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A TOPOLOGICAL APPLICATION OF THE ISOPERIMETRIC INEQUALITY

By M. Gromov* and V. D. Milman**

Many infinite dimensional topological spaces come with natural uniform structure, often associated to a metric. As an example one can take the Hilbert space H^∞, the sphere $S^\infty \subset H^\infty$ or the Grassmann manifold $G_k(H^\infty)$.

We show in this paper that passing from the category of continuous maps to the category of *uniformly* continuous maps has a non-trivial effect on the homotopy theory. In particular we exhibit some natural fibrations which have continuous sections but have no uniformly continuous (in particular Lipschitz) sections.

1. The Levy measure. For a set A in a metric space X we denote by $N_\epsilon(A)$, $\epsilon \geq 0$, its ϵ-neighborhood.

Consider a family $\{X_i, \mu_i\}$, $i = 1, 2, \ldots$, of metric spaces X_i with normalized (i.e. $\mu_i(X_i) = 1$) Borel measures μ_i. We call such a family *Levy* if for any sequence of Borel sets $A_i \subset X_i$, $i = 1, 2, \ldots$, such that $\liminf_{i\to\infty} \mu_i(A_i) > 0$, and for every $\epsilon > 0$ we have $\lim_{i\to\infty} \mu_i(N_\epsilon(A_i)) = 1$.

1.1. *Principal Example.* Let X_i be isometric to the Euclidean sphere $S^i \subset R^{i+1}$ of radius r_i. Take for μ_i the normalized i-dimensional volume element on S^i.

The family $\{X_i, \mu_i\}$ *is Levy iff* $r_i i^{-1/2} \to_{i\to\infty} 0$ (see [6]).

Proof. Let $A \subset S^i$ be an arbitrary Borel set and let $B \subset S^i$ be a ball (relative to the Riemannian metric in S^i) such that $\mu_i(B) = \mu_i(A)$. According to the isoperimetric inequality (see [12], [4]) one has $\mu_i(N_\epsilon(B)) < \mu_i(N_\epsilon(A))$, $\epsilon \geq 0$, and the general problem is reduced to the case when A_i are balls. A straight-forward calculation (see for instance, [6], [8]) yields now our assertion.

1.2. *Ricci curvature.* When X is a Riemannian manifold we always

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take for μ the normalized Riemannian volume element. We set $R(X) = \inf_\tau \operatorname{Ric}(\tau, \tau)$, where $\tau \in T(X)$ runs over all unit tangent vectors.

THEOREM. *Let $\{X_i\}$, $i = 1, 2, \ldots$, be closed Riemannian manifolds such that $R(X_i) \to_{i\to\infty} +\infty$. Then the family $\{X_i, \mu_i\}$ is Levy.*

Proof. An isoperimetric inequality from [5] reduces this theorem to the case of the above spheres.

Remark 1. For the sphere S^i of radius r one has $R(S^i) = (i-1)r^{-2}$ and, hence, our Riemannian theorem contains the case of spheres.

Remark 2. The isoperimetric inequality from [5] says that for any measurable $A \subset X$ and for any $\epsilon > 0$ the following inequality holds

$$\mu(N_\epsilon(A)) \geq \mu_k(N_\epsilon(B))$$

where B is a ball on the sphere S_r^k with $k = \dim X$, and with radius r such that $R(S_r^k) = R(X)$; here μ_k is normalized Haar measure on S_r^k and $\mu_k(B) = \mu(A)$.

