MESOSCOPIC CURVATURE AND HYPERBOLICITY

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Dedicated to the memory of Alfred Gray

We introduce in this paper a notion of $\mathrm{CAT}_\delta(\kappa)$-*spaces* mediating between $\mathrm{CAT}(\kappa)=\mathrm{CAT}_0(\kappa)$ and δ-hyperbolicity (essentially) corresponding to $\mathrm{CAT}_\delta(-\infty)$. This allows the local-to-global passage (in the spirit of Cartan-Hadamard) most convenient for applications to combinatorial group theory. There is little new about the proofs : everything follows by "δ-perturbing" similar arguments in the $\mathrm{CAT}(\kappa)$-case. This has been done in $[\mathrm{Gro}]_{\mathrm{HG}}$ for $\kappa=-\infty$ and we do more or less the same for finite κ with minor technical adjustments. We shall see, in particular, that $\mathrm{CAT}_\delta(\kappa)$-property of the σ-balls for a given σ in a *simply connected* space X implies δ_1-hyperbolicity of X with $\delta_1 < 2/\sqrt{-\kappa}$ for all $\kappa<0$, provided $\kappa>0$, $\sigma>0$ and $\delta \leq \delta_0(-\kappa,\sigma)>0$ for some universal *strictly* positive function δ_0 in two strictly positive variables. (This is standard for $\delta=0$).

0. Preliminary notations and definitions.

Given a metric space X we write

$$|x-y| = |x-y|_X = \mathrm{dist}_X(x,y)$$

(compare $[\mathrm{Gro}]_{\mathrm{HG}}$ and $[\mathrm{Gro}]_{\mathrm{CAT}}$). X is called a *path metric* space (see [G-L-P]) if $|x-y|$ equals the infimum of length of curves between x and y for all $x,y\in X$. We say that X is *geodesic* if this infimum is assumed where the minimal curve is called a *geodesic segment* between x and y, denoted $[x,y]\subset X$.

Recall (see [Bri-Hae] and references therein) that $\mathrm{CAT}(\kappa)$-spaces are introduced by improving certain inequalities on the six distances between four points $x_i \in X$, $i=1,\dots,4$, similar to those satisfied by (comparison) quadruples $\{x'_i\}$ in the *model* space X' that is the simply connected space X' with $\dim X'=2$ of constant curvature κ. We are mostly concerned here with negative curvature $\kappa<0$, where X' equals the hyperbolic plane H^2_κ. If $\kappa\to-\infty$, then H^2_κ (Hausdorff) converges to an infinitely branched tree (see [G-L-P], $[\mathrm{Gro}]_{\mathrm{AI}}$ serving as the model space for $\kappa=-\infty$.

The comparison inequality can be expressed by saying that for each quadruple $\{x_i\}$ in X there is a quadruple $\{x'_i\}\subset X'$ making a convex 4-gone for the cyclic order on x_i, such that the four edges of this 4-gone equal the corresponding 4 distances between x_i,

$$\left|x'_i - x'_{i+1}\right|_{X'} = |x_i - x_{i+1}|_X\,,\ i = 1, \dots, 4,$$

(with the convention $4+1=1$), while the remaining two distances satisfy

$$|x'_1 - x'_3|_{X'} \geq |x_1 - x_3|_X$$

and

$$|x'_2 - x'_4|_{X'} \geq |x_2 - x_4|_X\,.$$

This property for a quadruple $\{x_i\}$ is called $\mathrm{Cycl}_4^{(\kappa)}$-*inequality.* Similarly, given a cyclicly ordered k-tuple of points $\{x_i\} \subset X,\ i = 1, \dots k,\ k+1 = 1$, one says that $\{x_i\}$ satisfies $\mathrm{Cycl}_k(\kappa)$ if there exists $\{x'_i\} \subset X'(= H^2_\kappa)$, such that

$$\left|x'_i - x'_j\right|_{X'} \geq |x_i - x_j|_X$$

for all $i, j = 1, \dots, k$ and

$$\left|x'_i - x'_{i+1}\right|_{X'} \leq |x_i - x_{i+1}|_X$$

for $i = 1, \dots, k$, and where one additionally requires that $\{x'_i\}$ makes a *convex* k-gone in X'. One knows (Resketnyak's theorem) that

$$\mathrm{Cycl}_4(\kappa) \Rightarrow \mathrm{Cycl}_k(\kappa),\ k = 5, 6, \dots \text{ for all k},$$

for the *geodesic* spaces X (compare $[\mathrm{Gro}]_{\mathrm{CAT}}$).

One can express the inequality $\mathrm{Cycl}_4(\kappa)$ as

$$|x_1 - x_4| \leq F_\kappa(|x_i - x_j|), \qquad (\kappa)_4$$

where F_κ is a certain function in 5 variables $|x_i - x_j|,\ 1 \leq i < j \leq 4$, except $i = 1,\ j = 4$, that is monotone increasing in $|x_i - x_{i+1}|$ and decreasing in $|x_2 - x_4|$. Namely, one takes F_κ, such that $(\kappa)_4$ become *equality* for the model space X'_κ. This F_κ is defined for all *non-negative* values of the five arguments satisfying two *triangle inequalities* (for $\Delta(x_1, x_2, x)$ and $\Delta(x_1, x_3, x_4)$. Thus the domain of definition of F_κ is a convex subset, denoted $\Delta_5 \subset \mathbb{R}^5_+$. The function $F_\kappa : \Delta_5 \to \mathbb{R}_5$ is smooth in the interior of Δ_5 and 1/2-*Hölder* on all of Δ_5; but it is *not Lipschitz.* This causes some technical problems when we perturb this inequality by a small $\delta > 0$.

