PAUL LEVY'S ISOPERIMETRIC INEQUALITY

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Consider the standard $(n+1)$-dimensional sphere S^{n+1} and a domain $V_o \subset S^{n+1}$ with smooth boundary. The classical isoperimetric inequality provides the following lower bound for the n-dimensional volume $\mathrm{Vol}(\partial V_o)$ of the boundary ∂V_o.

Denote by α the ratio $\dfrac{\mathrm{Vol}(V_o)}{\mathrm{Vol}(S^{n+1})}$, where "Vol" means the $(n+1)$-dimensional volume, and take a ball $B_\alpha \subset S^{n+1}$ with $\mathrm{Vol}(B_\alpha) = \alpha \, \mathrm{Vol}(S^{n+1})$. Denote by $s(\alpha)$ the n-dimensional volume of the boundary sphere ∂B_α and denote by $\mathrm{Is}_{n+1}(\alpha)$ the ratio $\dfrac{s(\alpha)}{\mathrm{Vol}(S^{n+1})}$.

Classical Isoperimetric Inequality.

$$\frac{\mathrm{Vol}(\partial V_o)}{\mathrm{Vol}(S^{n+1})} \geq \mathrm{Is}_{n+1}(\alpha) \ , \quad \alpha = \frac{\mathrm{Vol}(V_o)}{\mathrm{Vol}(S^{n+1})} .$$

Observe, that by applying this inequality to domains with diameters coverging to zero we come to the isoperimetric inequality in the Euclidean space $\mathbb{R}^{n+1}$.

Back in 1919 Paul Levy extended the classical isoperimetric inequality to the convex hypersurfaces in $\mathbb{R}^{n+2}$ and he found a striking infinite dimensional application of his result. (See Chapter IV of the third part of his book [L]).

We show in this note that Levy's method works for all Riemannian manifolds. As an application we obtain some estimates for the eigenvalues of the

Laplacian on a Riemannian manifold.

1. Isoperimetric inequality. For a compact Riemannian manifold V we denote by $\mathrm{Vol}(V)$ its total volume and by Ric the Ricci tensor. We set

$$R(V) = \inf_{\tau} \mathrm{Ric}(\tau,\tau) ,$$

where τ runs over all unit tangent vectors of V.

Let V be a closed $(n+1)$-dimensional manifold and let $V_o \subset V$ be a domain with smooth boundary. If $R(V) \geq n = R(S^{n+1})$ then

$$\frac{\mathrm{Vol}(\partial V_o)}{\mathrm{Vol}(V)} \geq Is_{n+1}(\alpha) , \quad \alpha = \frac{\mathrm{Vol}(V_o)}{\mathrm{Vol}(V)} , \tag{*}$$

where $Is_{n+1}(\alpha)$ is the same function as in the classical isoperimetric inequality.

Remarks and corollaries. When $V = S^{n+1}$ the inequality (*) becomes the classical isoperimetric inequality.

Our proof (see §4) shows that equality in (*) holds only for the standard pair $(V,V_o) = (S^{n+1}, B_\alpha)$.

If one applies (*) to the ε-neighbourhoods U_ε of V_o and integrates over ε one gets

$$\frac{\mathrm{Vol}(U_\varepsilon)}{\mathrm{Vol}(V_o)} \geq \frac{\mathrm{Vol}(A_\varepsilon)}{\mathrm{Vol}(B_\alpha)} ,$$

where A_ε denotes the ε-neighbourhood of B_α in S^{n+1}.

2. The first eigenvalue. Denote by λ_1 the first eigenvalue of the Laplacian on V and by D the diameter of V.

If $R(V) \geq -n$, $0 < n = \dim V-1$, then

$$\lambda_1 \geq \exp(-2nD) \qquad (\substack{*\\*})$$

This inequality (in a sharper form) follows from the inequality (Is') of §4 and the following theorem of Cheeger (see [Ch]) :

$$\lambda_1 \geq \frac{1}{4} \left(\inf_{V_o} \frac{\mathrm{Vol}(\partial V_o)}{\mathrm{Vol}(V_o)}\right)^2$$

where V_o runs over all domains in V with smooth boundary such that $\frac{\mathrm{Vol}(V_o)}{\mathrm{Vol}(V)} \leq \frac{1}{2}$.