When $\mu(A) \geq 1/2$ this inequality implies (see [8]) the following:

$$\mu(N_\epsilon(A)) \geq 1 - \sqrt{2}\exp\left(-\frac{\epsilon^2}{2r^2}k\right)$$

$$= 1 - \sqrt{2}\exp\left(-\frac{\epsilon^2 k}{2(k-1)}R(X)\right) \simeq 1 - \sqrt{2}\exp\left(-\frac{\epsilon^2}{2}R(X)\right).$$

2. General properties of Levy families. The following three facts are immediate from the definition.

2.1. *Let $\{X_i, \mu_i\}$ be a Levy family, let $\{Y_i\}$ be a family of metric spaces and let $f_i: X_i \to Y_i$ be an equi-uniformly continuous family of maps. Then $\{Y_i, \nu_i = (f_i)_*(\mu_i)\}$ is a Levy family.*

2.2. *Let $\{X_i, \mu_i\}$, $\{Y_i, \nu_i\}$ be Levy families. Then $\{X_i \times Y_i, \mu_i \times \nu_i\}$ is a Levy family.*

2.3. *Let $\{X_i, \mu_i\}$ be a Levy family and $A_i \subset X_i$ be Borel sets such that* $\lim_{i\to\infty}$ *inf $\mu_\iota(A_i) > 0$. Then $\{A_i, \mu_{\iota|A_i}\}$ is a Levy family where $\mu_{|A}$ is defined as follows: $\mu_{|A}(B) = \mu(B)\mu^{-1}(A)$, $B \subset A$.*

We call a family $\{X_i, \mu_i\}$ *degenerate* if there are balls $B_i \subset X_i$ of radii r_i such that $r_i \to_{i\to\infty} 0$ and $\mu_i(B_i) \to_{i\to\infty} 1$.

2.4. *If $\{K, \mu_i\}$ is a Levy family for a fixed compact K, then $\{K, \mu_i\}$ is degenerate.*

Proof. It is enough to show that there exist subsequences i_n, $r_n \to 0$ and $x_0 \in K$ such that $\mu_{i_n}(B_{r_n}(x_0)) \to 1$ where $B_{r_n}(x_0)$ is the ball of radius r_n and center x_0. Take some finite covering $\{B_\epsilon(x_j)\}_{j=1}^N$ of K for $\epsilon > 0$. There exist a point x_ϵ and a subsequence $\{i_n\}$ such that $\lim \mu_{i_n}(B_\epsilon(x_\epsilon)) \geq a > 0$ and therefore $\lim \mu_{i_n}(B_{2\epsilon}(x_\epsilon)) = 1$. Let $\epsilon \to 0$. Then one can take for x_0 any accumulation point of $\{x_\epsilon\}_{\epsilon \to 0}$. Q.E.D.

2.5. *Levy mean.* Let (X, μ) be as in section 1 and let $f: X \to R$ be a continuous bounded function. We say that the number L_f is the Levy mean (or median of f iff $\mu\{x \in X: f(x) \leq L_f\} \geq 1/2$ and $\mu\{x \in X: f(x) \geq L_f\} \geq 1/2$. Let $\{X_i, \mu_i\}$ be a Levy family. We consider a family of continuous functions f_i: $X_i \to \mathbf{R}$. Let $A_i^+ = \{x \in X_i: f_i(x) \geq L_{f_i}\}$ and $A_i^- = \{x \in X_i: L_i(x) \leq L_{f_i}\}$. It is obvious from the definition of Levy family that for any $\epsilon > 0$ one has $\mu_i(N_\epsilon(A_i^+) \cap N_\epsilon(A_i^-)) \to_{i \to \infty} 1$. In particular, for uniformly bounded and equi-uniformly continuous family of functions f_i we have

$$\left| \int_{X_i} f_i d\mu_i - a_i \right| \to 0, \qquad i \to \infty.$$

3. Further examples. We say that a family $\{X_i\}$ of complete Riemannian manifolds has *uniformly bounded geometry* if the sectional curvatures $K(X_i)$ and the injectivity radii $\mathrm{Rad}(X_i)$ satisfy: $|K(X_i)| \leq$ const, $\mathrm{Rad}(X_i) \geq (\text{const})^{-1}$, "const" > 0.

We say that submanifolds $Y_i \subset X_i$ have uniformly bounded relative geometry if all principal curvatures $\sigma(Y_i)$ satisfy: $\sigma_\tau(Y_i) \leq$ const where τ runs over all unit vectors normal to Y_i.