If $\kappa = -\infty$, then $(\kappa)_4$ becomes an *equality* and its δ-perturbation leads to the δ-*hyperbolicity* of X expressed by the following $\mathrm{CAT}_\delta(-\infty)$-*relation* between the three numbers

$$m_i = |x_1 - x_i| + |x_j - x_k|,$$

for $i \neq j \neq k \neq i = 2, 3, 4$, as follows

$$m_2 \leq \max(m_3, m_4) + 2\delta \qquad \boxtimes_\delta$$

for all quadruples $\{x_i\} \subset X$. (See [Bri-Hae] and references therein). Similarly, one can introduce $\mathrm{CAT}_\delta(\kappa)$-inequality to be

$$|x_1 - x_4| \leq F_x(|x_i - x_j|) + \delta$$

expressing the idea that the quadruple $\{x_i\}$ is δ-close (in the Hausdorff sense) to $\{x'_i\}$ satisfying $(\kappa)_4$. Unfortunately this does not work as well for $\kappa > -\infty$ as for

$\kappa = -\infty$ due to non-Lipschitz behaviour of F_κ on the boundary of Δ_5 and leads to somewhat heavier notations and inequalities than those for $\kappa = -\infty$.

COMPARISON MAPS IN GEODESIC SPACES. If X is a geodesic space, then one can express the CAT(κ)-property in terms of the *comparison maps.* Such a map c is defined with a triple $\{x_i\} \subset X$ and sends the *comparison triangle* $\Delta' \subset X' = H^2_\kappa$, spanned by $\{x'_i\} \subset H^2_\kappa$ with $\left|x'_i - x'_j\right|_{X'} = |x_i - x_j|_X$, to X, such that $x'_i \mapsto x_i$, $i = 1, 2, 3$ and the geodesic segments $[x'_i, x'_j]$ *isometricly* go to (possibly non-unique) segments $[x_i, x_j] \subset X$. The Alexandrov comparison inequality reads

$$|c(x') - c(y')|_X \leq |x' - y'|_{X'}$$

for all x', y' on the perimeter of Δ' (and it is not hard to extend this c from the boundary of Δ to a 1-*Lipschitz* map on all of Δ, see [La-Sch]).

1. Broken Triangles and $\mathrm{CAT}_\delta(\kappa)$.

It is slightly more convenient to depart from Cycl_5 rather than from Cycl_4 and work with 5-tuples of points in X, denoted by $\{x_1, y_2, x_2, x_3, y_3\}$, thought of as a broken (geodesic) triangle $\Delta(x_1, x_2, x_3)$ with the "breaks" y_2 and y_3, see Fig. 1.

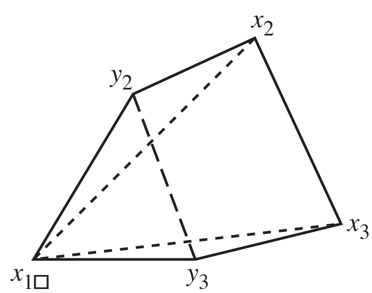

FIGURE 1

where we are concerned with an upper bound on $|y_2 - y_3|_X$ in terms of the five boundary edges and $|x_1 - x_i|$, $i = 2, 3$. We consider the comparison triangle (x'_1, x'_2, x'_3) in the model space $X' = H^2_\kappa$, where

$$|x'_1 - x'_i|_{X'} = |x_1 - y_i|_X + |y_i - x_i|_X \text{ for } i = 2, 3. \tag{$*$}$$

and

$$|x'_2 - x'_3|_{X'} = \max(|x_2 - x_3|_X\,,\ |x'_1 - x'_2| - |x'_1 - x'_3|) \tag{${*\atop *}$}$$

and where $|x'_1 - x'_i|$ are computed according to $(*)$. (In other words, if the numbers $|x'_i - x'_j|$ violate the triangle inequality, we use the degenerate triangle in X' with $|x'_2 - x'_3| = ||x'_1 - x'_2|_{X'} - |x'_1 - x'_3|_{X'}|$). We measure non-straightness of the "broken edges" $[x_1, x_i]$, $i = 2, 3$ by the errors $\beta_i = |x_1 - y_i| + |y_i - x_i| - |x_1 - x_i|$ for $i = 2, 3$, where the equalities $\beta_i = 0$ signify that there is no breaks in these "edges".

We look for a bound of the form

$$|y_2 - y_3|_X \leq |y'_2 - y'_3|_\kappa + \delta(\beta), \tag{$\square$}$$

where y_i' are points on the segments $[x_1', x_i'] \subset X' = H_\kappa^2$ corresponding to y_i with the distance $|y_2' - y_3'|_\kappa \overset{\text{def}}{=} |y_2' - y_3'|_{H_\kappa^2}$, where $e = \beta_2 + \beta_3$ and $\delta = \delta(\beta)$ some universal function. We observe, that for X *isometric* to H_κ^2, the bound $(\pentagon)$ is valid with

$$\text{some } \delta(\beta) \leq 10(\sqrt{\beta} + \beta)/\sqrt{-\kappa}$$

we denote by $\delta_\kappa(\beta)$ the infimum of monotone increasing functions $\delta(\beta)$ satisfying $(\pentagon)$ for all 5-tuples in this $X = H_\kappa^2$. Then we conclude that $(\pentagon)$ holds true with this very $\delta_\kappa(\beta)$ for all $\mathrm{CAT}(\kappa)$-spaces as follows from the Cycl_5-comparison inequality. It is also clear that every δ'-hyperbolic space satisfies $(\pentagon)$ with $\kappa = -\infty$ (i.e all $\kappa < 0$ and $\delta(\beta) \leq 20\delta'$ for all $\beta \geq 0$.

