fiber, then there is a section $S^q \to X$.*

Proof. We must show that the boundary homomorphism $\partial: \pi_q(S^q) \to \pi_{q-1}(S^p)$ from the homotopy sequence of the fibration is trivial.

Take an orbit $T \subset S^p$ of a point under the action of G and consider the corresponding subfibration $Y \subset X$ with the fiber T. The homomorphism ∂ can be decomposed into the boundary homomorphism $\pi_q(S^q) \to \pi_{q-1}(T)$ and the inclusion homomorphism $\pi_{q-1}(T) \to \pi_{q-1}(S^p)$. Since $T \neq S^p$, the inclusion homomorphism is trivial, and hence, ∂ is also trivial.

4.2. **Corollary.** *Consider the unit tangent bundle of the $(n-1)$-dimensional sphere $St_2^n \to S^{n-1}$. If n is odd, then the structure group of this bundle cannot be reduced to a subgroup which is not transitive on the fiber $(=S^{n-2})$.*

This follows from Lemma 4.1 and from the classical fact of the non-existence of vector fields on the even dimensional spheres.

We are able now to prove the Main Theorem for $k=2$.

4.3. **Proposition.** *The 2-frame flow on an odd dimensional manifold V of negative curvature is ergodic.*

Proof. By restricting the bundle $St_2(V) \to St_1(V)$ to the tangent sphere at a point in V, we get the bundle $St_2^n \to S^{n-1}$. Corollary 4.2 says that its structure group must be transitive on the fiber $(=S^{n-2})$, and hence, this is also true for the ambient fibration $St_2(V) \to St_1(V)$. Lemma 2.1 concludes the proof.

Kähler Manifolds. The ergodicity of the unitary 2-frame flow can be proved by the same argument as above in view of the following fact.

4.4. **Proposition.** *Consider the fibration $Sc_2^m \to S^{2m-1}$, where Sc_2^m denotes the manifold of the unitary 2-frames in $\mathbb{C}^m$. If m is odd, then there is no section $S^{2m-1} \to Sc_2^m$.*

For the proof see [15].

5. Proof of the Ergodicity of the $(n-1)$-frame Flows

Using the Ergodicity Lemma and the results of the previous section we reduce the main theorem to the following fact.

5.1. **Proposition.** *The structure group of the fibration $SO(n) = St_{n-1}^n \to S^{n-1}$ cannot be reduced to a subgroup $G \subset SO(n-1)$ which acts transitively on S^{n-2}, provided n is odd and different from 7.*

Proof. According to the table from Sect. 3 we must consider the following cases.

Case 1. $G=U(q)$, $2q=n-1$, $n>3$. In this case the proposition is equivalent to the non-existence of quasi-complex structures on S^{2q}, $q \neq 1, 3$, but this is a known fact (see [3]).

Case 2. $G=Sp(q)\times Sp(1)$, $4q=n-1$, $n\geqq 9$. Denote by $\chi\in\pi_{4q-1}(SO(4q))$ the element corresponding to the bundle $SO(4q+1)\to S^{4q}$. We must show that χ is not contained in the image $\bar{I}\subset\pi_{4q-1}(SO(4q))$ of $\pi_{4q-1}(Sp(q)\times Sp(1))$ under the homomorphism corresponding to the inclusion $Sp(q)\times Sp(1)\subset SO(4q)$. Observe that $\pi_{4q-1}(\mathrm{SO}(4q))=\mathbb{Z}+\mathbb{Z}$ and $\pi_{4q-1}(Sp(1))$ is finite (see [4], [9]). It follows that $\bar{I}$ is equal to the image of $\pi_{4q-1}(Sp(q))$ alone. But the inclusion $Sp(q)\to SO(4q)$ factors as $Sp(q)\to U(2q)\xrightarrow{j} SO(4q)$, and by the previous case, the image $j_*(\pi_{4q-1}(U(2q)))\subset\pi_{4q-1}(SO(4q))$ does not contain χ.

Case 3. $G=\mathrm{Spin}(7)$, $n=9$. Suppose that the structure group of the fibration $SO(9)\to S^8$ is reduced to $\mathrm{Spin}(7)\overset{i}{\subset} SO(8)$. In this case the element $\chi\in\pi_7(SO(8))$ corresponding to the fibration $SO(9)\to S^8$ must be contained in the image $\bar{I}\subset\pi_7(SO(8))$ of the inclusion homomorphism. Consider the commutative diagram:

$$\begin{tikzcd}[column sep=small]
\mathrm{Spin}(7) \arrow[rr, phantom, "\overset{i}{\subset}"] \arrow[dr, "f"'] & & O(8) \arrow[dl, "g"] \\
& S^7 &
\end{tikzcd}$$