We say that a family of sets $Y_i \subset X_i$ is essential if for any $\epsilon > 0$ we have $\lim_{i \to \infty} \inf \mu_i(N_\epsilon(Y_i)) > 0$.

3.1. *Let $\{X_i \mu_i\}$ be a Levy family, where X_i are closed Riemannian manifolds with uniformly bounded geometry. Let $Y_i \subset X_i$ be an essential family of closed connected submanifolds of fixed codimension and with uniformly bounded relative geometry. Then $\{Y_i, \nu_i\}$ is a Levy family where ν_i denote the normalized volume associated to the Riemannian metrics induced by the inclusions $Y_i \to X_i$.*

Proof. When geometry is bounded, then there exists $\epsilon > 0$ such that the normal projections $N_\epsilon(Y_i) \to Y_i$ are well defined and uniformly continuous. Applying 2.3 and 2.1 we get 3.1.

Let us give a criterion for a family $Y_i \subset X_i$ to be essential. Let codim $Y_i = 1$ and let each Y_i divides X_i into two pieces X_i^0, X_i^1 with common boundary Y_i.

3.2. *If $\{X_i, \mu_i\}$ is a Levy family, X_i are connected and*

$$\liminf_{i\to\infty} \min(\mu_i(X_i^0), \mu_i(X_i^1)) > 0,$$

then Y_i are essential.

Proof. In fact, $N_\epsilon(Y_i) \supset N_\epsilon(X_i^0) \cap N_\epsilon(X_0^1)$; hence $\mu(N_\epsilon(Y_i)) \to_{i\to\infty} 1$, $\epsilon > 0$. Q.E.D.

3.3. Let Y_i be the Stiefel manifolds $W_2(\mathbf{R}^i)$. By using the natural imbeddings $Y_i \to S^{i-1} \times S^{i-1}$ and applying 2.2, 3.1 and 3.2 we conclude that Y_i *form a Levy family.* Repeating this argument we conclude that for each k *the family* $\{W_k(\mathbf{R}^i)\}$ *is also* Levy. By using the natural projections $W_k(\mathbf{R}^i) \to G_k(\mathbf{R}^i)$ we get the *Levy property for the Grassmann manifolds* as well (see [9] for further information).

3.4. The following examples of Levy family are obtained using 1.2 and the known values (see [3]) of Ricci curvature for these families (for the standard bi-invariant metric):

$$R(SO(n)) = \frac{n}{4}$$

$$R(S^n \times \cdots \times S^n) \geq n - 1;$$

$$R(X_1 \times \cdots \times X_k) \geq \min_i R(X_i)$$

if X_i are of non-negative curvatures;

$$R(W_k(\mathbf{R}^n)) \geq \frac{n}{4}.$$

3.5. Next two examples are combinatorial.

Let E_2^n be an n-dimensional linear space over the field $\mathbf{Z}_2$ with $\mu_n(A \subset E_2^n) = {}^{|A|}/2^n(|A|$ is a number of elements of $A)$ and with the

Hamming metric $\rho(x, y) = |\{i : x_i \neq y_i\}|$ where $x = (x_1, \ldots, x_n) \in E_2^n$ and $y = (y_1, \ldots, y_n) \in E_2^n$.

$\{E_2^n, \mu_n, \rho/n\}$ is a Levy family. Moreover, if $\mu_n(A \subset E_2^n) = 1/2$ and $\epsilon = \rho/n$ then $\mu_n(N_\rho(A)) \geq 1 - 1/2 \exp(-2n\epsilon^2)$.

This follows from a solution of the isoperimetric problem for (E_2^n, μ_n, ρ) [13]; see details and some applications in [1].