Definition. We say that a (not necessarily geodesic) metric space is $\mathrm{CAT}_\delta(\kappa)$ for a given monotone increasing function $\delta = \delta(\beta)$ if it satisfies $(\pentagon)$ with this δ.

Remark. As far as the essential geometry of an *individual geodesic* space X is concerned, we shall only need and use $(\pentagon)$ for "unbroken" triangles where $\beta = 0$. But if we want to compare spaces X and Y with some correspondence between their points, where $||x_1 - x_2|_X - |y_1 - y_2|_Y| \leq \epsilon$ for the corresponding points, then it is convenient to have $\delta(\beta)$ defined for all $\beta > 0$, as the CAT_δ-inequality with a given $\delta(\beta)$ for one space implies such an inequality with $\delta(\beta + 2\epsilon) + \epsilon$ for the other.

In most of what follows, we deal with geodesic spaces X and with $(\pentagon)$ for a *number* $\delta = \delta(0)$ where $\delta(\beta)$ is undefined (or assumed $= +\infty$) for $\beta > 0$. Such $(\pentagon) = (\pentagon)_0$) amounts to the δ-*perturbation* of the comparison inequality for *geodesic triangles* $\Delta \subset X$ (see [Bri-Hae]) which requires that every comparison map c satisfies

$$|c(x') - c(y')|_X \leq |x' - y'|_{X'} + \delta \qquad (*_\delta)$$

for all $x', y' \in \Delta' \subset X'$.

Observe that this $\mathrm{CAT}_\delta(-\infty)$ is equivalent to hyperbolicity:

$$\mathrm{CAT}_\delta(-\infty) \Rightarrow 4\delta - \text{hyperbolicity} \Rightarrow \mathrm{CAT}_{20\delta}(-\infty).$$

It is also clear that

$$\mathrm{CAT}_\delta(\kappa) \Rightarrow (4\kappa^{-\frac{1}{2}} + 4\delta) - \text{hyperbolicity}$$

for all $\kappa < 0$. Let us remind that here and in the rest of §3, δ stands for the number $\delta = \delta(0)$, while $\delta(\beta)$ remains undefined (or set $= +\infty$) for $\beta > 0$.

2. Propagation of convexity.

We shall give below a definition of convexity of subsets Y in $\mathrm{CAT}_\delta(\kappa)$-spaces X which is preserved undertaking the ϵ-neighbourhoods, $Y \mapsto Y + \epsilon \subset X$ (compare $[\mathrm{Gro}]_{HG}$).

A subset Y in a geodesic metric space X is called *geodesicly ϱ-convex* if every geodesic segment $[y_1, y_2] \subset X$ with $y_1, y_2 \in Y$ and $|y_1 - y_2| > \varrho$ contains a point $y \in Y$ different from y_1 and y_2. Notice that the geodesic 0-convexity amounts to the ordinary geodesic convexity for *closed* subsets $X \subset Y$.

2.A. Proposition. *If $\varrho, \epsilon \geq 1000(-\kappa)^{-\frac{1}{2}} \max(\delta\sqrt{-\kappa}, (\delta\sqrt{-\kappa})^{\frac{1}{4}})$, the ϵ-neighbourhood implies that for $Y + \varrho \subset X$ for all $Y \subset X$ and all $\mathrm{CAT}_\delta(\kappa < 0)$-spaces X.*

SKETCH OF THE PROOF. If $\delta \geq 1/\sqrt{-\kappa}$, then X is δ'-hyperbolic with $\delta' \approx 11\delta$ and everything follows along the lines in $[\mathrm{Gro}]_{HG}$. An interesting case here is $\delta \leq 1/\sqrt{-\kappa}$ where we should show that the gain in convexity due to $\kappa < 0$ overcomes δ. In fact the comparison inequality reduces our problems to the following

2.B. LEMMA. *Let Y' be a geodesicly ϱ-convex subset in the hyperbolic plane $X' = H_\kappa$ contained in some geodesic in this plane. Then $Y' + \varrho$ is geodesicly ϱ'-convex for*

$$\varrho' \leq \varrho - 10^{-3}(-\kappa)^{-\frac{1}{2}} \min(\epsilon\sqrt{-\kappa}, \varrho\sqrt{-\kappa}, (\epsilon\sqrt{-\kappa})^{\frac{1}{4}}, (\varrho\sqrt{-\kappa})^{\frac{1}{4}}).$$

Our generous assumptions make the proof rather obvious. In fact, one can, probably, make it work with 10 instead of 1000 and (this is more relevant) with $\frac{1}{2}$ instead of $\frac{1}{4}$.

2C. The above shows, that there is some gain in convexity as ϵ-growth. It is also clear that *"normal projection" $X \to Y$ is distance decreasing up to an error* $\approx \delta + \varrho$. Namely,

If Y is geodesicly ϱ-convex and $y_i \in Y$ are nearest points in Y to some points $x_i \in X,\ i = 1, 2$, then

$$|y_i - y_j| \leq |x_i - x_j| + 5(\delta + \varrho),$$

and, moreover, if $\epsilon \overset{\text{def}}{=} \max\limits_{i=1,2} \operatorname{dist}(x_i, Y) \leq 0,1(-\kappa)^{\frac{1}{2}}$ and $|x_i - x_j| = d > 10\epsilon$, then

$$|y_i - y_j| \leq d - 0.01(\sqrt{-\kappa}\epsilon)^4 + 5(\delta + \varrho).$$

3. Local ϱ-convexity.

We say that Y is *σ-locally (geodesicly) ϱ-convex* if every segment $[y_1, y_2] \subset X$ with $y_1, y_2 \in Y$ and $\varrho \leq |y_1 - y_2| \leq \sigma$ contains an interior interior point in Y.