The map f is a fibration with the fiber $\mathbb{G}_2$, and the image of the characteristic homomorphism $\partial\colon \pi_7(S^7)\to\pi_6(\mathbb{G}_2)$ has a non-trivial 3-component (see [3]; in fact, this image coincides with $\pi_6(\mathbb{G}_2)=\mathbb{Z}_3$, see [10]). Using the exact sequence

$$\mathbb{Z}=\pi_7(\mathrm{Spin}(7))\xrightarrow{f^*}\pi_7(S^7)=\mathbb{Z}\to\pi_6(\mathbb{G}_2)$$

we conclude that the image $f_*(\mathbb{Z})\subset\mathbb{Z}=\pi_7(S^7)$ has at least index 3. On the other hand, $g_*(\chi)=2\in\mathbb{Z}=\pi_7(S^7)$, and the contradiction finishes the proof.

Case 4. $G=\mathrm{Spin}(9)$, $n=17$. Suppose that the structure group of the fibration $SO(17)\to S^{16}$ is reduced to $\mathrm{Spin}(9)\subset SO(16)$. We derive the contradiction as before from the fact that the image of the characteristic homomorphism $\pi_{15}(S^{15})\to\pi_{14}(\mathrm{Spin}(7))$ of the fibration $\mathrm{Spin}(9)\to S^{15}$ (with the fiber $\mathrm{Spin}(7)$) has a non-trivial 3-component (see [3]; notice that the group $\pi_{14}(\mathrm{Spin}(7))$ is of order $2^7\cdot 3^2\cdot 5\cdot 7$, see [10]).

Cases 1 and 2 imply the proposition for the remaining groups SU, Sp and $Sp\times U(1)$, and the proof is finished.

6. Proof of the Ergodicity of the *m*-frame Flows on the Kähler Manifolds

We consider now a Kähler manifold V of complex dimension m and the bundle $Sc_m(V)\to Sc_1(V)$ from Section 1. Suppose that the structure group is reduced to a proper subgroup $G\subset U(m-1)$. When $m-1$ is odd, the table from Section 3 shows that G cannot act transitively on S^{2m-3}, and together with the complex version of the Ergodicity Lemma this implies Remark 1.4.

Observe further that *the connected component of the identity in G cannot be simply connected.* Otherwise, the real Chern class $c_1(V)$ would vanish, but on a Kähler manifold of negative curvature c_1 is represented by a negative (1, 1)-form. This shows that for $m=2, 3$ the group G must coincide with $U(m-1)$.

Turn now to the case when $m \geqq 5$ is of the form $2q+1$. The group G must be transitive on S^{4q-1} (see Sect. 4), and the only remaining possibility is the one of $G=Sp(q)\times U(1)\subset U(2q)$. But this cannot happen because of the following.

6.1. **Lemma.** *The structure group of the fibration $U(2q+1)\to S^{4q+1}$ cannot be reduced to $Sp(q)\times U(1)\subset U(2q)$.*

Proof. We must show that the characteristic element $\chi\in\pi_{4q}(U(2q))$ of the fibration $U(2q+1)\to S^{4q+1}$ is not contained in the image of the inclusion homomorphism $\pi_{4q}(Sp(q)\times U(1))=\pi_{4q}(Sp(q)\to\pi_{4q}(U(2q))$. It follows from Proposition 4.4 that the image of χ under the projection $\pi_{4q}(U(2q))\to\pi_{4q}(S^{4q-1})=\mathbb{Z}_2$ does not vanish, and hence, we must only prove that the homomorphism $\pi_{4q}(Sp(q))\xrightarrow{p_*}\pi_{4q}(S^{4q-1})$, corresponding to the fibration $Sp(q)\xrightarrow{p}S^{4q-1}$, vanishes.

When q is even $\pi_{4q}(Sp(q))=0$ (see [4]), and the proof is finished.

Let q be odd and consider the exact homotopy sequence of the last fibration

$$\begin{array}{ccccccc}\pi_{4q}(Sp(q)) & \xrightarrow{p^*} & \pi_{4q}(S^{4q-1}) & \xrightarrow{\partial} & \pi_{4q-1}(Sp(q-1)) & \xrightarrow{i^*} & \pi_{4q-1}(Sp(q)) \\ & & \| & & & & \\ & & \mathbb{Z}_2 & & & & \end{array}$$

By using the facts that $\pi_{4q}(Sp(q))=\mathbb{Z}_2$, $\pi_{4q-1}(Sp(q))=\mathbb{Z}$ (see [4]) and $\pi_{4q-1}(Sp(q-1))=\mathbb{Z}_2$ (see [11]) we conclude that p_* vanishes. The proof is finished.

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