Remark. Let us consider the natural embeddings $\ldots E_2^n \subset E_2^{n+1} \subset \ldots$ and let $E_2^\infty = \cup_1^\infty E_2^n$. For $x, y \in E_2^\infty$, let $f(x, y) = \rho(x, y)/\max(\rho(x, 0), \rho(y, 0))$ except at $x = y = 0$ and let $f(0, 0) = 0$. We define the metric $\hat{\rho}$ by

$$\hat{\rho}(x, y) = \inf\left\{f(x, z_1) + \sum_1^{k-1} f(z_i, z_{i+1}) + f(z_k, y) : \{z_i\}_1^k \subset E_2^\infty\right\}.$$

It is not difficult to check that $f(z_1, z_k) \leq 2 \Sigma_1^{k-1} f(z_i, z_{i+1})$ for any $\{z_i\}_1^k \subset E_2^\infty$ and so $\hat{\rho}(x, y) \leq f(x, y) \leq 2\hat{\rho}(x, y)$ for all x, y in E_2^∞.

As before, *$(E_2^\infty, \mu_n, \hat{\rho})$ is a Levy family*, where μ_n denotes the same measure as in 3.5 (note that supp $\mu_n = E_2^n$).

3.6. Let $\mathfrak{S}_n$ be the permutation group of the numbers $\{1, \ldots, n\}$ $\mu_n(A \subset \mathfrak{S}_n) = {}^{|A|}/n!$. We give $\mathfrak{S}_n$ the Hamming metric $d(\sigma, \tau) = |\{i : \sigma(i) \neq \tau(i)\}|$. B. Maurey [7] has proved that if $\mu_n(A) \geq \alpha$ then $\mu_n(N_d(A)) \geq 1 - \exp(-(d - \sqrt{n \log 1/\alpha})^2/16n) \simeq 1 - \exp(-\epsilon^2 n/16)$ for $d \gg \sqrt{n}$ and $\epsilon = d/n$.

It means that *$(\mathfrak{S}_n, \mu_n, d/n)$ is Levy family.*

Remark. In the same way as in the Remark to 3.5 we introduce

— the group $\mathfrak{S}_\infty = \cup_1^\infty \mathfrak{S}_n$ (considering each $\mathfrak{S}_n$ as a subgroup of $\mathfrak{S}_{n+1}$ with natural embedding for every $n = 1, 2, \ldots$),

— the function $\varphi(\sigma, \eta) = d(\sigma, \eta)/\max(d(\sigma, \mathfrak{e}), d(\eta, \mathfrak{e}))$ for $\sigma, \eta \in \mathfrak{S}_\infty$ where $\mathfrak{e}$ is the identity element of $\mathfrak{S}_\infty$,

— and the metric $\hat{d}(\sigma, \eta)$ such that $\hat{d}(\sigma, \eta) \leq \varphi(\sigma, \eta) \leq 2\hat{d}(\sigma, \eta)$.

Then *$(\mathfrak{S}_\infty, \mu_n, \hat{d})$ is a Levy family.*

4. The Laplace operator. Let M be a compact connected Riemannian manifold. Then the Laplace operator $-\Delta$ on M has its spectrum consisting of eigenvalues $\lambda_0 = 0 < \lambda_1(M) \leq \lambda_2(M) \leq \ldots$. The first nontrivial eigenvalue λ_1 may be represented as the largest constant such that

$$\lambda_1 \|f\|_{L_2}^2 \leq (-\Delta f, f) = \int_M |\nabla f|^2 \tag{1}$$

for any continuous and "sufficiently smooth" function f on M such that $\int_M f = 0$ (see some consequences and references [11]).