3.A. *If $\kappa, \epsilon, \delta, \varrho$ satisfy the assumptions of* 2.A. *then σ-local ϱ-convexity possess from Y to $Y + \epsilon$ for all $\sigma > 10(\varrho + \epsilon)$, provided Y is connected.*

In fact, the σ-local geometry of $Y + \epsilon$ is determined by the σ'-local properties of Y for σ' somewhat smaller than σ and so the local convexity is preserved as ϵ grows. The growing locally convex domain $Y + \epsilon$ can, a priori, meet itself at some point as happens, for example for growing balls in flat tori, see Fig. 2).

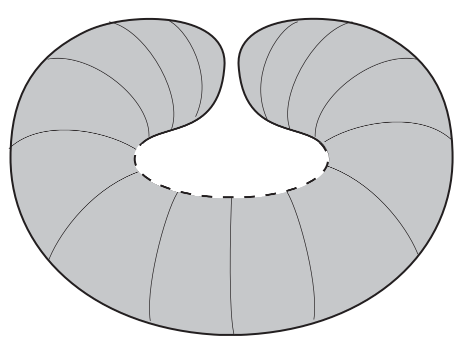

FIGURE 2

If this happens, we still can continue to increase ϵ, but now our $Y+\epsilon$ becomes a *multi domain*, i.e. a metric space with a locally isometric map into X. Eventually, for $\epsilon=\infty$, we obtain a space $\widetilde{X}$ with such a map into X, and then we conclude to the injectivity of this map by confronting completeness of $\widetilde{X}$ with simply connectedness of the *Ribs complex* of X (see $[\mathrm{Gro}]_{\mathrm{HG}}$, $[\mathrm{Gro}]_{\mathrm{CAT}}$, [Bri-Hae] and references therein).

There is an additional bonus to this argument.

3.3. *If Y is σ-locally ϱ-convex and connected, then $Y+\epsilon$ is (globally!) geodesicly ϱ-convex.*

4. $\mathrm{CAT}_\delta(\kappa|\sigma)$-spaces.

The above argument needed the $\mathrm{CAT}_\delta(\kappa)$-properly only for subsets of the size $\le\sigma$. With this in mind, we say that Y *is* $\mathrm{CAT}_\delta(\kappa)$ *on the scale* $\le\sigma$ and denote the corresponding class of spaces by $\mathrm{CAT}_\delta(\kappa|\sigma)$ if the comparison inequality $(*_\delta)$ is satisfied by the triangles of diameter $\le 10\sigma$. Then we have the following

4.A. CAT_δ Hadamard-Cartan theorem. *Let X be $\mathrm{CAT}_\delta(\kappa|\sigma)$ with $\sigma\ge 10^5(-\kappa)^{\frac{1}{2}}\max(\delta\sqrt{-\kappa},(\delta\sqrt{-\kappa})^{\frac{1}{4}})$. If X is locally and globally simply connected, then the metric balls in X are geodesicly ϱ-convex for some $\varrho\le 1000(-\kappa)^{\frac{1}{2}}$ $\max\delta\sqrt{-\kappa},(\delta\sqrt{-\kappa})^4)$. Furthermore, X is δ'-hyperbolic for δ' depending only on $\kappa<0$ and δ.*

Proof. The only point requiring an explanation is the δ'-hyperbolicity. But this follows from the contraction property of the normal "projections" of X to balls (compare $[\mathrm{Gro}]_{\mathrm{HG}}$).

4.B. Remark. It is not hard to show that $\delta'\le 10^5((-\kappa)^{\frac{1}{2}}+\delta)$ and that X is $\mathrm{CAT}_{\delta'}(\kappa)$ with the same bound on δ'.

5. Definition of $K_\delta(X|\sigma)$.

One could define $K_\delta(X|\sigma|\le\kappa)$ as the $\mathrm{CAT}_\delta(\kappa|\sigma)$-property but then one would loose, for example spaces of negative curvature with cusps. To take care of this, we have to require $\mathrm{CAT}_\delta(\kappa|\sigma)$ not for balls but for suitable *multi-balls* $\widetilde{B}(R)\to X$. So we say that X has *curvature* $\le\kappa$ *with error* $\le\delta$ *in the σ-multivicinity of* $x_0\in X$, and write $K_\delta(X,x_0|\sigma)\le-\kappa$ if there exists a multidomain $(\widetilde{U},\widetilde{x}_0)\to X$ with $\widetilde{x}_0\mapsto x_0$ (compare 3.A.) such that

(i) $\mathrm{dist}(\widetilde{x}_0,\partial\widetilde{U})\ge 30\sigma$, where the distance in $\widetilde{U}$ is defined via the *path* metric induced from X and where $\partial\widetilde{U}\subset\widetilde{U}$ is defined as the subset where the (locally isometric!) map $\widetilde{U}\to X$ fails to be locally homeomorphic.

(ii) Every geodesic triangle in $\widetilde{U}$ of diameter $\le 10\sigma$ and lying 10σ far from the boundary $\partial\widetilde{U}\subset\widetilde{U}$ satisfies the comparison inequality $(*_\delta)$.