4.1. THEOREM. *Let M and $\lambda_1(M) = \lambda_1$ be as in 4 and let $A \subset M$ be a closed set, with $\mu(A) = a > 0$, where μ is the normalized Riemannian volume element of M. Then for $\epsilon > 0$*

$$\mu(N_\epsilon(A)) \geq 1 - (1 - a^2)\exp(-\epsilon\sqrt{\lambda_1} \cdot \ln(1 + a)).$$

Proof. Let A and B be closed subsets of M, $\mu(A) = a > 0$, $\mu(B) = b > 0$, such that the distance $\rho(A, B) = \rho > 0$. We consider a function

$$f(x) = \frac{1}{a} - \frac{1}{\rho}\left(\frac{1}{a} + \frac{1}{b}\right)\min(\rho(x, A), \rho).$$

Let $\int_M f d\mu = \alpha$. Then by (1) we have

$$\lambda_1 \| f - \alpha \|^2 \leq \int | \nabla f |^2 d\mu \leq \frac{1}{\rho^2}\left(\frac{1}{a} + \frac{1}{b}\right)^2 (1 - a - b)$$

and

$$\lambda_1 \| f - \alpha \|^2 \geq \lambda_1 \left[\int_A (f - \alpha)^2 d\mu + \int_B (f - \alpha)^2 d\mu \right]$$

$$= \lambda_1 \left(\left(\frac{1}{a} - \alpha\right)^2 a + \left(\frac{1}{b} + \alpha\right)^2 b \right)$$

$$\geq \lambda_1 \left(\frac{1}{a} + \frac{1}{b}\right).$$

Therefore $\rho^2\lambda_1 \leq ({}^1/a + {}^1/b)(1 - a - b) \leq (1 - a - b)/ab$, and

$$b \leq \frac{1 - a}{1 + \lambda_1 \rho^2 a}. \tag{2}$$

Take some $\delta > 0$ and consider a sequence of pairs (A_i, B_i) of subsets of M: $A_0 = A$ and $B_0 = (N_\delta(A))^C \equiv M \backslash N_\delta(A)$; $A_1 = N_\delta(A_0)$ and $B_1 = (N_\delta(A_1))^C$ and so on. After p-steps we have by (2)

$$b_p \leq \frac{1 - a_p}{1 + \lambda_1 \delta^2 a_p}$$

where $b_p = \mu(B_p)$, $a_p = \mu(A_p) = \mu(N_{p\delta}(A))$ and $a_{p+1} = 1 - b_p$.

Therefore

$$1 - a_{p+1} \leq (1 - a_p)\frac{1}{1 + \lambda_1 \delta^2 a} \tag{3}$$

(we used $a_p \geq a$). Take $\delta = 1/\sqrt{\lambda_1}$ and define $\varphi(\epsilon) = \mu(N_\epsilon(A))$. Then apply (3) inductively p times to obtain

$$1 - \varphi(p \cdot \delta) \leq (1 - a)\exp(-p \ln(1 + a)).$$

Let $p\delta \leq \epsilon < (p + 1)\delta$. So $(\epsilon - p\delta) \cdot \sqrt{\lambda_1} \leq 1$ and $\varphi(\epsilon) \geq \varphi(p\delta)$. Finitely we have

$$\begin{aligned} 1 - \varphi(\epsilon) \leq 1 - \varphi(p \cdot \delta) &\leq (1 - a)\exp(-p\delta\sqrt{\lambda_1} \ln(1 + a)) \\ &\leq (1 - a)\exp(-\epsilon\sqrt{\lambda_1} \ln(1 + a) + (\epsilon - p\delta)\sqrt{\lambda_1} \ln(1 + a)) \\ &\leq (1 - a^2)\exp(-\epsilon\sqrt{\lambda_1} \ln(1 + a)). \end{aligned}$$

COROLLARY. *If X_n are compact Riemannian manifolds and $\lambda_1(X_n) \to \infty$ $(n \to \infty)$ then the family (X_n, μ_n) is Levy* (where *μ_n is the normalized Riemannian volume element of X_n).*

4.2. *Examples.* (a) $\lambda_1(S^n) = n$ and we once more obtain 1.1.

(b) Let $T^n = S^1 \times \cdots \times S^1$ be the n-dim. torus; $R(T^n) = 0$ and therefore we cannot apply 1.2 to this manifold. However $\lambda_1(S^1) = 1$ and so $\lambda_1(T^n) = 1$. Let $A_n \subset T^n$ and $\mu_n(A_n) \geq a > 0$.

Then, by 4.1, $\mu_n(N_{\epsilon_n}(A_n)) \to 1$ if $\epsilon_n \to \infty$. (It is worthwhile in this example to consider $\epsilon_n \to \infty$ because the diameter $d(T^n) = \pi\sqrt{n}$). So we have the following statement:

Let $0 < a_n \to \infty$ and let *$\rho(x, y)$ be the usual Riemannian metric* on *T^n. Then $(T^n, \mu_n, \rho_n = \rho/a_n)$ is Levy family.*

Example (b) returns us to the first observation of this kind, due to E. Borel which preceded the Theorem of Levy 1.1. E. Borel gave the following geometrical interpretation of the Law of large numbers for a sum of uniformly distributed on the $[-1, 1]$ independent random variables: Let $M^n = [-1, 1]^n$ be an n-dim. cube and let μ_n be the normalized Lebesgue measure on M^n. Let H be a hyperplane which is orthogonal to a principal diagonal of M^n at the point 0. Then $\mu_n\{x \in M^n \text{ s.t. } \rho(x, H) \leq \epsilon\sqrt{n}\} \to 1$ if $n \to \infty$ for any fixed $\epsilon > 0$ where $\rho(x, H)$ is the distance from

x to the set H. Obviously, example (b) contains not only the observation of E. Borel for the hyperplane H and $\epsilon_n = \epsilon\sqrt{n}$ but it claims a stronger statement:

Let $f_n : M^n \to \mathbf{R}$ be a family of continuous functions, let L_n be the Levy mean of f_n (see 2.5) *and let $H_n = \{x \in M^n : f_n(x) = L_n\}$. Then $\mu_n\{N_{\epsilon_n}(H_n)\} \to 1$ for any $\epsilon_n \to \infty$.*

5. *G-spaces.* Let X be a metric space with a uniformly continuous action of a group G and $G = \bigcup_{i=1}^{\infty} G_i$ where $G_i \subseteq G_{i+1}$ are subgroups of G. We call the G-space (X, G) *Levy* if there is a sequence G_i invariant measures μ_i on X, such that $\{X_i = X, \mu_i\}$ is a Levy family.

5.1. *Examples.* Let G be a compact group which acts by isometries on the Hilbert sphere S^∞. *The G-space (S^∞, G) is Levy* (here $G_i \equiv G$).

Proof. Take a sequence of great spheres $S^{d_i} \subset S^\infty$ such that they are G invariant and $d_i \to \infty$ as $i \to \infty$. Take for μ_i the normalized Riemannian volume supported on S^{d_i} and use 1.1. Q.E.D.

Take a subgroup $G \subset 0(k)$. This group naturally acts on the disjoints union $X = \bigcup_{i=k}^{\infty} W_k(\mathbf{R}^i)$ and this action is Levy due to 3.3.

5.2. Let (X, G), (Y, G) be G-spaces and $f: X \to Y$ be a G-invariant uniformly continuous map.

If (X, G) is Levy then (Y, G) is also Levy.

Proof. Use 2.1.

5.3. THEOREM. *If X is compact, (X, G) is Levy and G acts equicontinuously on X then there is a point $x \in X$ which is fixed under the action of G.*

Proof. Use 2.4.

Remark. Theorem 5.3 is true even if we don't assume that G_i (and G) are groups.

5.4. MAIN THEOREM. *Let (G, X) be a Levy G-space as in* 5 *and let Y be an arbitrary compact G-space without fixed points (i.e. fixed under all elements of G). Then there is no G-invariant uniformly continuous maps $X \to Y$.*

Proof. Use 5.2 and 5.3.

Remark. When Y is contractible there usually is a *continuous*

G-invariant map $X \to Y$. Examples of a fixed-point-free action on balls can be found in [2].