(iii) $\widetilde{U}$ is (δ,σ) *regular* in the sense that every 100δ-small in diameter closed curve in U staying 10σ-from ∂U bounds a disk of diameter $\le\sigma/100$ in $\widetilde{U}$ (compare $[\mathrm{Gro}]_{\mathrm{HG}}$).

5.A. K_δ-Cartan-Hadamard theorem. *If $K_\delta(X,x|\sigma)\le-\kappa$ at all point $x\in X$, the space X is simply connected and $\sigma\ge 10^5(-\kappa)^{\frac{1}{2}}\max(\delta\sqrt{-\kappa},(\delta\sqrt{-\kappa}^{\frac{1}{4}})$, then X is δ'-hyperbolic for some $\delta'=\delta'(\kappa,\delta)<\infty$. Moreover, $\delta'\le 10^5((-\kappa)^{\frac{1}{2}}+\delta)$ and X is $\mathrm{CAT}_{\delta_\bullet}(\kappa_\bullet)$ for $\delta_\bullet\le 10^5\delta$.*

The proof is quite similar to the case of $K(X) \leq \kappa$ as well as of the the δ-hyperbolic case (see $[\mathrm{Gro}]_{\mathrm{HG}}$, $[\mathrm{Gro}]_{\mathrm{CAT}}$ and [Bri-Hae].

5.B. REMARK. One can define K_δ with suitable fillings of closed contractible curves $S = S_\ell$ in X by κ-disks (D_μ) with length $(\partial D) = \ell$ (compare $[\mathrm{Gro}]_{\mathrm{CAT}}$. Here the relevant maps : $(D, \mu) \to X$ are required to be distance decreasing up-to an additive error δ, i.e.

$$|\varphi(x) - \varphi(y)|_X \leq |x - y|_\mu + \delta,$$

where we additionally insist on the continuity of φ (which is unnecessary under the above regularity assumption). In fact, one can localize this definition by applying it only to short curves S, i.e. of length $\ell \leq \sigma$ and show that this suffices for the conclusion of the K_δ-Cartan-Hadamard theorem to hold true.

6. Evaluation of K_δ for coned spaces.

The *hyperbolic cone* $C_{\kappa,r}(U)$ over a path metric space U is defined as the topological cone $U \times [0,r]/U \times 0$ with the path metric, such that for each (short) curve $S \subset U$ the metric on $S \times [0,r]/S \times 0 \subset \mathrm{Cone}_{\kappa,r}(U)$ equals that of the segment in C' the r-disk $D(r)$ in H^2_κ over an ark $S' \subset \partial D(r)$ with $\mathrm{length}(S') = \mathrm{length}(S)$, where S' is (obviously) identified with S and C' with $S \times [0,r]/S \times 0$.

6.A. LEMMA. *If U is $\mathrm{CAT}(\kappa_0 \leq 0)$ then $C_{\kappa,r}(U)$ is $\mathrm{CAT}(\kappa)$.*

This is obvious (compare [Bri-Hae]). In fact, it suffices to assume that U is $\mathrm{CAT}(\kappa_+)$ where κ_+ is the intrinsic curvature of the r-sphere in H^3_κ.

6.A$'$. COROLLARY. *If U is $\mathrm{CAT}_\delta(\kappa_0 \leq 0)$ then $C_{\kappa,r}(U)$ is $\mathrm{CAT}_\delta(\kappa)$.*

REMARK. All of the above generalizes to *non*-path metric spaces.

6.B. Given subsets U_i in a geodesic space X we consider the cones $C_{\kappa,r}(U_i)$, attach each $C_{\kappa,r}(U_i)$ to X accross $U_i = U_i \times r \subset C_{\kappa,r}$, and let $X^\bullet \supset X$ denote so multi-coned spaced X with the natural path metric (compare $[\mathrm{Gro}]_{\mathrm{CAT}}$).

We want to evaluate $K_\delta(X^\bullet)$ for the resulting space $X^\bullet$, where we are concerned with the points away from the apices of the cones as the latter has been taken care of in the above lemmas. Start with the case where X is a tree, i.e. a $\mathrm{CAT}(-\infty)$-space and U_i, $i \in I$ are subtrees. Here an important characteristics of $\{U_i\}$ is the *(maximal) overlap* between U_i, i.e. $\ell = \ell\{U_i\} = \mathrm{supdiam}(U_i \cap U_i)$.

6.C. CONING LEMMA. *Let $\sigma \leq r/2$ and $\ell \leq 0.01 \min(1/\sqrt{-\kappa}, \sigma)$. Set*

$$\delta = \delta(\beta) = \delta_\kappa(\beta + \epsilon) + \epsilon$$

for δ_κ defined in §1 and

$$\epsilon = 100(\ell\sqrt{-\kappa})^{\frac{2}{3}} \sigma \sqrt{-\kappa}.$$

Then $X^\bullet$ is $\mathrm{CAT}_\delta(\kappa)$ within the σ ball around each point $x \in X \subset X^\bullet$. Notice that we returned to *functions* $\delta(\beta)$ and $\delta_\kappa(\beta)$ from §1).