5.5. *The bundle sections.* When G acts on X freely and (Y, G) is any G-space one has a natural fibration $(X \times Y)/G \to X/G$ with fiber Y. (We use the diagonal action in $X \times Y$.) There is a canonical correspondence between the equivariant maps $X \to Y$ and the sections $X/G \to (X \times Y)/G$. This correspondence preserves uniform continuity and hence we get examples of bundles which admit continuous, but not uniformly continuous sections.

6. A spectrum of functions on X. Let again (X, G) be a G-space as in 5. Let G_i act transitively on $X_i \subset X$, let μ_i be supported by X_i and let X be the closure of $\cup_i X_i$. The following examples are typical: (a) $X_i = S^{i-1}$ and $X = S^\infty$; (b) $X_i = W_k(\mathbf{R}^i)$ and $X = W_k(H^\infty)$; (c) $X_i = G_k(\mathbf{R}^i)$ and $X = G_k(H^\infty)$. In all these examples we have $G_i = SO(i)$.

6.1. Let $f(x)$ be a uniformly continuous function on $X(f: X \to \mathbf{R})$. We say (see [8], [10]) that a number $a \in \mathbf{R}$ is from *the spectrum* of f iff for every $\epsilon > 0$ and for every i there exists $g \in G$ such that $|f(x) - a| < \epsilon$ for every $x \in gX_i$.

6.2. *If (X, G) is Levy and $\{G_i\}_1^\infty$ are compact then for any bounded uniformly continuous function $f: X \to \mathbf{R}$ the spectrum* of f is not empty. (A number of examples and applications can be found in [8], [10].)

We shall prove a formally stronger statement:

6.3. *Let $\{a_i\}_1^\infty$ be the Levy means of $f_{|X_i}$ as in* 2.5. *For every integer k and any $\epsilon > 0$ there exists an integer N such that for any $i \geq N$ and* any set $M_i \subset X_i$ containing k points ($|M_i| = k$) we have $|f(g_i x) - a_i| < \epsilon$ for every $x \in M_i$ and for some $g_i \in G_i$.

Proof. It is clear that μ_i are induced by the Haar measures ν_i on G_i: for some fixed $x_0 \in X_i$ and $Y \subset X_i$ we have $\mu_i(Y) = \nu_i(T = \{g \in G_i : gx_0 \in Y\}$.

Let $T_x = \{g \in G_i\colon |f(gx) - a_i| < \epsilon\}$ for fixed $x \in X_i$. By 2.5 $\mu_i(T_x) \to 1$ when $i \to \infty$. We apply this argument k times for every $x \in M_i$ and then consider an intersection of k subsets $T_x \subset G_i$ (for i large enough) which is not empty because it has asymptotically (when $i \to \infty$) full measure in G_i. Q.E.D.

To prove 6.2 we consider an ϵ-net of X_i and apply 6.3.

Remark 1. Of course, we can consider in 6.2 any uniformly continuous map $f: X \to Y$ where Y is a compact space with a metric $\rho = \rho_Y$ and we obtain the same kind of result, namely, instead of $a \in \mathbf{R}$ in 6.1 we have $a \in Y$ and $|f(x) - a| < \epsilon$ is replaced by $\rho_Y(f(x), a) < \epsilon$.

Remark 2. Let $f(x, y)$ be a uniformly continuous real function on the product $X \times Y$ where X is as in 6.2 and Y is a compact metric space. Then there exists a continuous function $a(y): Y \to \mathbf{R}$ such that for every $y \in Y$ the number $a(y)$ is from the spectrum of $f: X \times \{y\} \to \mathbf{R}$, and moreover for every $\epsilon > 0$ and every i there exists $g \in G$ such that

$$|f(x, y) - a(y)| < \epsilon$$

for every $x \in gX_i$ and for every $y \in Y$.

6.4. *An additional example.* Let $\Sigma_k = \{(\xi, x): \xi \in G_k(H^\infty)$ and $x \in \xi \subset H^\infty, \|x\| = 1\}$ be the canonical sphere bundle over the Grassmann manifold $G_k(H^\infty)$. If $f: \Sigma_k \to \mathbf{R}$ is a uniformly continuous function then *there exists* $a \in \mathbf{R}$ *such that for any integer* $n > k$ *and any* $\epsilon > 0$ *one can find an n-dimensional subspace* $E^n \subset H^\infty$ *such that*

$$|f(\xi; x) - a| < \epsilon \quad \text{for} \quad \forall \xi \in G_k(E^n) \quad \text{and} \quad \forall x \in \xi \subset E^n.$$

The proof is based on a simple interpretation of 6.2 for $X = \Sigma_k$ and $G = \bigcup_{n \geq k}^{\infty} G_n$, $G_n = SO(n)$.

Nonexistence of a uniformly continuous section $G_k(H^\infty) \to \Sigma_k$ is an immediate consequence of this fact (of course, it is well known that even continuous section does not exist in this example). Indeed, if such a section had existed then, by Uryssohn Theorem, one could construct a uniformly continuous function $f: \Sigma_k \to \mathbf{R}$ which takes the values one and zero for every k-dimensional subspace $\xi \subset H^\infty$ which contradicts the statement of 6.4.

6.5. We consider now a fibration $X \times Y/G_k \to X/G_k$ as in 5.5 where Y is a compact, (Y, G_k) is a G_k-space and X and G as in 6 and 6.2. The next statement follows from 6.3 (see Remark 2):

Let $f: X \times Y/G_k \to \mathbf{R}$ *be a uniformly continuous bounded function.*

For any $\epsilon > 0$ *and for any integer* $\ell > k$ *there exists* $g \in G$ *such that the function* $f(\xi)_{|\xi \in X_\ell \cdot g \times Y/G_k}$ *depends (up to* ϵ*) only on orbits* Y/G_k.

6.6. We are ready to obtain once more the result 5.5 (with some additional condition on X) from 6.5 in the same way as we did in 6.4.

Remark. In all previous results we don't need, in fact, to consider uniformly continuous maps and sections. It is enough to consider even not continuous map f but which is "uniformly continuous on an essential subset." It means that there exists $\delta > 0$ and a continuous function $w(\epsilon) \to 0$, $\epsilon \to 0$, such that

$$\overline{\lim_{i\to\infty}}\, \mu_i\{x: w_f(x; \epsilon) \leq w(\epsilon)\} \geq \delta$$

where $w_f(x; \epsilon)$ is the modulus of continuity f at the point $x \in X$.

7. Generalization; A fixed-point theorem. Most of the previous results hold not only for metric space X but may be reformulated for X with an arbitrary Hausdorff topology.

Particularly, 2.4 and 5.3 hold for any Hausdorff compact space X.

7.1. THEOREM. *Let (X, G) be a G-space as in* 5 *and let (G_i, μ_i) be a Levy family where μ_i is the normalized Haar measure on G_i. Let G act on X as an equicontinuous transformation group* (it means, particularly, that $\psi(g; x) = gx$ is a uniformly continuous map $G \times X \to X$). *If X is a compact then there exists a common fixed point under the action of G.*

Proof. Let $x_0 \in X$. We construct a map $\varphi: G \to X$, $\varphi(g) = gx_0$, and we consider a family ν_i of measures on X such that $\nu_i = \varphi_*(\mu_i)$. By 5.2 (X, ν_i) is a Levy family. So we can use 5.3.

7.2. One can apply 7.1 for

(i) $G = \{\cup_{i=1}^{\infty} SO(i)\}$ using 3.4,
(ii) $G = \mathfrak{S}_\infty = \cup_{i=1}^{\infty} \mathfrak{S}_n$ taken in the sense of 3.6 with the metric $\hat{\rho}$,
(iii) $G = E_2^\infty = \cup_{n=1}^{\infty} E_2^n$ as in 3.5 with the metric $\hat{d}$.

Remark. 7.2 (i) and (ii) answer a question of H. Furstenberg. However, H. Furstenberg wanted to prove these statements as another approach to Theorem 6.2, different from that of [8] and this paper.

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