PROOF. To grasp the idea, let first $\ell = 0$. In this case $X^\bullet$ is obviously (compare $[\mathrm{Gro}]_{\mathrm{HG}}$, $[\mathrm{Gro}]_{\mathrm{CAT}}$ and [Bri-Hae]) $\mathrm{CAT}(\kappa)$. Next we observe that $\mathrm{CAT}_\delta(\kappa)$ (including $\mathrm{CAT}_\delta(-\infty)$ corresponding to hyperbolicity) is stable under perturbations of spaces in the Hausdorff metric (with $\frac{1}{2}$-Hölder dependence of δ on the Hausdorff distance). This, and the essential locality of $\mathrm{CAT}_\delta(\kappa)$ (by Hadamard-Cartan) imply that $X^\bullet$ is $\mathrm{CAT}_\delta(\kappa)$, where $\delta = \delta(\ell) \to 0$ for $\ell \to 0$.

To get a feeling for an actial bound on δ, one may look at the simplest tree, namely $X = \mathbb{R}$ where U_i are just subsegments $\subset \mathbb{R}$. The major problems in understanding the geometry of the corresponding $X^\bullet$ consists in evaluating by how much the distance $|x-y|_{X^\bullet}$ is smaller than $|x-y|_{\mathbb{R}}$ for $x, y \in \mathbb{R}$. The small (i.e. $\ll \ell$) segments $U_i \subset \mathbb{R}$ do not have a significant effect on the shortening $|x-y|_{\mathbb{R}} \rightsquigarrow |x-y|_{X^\bullet}$ and one may concentrate on the large ($\gg \ell$) U_i's. These can be assumed to have only double intersections, where the combinatorics is transparent and $|x-y|_{X^\bullet}$ is easily computable.

For general trees X, the problem is complicated by the presence of brance points, but the number of these can be, a priori, bounded. Thus one essentially reduces the general case to that of the Y-shaped tree, where the branched point is treated as if it is covered by a large U_i.

Keeping all this in mind we start by observing, now for a general tree X, that the radical projections from the apices of the cones send every geodesic segment $[x_1^\bullet, x_2^\bullet]$ in $X^\bullet$ missing the apices to *geodesic* segments in X, i.e. these projection *biject* $[x_1^\bullet, x_2^\bullet]$ to a segment $[x_1, x_2]$ in the tree X, as an obvious argument shows. Since the CAT_δ-property concerns 5-tuples of points $\{x_i^\bullet\} \subset X^\bullet$, $i = 1, \dots, 5$, we may assume that our X is spanned by five extreme points (leaves) $x_i \in X$. Now X contains at most three interior branch points of valency (degree) at most five and to simplify the computation, we also assume $\kappa = -1$. (This can always be achieved by scaling $X^\bullet$ to $\sqrt{-\kappa} X^\bullet$). We divide U_i into two groups : those with $\operatorname{diam} U_i \geq \ell^{\frac{1}{3}}$ and the remaining short ones with $\operatorname{diam} U_i < \ell^{\frac{1}{2}}$. The intersection between two large U_i, say $U_i \cap U_j$, consists of at most $14 = 7 + 7$ disjoint subtrees of diameters $\geq \ell$, where each of these meets a vertex (i.e. an extreme or a branch point) either in U_i or in U_j and no more than five large U_i ever meet at a single point. In fact, the union of all these double intersections may consist of at most three subtrees meeting the branch points of X and having diameters $\leq 3\ell$ and several disjoint segments of diameters $\leq \ell$, where each of the segments meets exteme points of exactly two U_i.

Now, we perform two operations over $X^\bullet$: first we remove the cones over all short U_i with the exception of those containing our points $x_i^\bullet$, $i = 1, \dots, 5$. Clearly the distances between x_i' in the resulting space may be greater than they were in $X^\bullet$ but not by face. In fact, as a geodesic in $X^\bullet$ passes through a cone C_j with short base U_j, it remains within distance $\leq 5(\ell^{\frac{1}{3}})^2$ from U_j (which was *not* at all removed along with C_j by our construction) and its length is only slightly shorter at most by the factor $1 - 5\ell^{\frac{2}{3}}$. Thus the total enlargement in the distances does not exceed $\sigma(20\ell^{\frac{2}{3}} + 30(\ell^{\frac{1}{3}})^2) \leq \frac{\epsilon}{2}$. Next, after removing the above C_j we shrink to points the connected components of the pairwise intersections of the remaining U_i in X (i.e. the great ones and those containing the ends) and conicly extend this shrinking to the cones C_i. Dispite the fact that the total length of U_i can be quite large compared to σ (since the apex projections shorten *long* geodesic segments in C_i near their ends where they meet U_i, while the total length of these segments strongly increases under these projections), the shortening effect to each C_i is bounded by 2ℓ since our shrinking is conical. On the other hand, the total length of these segments does not exceed 10σ and so the enlargement is bounded by $\epsilon/2 = 50\sigma\ell/\ell^{\frac{1}{3}}$. Now the lemma follows easily with the local-to-global transition provided by the Hadamard-Cartan theorem.

Corollary. *If $\ell \geq 10^{-11}/\sqrt{-\kappa}$ and $r \geq 0.2/\sqrt{-\kappa}$, then $X^\bullet$ is $(10^3/\sqrt{-\kappa})$-hyperbolic.*

6.D. On the role of CAT_δ and K_δ. One could prove the Corollary within a purely δ-hyperbolic framework and essentially by the same argument without bothering with the notions of CAT_δ and K_δ. In fact, as far as applications to random groups are concerned, one could go without these spaces. However, the lemma carries extra geometric information, such a (better than just δ-hyperbolic) bound on the distance between "parallel" geodesics that is needed in the combinatorial group theory. Actually, one needs this mesoscopic curvature not so much for $X^\bullet$ itself, but rather for quotient spaces $X^\bullet/\Gamma$ for (rotation) isometry groups Γ acting on $X^\bullet$ where the action is free apart from the apices of the cones. If the curvature remains $\leq \kappa$ at the appices, then one can prove (compare $[\mathrm{Gro}]_{\mathrm{CAT}}$) that $X^\bullet/\Gamma$ is $\mathrm{CAT}_\delta(\kappa)$ which leads to the $\mathrm{CAT}_\delta(\kappa)$-refinement of the hyperbolicity (i.e. $\mathrm{CAT}_{\delta'}(-\infty)$ with $\delta' \gg \delta$) of small cancellation groups. Furthermore, this allows a similar sharpening of the *hyperbolic small cancellation* theory as we shall see below.

Remark. If the subtrees U_i in the tree X are homeomorphic to $\mathbb{R}$, then $X^\bullet$ can be (canonicly) Hausdorff approximated by a $\mathrm{CAT}(\kappa)$ space (see $[\mathrm{GRO}]_{\mathrm{CAT}}$).

7. Cones over hyperbolic spaces.

Let X be a δ_0-hyperbolic space and let us look first at an individual cone $C = C_{\kappa,r}(X)$, where the important case is where $|\kappa|^{\frac{1}{2}}$ is small compared to $1/\delta_0$. Since X behaves like a tree on the scale $\gg \delta_0$, we see that C has curvature $\leq \kappa$ up to a δ_0-error; more precisely we have the following

7.A. Lemma. *The cone C is $\mathrm{CAT}_\delta(\kappa)$ for*

$$\delta = \delta(\beta) = \delta_\kappa(\beta) + 10\delta_0.$$

In particular, if $100\sqrt{-\kappa} \leq \delta_0$, then C is δ_1-hyperbolic for $\delta_1 = 10/\sqrt{-\kappa}$.

Notice that this hyperbolicity less informative than the $\mathrm{CAT}_\delta(\kappa)$-property since the former (non-trivially) controls the geometry of the balls of radii $\gamma\sigma$ for σ starting from something like $100\sqrt{-\delta_0/\kappa}$ while the hypbolicity of C emerges only on the scale of $\sigma \approx 10/\sqrt{-\kappa}$.

Take ϵ-convex (see $[\mathrm{Gro}]_{\mathrm{HG}}$) subsets $U_i \subset X$ and obtain $X^\bullet$ by attracting the cones $C_{\kappa,r}(U_i)$ to X as earlier. The definition of the overlap ℓ defined via the intersections $U_i \cap U_j$ is no good anymore. For example, if we have two distinct geodesics U_1 and U_2 in H^2_κ going close to each other along segments of length $\approx \ell$ and then going far apart, we should assign the overlap ℓ to

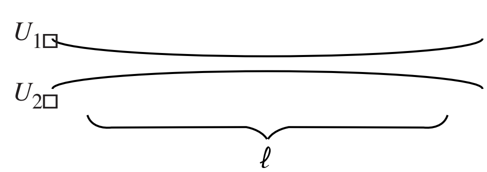

Figure 3

them rather than zero.

This can be taken care of by either considering $(2\epsilon + 10\delta)$-neighbourhoods of U_i and taking

$$\ell_1 = \sup_{i \neq j} \mathrm{Diam}((U_i + 2\epsilon + 10\delta) \cap (U_j + 2\epsilon + 10\delta))$$

or by using the supremum ℓ_2 of the diameters of the normal (i.e. nearest point(s)) projections $U_i \to U_j$ for all i and $j \neq i$ with either of these ℓ's one has a generalization of the tree coning lemma, where we stick below to $\ell = \ell_1$.

7.B. Hyperbolic Coning Lemma. *If* $1/\sqrt{-\kappa} \geq 1000\delta_0$ *and* $\ell \leq 10^{-3}/\sqrt{-\kappa}$ *then* $X^\bullet$ *is* $\mathrm{CAT}_\delta(\kappa)$ *on the* σ*-balls in* $X^\bullet$ *for*

$$\delta = \delta(\beta) = \delta_\kappa(\beta - \epsilon + 10\delta_0) + \epsilon_0 + 10^4\delta_0$$

with

$$\epsilon_\bullet = 10^{10}(\ell\sqrt{-\kappa})^{\frac{2}{3}} \min(1, \sigma\sqrt{-\kappa}),$$

where δ_κ *is from* §1. *In particular, if* $r \geq 1/\sqrt{-\kappa} \geq 10^5\delta_0$ *and* $\ell \leq 10^{-50}/\sqrt{-\kappa}$*, then* $X^\bullet$ *is* δ_1*-hyperbolic for* $\delta_1 = 10^{20}/\sqrt{-\kappa}$*.*

Proof. The idea is that the space X is tree-like on δ_0-scale and so 7.A applies with a δ_0-error. Actually, one could go to the limit for $\delta_0 \to 0$, where X converges to a tree, and thus derive the qualitative form of the present lemma from 7.A. Since it is convenient (at least notationally) to keep track of the constants, we do it differently. Take five points x_i in $X^\bullet$, join them by ten segments in $X^\bullet$ and project these to X from the apices of the cones in $X^\bullet$. (This applies only to points within σ-balls around points $x \in X \subset X^\bullet$ with $\sigma \leq \min(r/4,\ 1/\sqrt{-\kappa})$, while the balls of radii $\leq r/2$ away from $X \subset X^\bullet$ are already taken care of. Then the properties of larger balls follow with our generous constants). Segments project to segments and the union of the ten projected segments can be δ_0'-approximated by a tree Y in X built of seven segments with $\delta_0' \leq 1000\delta$ (see $[\mathrm{Gro}]_{\mathrm{HG}}$). Now we apply the previous proof to this subtree with minor adjustments related to the position of Y relative to U_i. A little difficulty comes from the fact, that U_i are *not* geodesicly convex in the strict sense and so some edges of Y with vertices in some U_i may go outside U_i. The easiest way to help it is to stabilize U_i by taking their $4\delta_0$-neighbourhoods. This only slightly changes the geometry of $X^\bullet$ and brings a minor (by far less than we presented) modification of the constants. (Recall that ℓ refers to the geometric overlaps of neighbourhoods of U_i anyway and with this convention the constants hardly change at all).

7.C. Applications. The main case of interest is where X is acted upon an isometry group Γ generated by Γ_i where each Γ_i keeps U_i invariant and has "large" displacements outside U_i. Then Γ naturally acts on $X^\bullet$ with the apices of the cones kept fixed by Γ_i and, under suitable (and easily verifiable) conditions each quotient space $X^\bullet/\Gamma_i$ is $\mathrm{CAT}_\delta(\kappa)$. Then by a straightforward generalization of the argument in $[\mathrm{Gro}]_{\mathrm{CAT}}$ one obtains the $\mathrm{CAT}_{\delta'}(\kappa)$ property for $X^\bullet/\Gamma$ with something like $\delta' = 4\delta$ (or slightly worse but yet controlled on large scale σ). This shows, that the hyperbolic small cancellation groups are not only hyperbolic, but behave as if they were acting on spaces of negative curvature $\kappa < 0$ where "as if" corresponds to δ in $\mathrm{CAT}_\delta(\kappa)$ that can be made arbitrarily small (for a fixed $\kappa < 0$) in many cases of interest.

8. Coning $\mathrm{CAT}^+_\delta(\kappa, r)$-spaces.

When we compare a space X_+ with the sphere $X'_+ = S^2_{\kappa_+}$ of radius $1/\sqrt{x_+}$ we must be aware of the fact that some configurations of points in X_+ have no counterparts in X'_+ as the distances between these points may exceed $\pi/\sqrt{\kappa_+} = \mathrm{diam}\, X'_+$; Then, defining $\mathrm{CAT}_\delta(\kappa_+)$-property, we require the relevant inequality between distances only from those 5-tuples $\{x_1, \dots, y_3\} \subset X_+$ which are suitably realizable in X'. As we are eventually concerned with $C_{r,\kappa}$-cones over these spaces we just *define* the relevant property via these cones as follows.

A space X_+ is *called* $\mathrm{CAT}^+_\delta(\kappa, r)$ for some $\kappa < 0$ and $\delta \geq 0$ (which may be understood either as a function $\delta(\beta)$ or just a single number $\delta = \delta(0)$) if the cone $C_{\kappa,r}(X_+)$ is $\mathrm{CAT}_\delta(\kappa)$.

For example, $\mathrm{CAT}^+_{\delta=0}(0,1)$ is obviously equivalent to $\mathrm{CAT}(1)$. It is easy to show that

$$\mathrm{CAT}^+_\delta(\kappa, r) \Rightarrow \mathrm{CAT}^+_\delta(\kappa', r')$$

for all $\kappa' \leq \kappa$ and $r' \leq r$.

What we shall actually need is the following obvious

8.A. Lemma. *Let X_+ be a geodesic metric space that is δ_0-hyperbolic in every ball of radius R in X_+. Then X_+ is $\mathrm{CAT}^+_{\delta_1}(\kappa, r)$, for $\delta_1 = \delta_1(\beta) = \delta_\kappa(\beta) + 10\delta_0$,*

$$r \leq 0.1(\log \sqrt{-\kappa R}/\sqrt{-\kappa}.$$

Notice, that this inequality needs to be confronted with the lower bound $r \geq 1/\sqrt{-\kappa}$. Also observe, that if $X_+ = \widetilde{X}_+/\Gamma_+$ for a (now globally) δ_0-hyperbolic space $\widetilde{X}_+$ and a free isometry group Γ acting on $\widetilde{X}_+$ with $|x - \gamma(x)|_{\widetilde{X}_+} \geq 3R \geq 100\delta_0$ for all $x \in \widetilde{X}_+$ and all non-identity elements $\gamma \in \Gamma_+$, then X_+ is δ_0-hyperbolic on the R-balls in X_+. This is exactly where the above Lemma becomes useful.

9. Final Remarks and open questions.

(a) It is unclear if the $\mathrm{CAT}_\delta(\kappa)$-spaces can be Hausdorff δ'-approximated by $\mathrm{CAT}(\kappa)$-spaces with $\delta' = \delta'(\delta) \to 0$ for $\delta \to 0$. Probably, to have such approximation, one should strengthen the definition of $\mathrm{CAT}_\delta(\kappa)$ by incorporating more (δ-perturbed) inequalities, such as all $\mathrm{Cycl}_k(\kappa)$, for example.

(b) There is an obvious δ-perturbed version of the inequality defining Alexandrov's spaces with $K > 0$, and it may be interesting to look at the corresponding spaces with $K > \kappa$ up-to an error δ.

(c) The above mesoscopic extension (δ-perturbation) of locally defined classes of spaces goes along with the Hausdorff distance between metric spaces. If we look from the angle of *metric measure* spaces (see [G-L-P]) we come up with other classes where the relevant (e.g. 4-point) inequalities are satisfied up-to δ-error understood in some *measure theoretic* (integral) sense. What happens, for example, to Ricci > 0 in this setting ?

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