Some other relations of this type can be found in [M] and [B-M] .

The inequality ($\substack{*\\*}$) was conjectured by Cheeger and it was proven in some cases by Yau [Y] and Li [Li] .

I was recently informed that Yau found an independent proof (†) of ($\substack{*\\*}$)

We refer to [B-G-M] and [O] for the further information on the spectrum of the Laplacian. In the §5 we generalize this inequality to all eigenvalues λ_i.

3. Levy-Heintze-Karcher comparison theorem. Let $H \subset V$ be a smooth hypersurface with a normal unit vector field ν . Denote by $\exp : H \times \mathbb{R}_+ \to V$ the normal exponential map in the direction ν and denote by $J = J(h,t)$, $h \in H$, $t \in \mathbb{R}_+$, the Jacobian of this map.

Consider also a model pair $(\overline{V},\overline{H})$, where $\overline{V}$ has constant sectional curvature δ and $\overline{H}$ is a totally umbilic hypersurface of mean curvature η (relative to a prefered direction $\overline{\nu}$) . Denote by $\overline{J}(t) = \overline{J}_{\eta\delta}(t)$ the Jacobian of the corresponding exponential map.

Take a point $h \in H$ such that at this point the mean curvature of H (in the direction ν) is not less than η . Take a point $t \in \mathbb{R}_+$, such that the

(†) See [Li-Y], where the treatment is, in many respects, more complete and precise than ours.

segment $h \times [0,t]$ is sent by the exponential map to a distance minimizing geodesic segment, i.e. $\mathrm{dist}(H, \exp(h,t)) = t$.

If $R(V) \geq n\delta = R(\overline{V})$, $n = \dim H = \dim \overline{H}$, then $|J(h,t)| \leq \overline{J}(t)$.

The proof can be found in [H-K] . See also [Bu] for some related results.

The explicit formula for $\overline{J}(t)$ is as follows. Set $s_\delta(t) = \delta^{-\frac{1}{2}}\sin(\delta^{\frac{1}{2}}t)$, for $\delta > 0$, $s_o(\delta) = t$, and $s_\delta(t) = |\delta|^{-\frac{1}{2}}\sinh(|\delta|^{\frac{1}{2}}t)$ when $\delta < 0$. Using these notations we have (see [H-K])

$$\overline{J}_{\eta\delta}(t) = \left(\frac{ds_\delta(t)}{dt} - \eta s_\delta(t)\right)^n$$

It is convenient to use another function $\overline{J}^+(t)$ defined as followes : $\overline{J}^+(t) = \overline{J}(t)$ when t is less than the first zero t_o of $\overline{J}(t)$ and $\overline{J}^+(t) = 0$ for $t \geq t_o$.

Corollary. (See [H-K]) . Let a domain $V_o \subset V$ be bounded by a smooth hypersurface $H = \partial(V_o)$ with the mean curvature everywhere not less than η (relative to the inward normal). Then

$$\mathrm{Vol}(V_o) \leq \mathrm{Vol}(\partial V_o) \int_o^d \overline{J}^+_{\eta\delta}(t)\,dt ,$$

where $d = \sup_{v \in V_o} \mathrm{dist}(v, \partial V_o)$. (Observe that $d \leq D = \mathrm{Diam}(V)$) .

4. Main inequalities. Let V be a closed $(n+1)$-dimensional manifold with $R(V) = n\delta$, $-\infty < \delta < \infty$, and let $H \subset V$ be an arbitrary smooth hypersurface which divides V into two domains V_o and V_1 with common boundary H .

There exist positive numbers d_o and d_1 with $d_o + d_1 \leq D = \mathrm{Diam}\, V$ and an $\eta \in (-\infty,\infty)$ such that

$$\mathrm{Vol}(V_o) \leq \mathrm{Vol}(H) \int_o^{d_o} \overline{J}^{\,+}_{\eta\delta}(t)\,dt \quad ,$$
$$\mathrm{Vol}(V_1) \leq \mathrm{Vol}(H) \int_o^{d_1} \overline{J}^{\,+}_{-\eta\delta}(t)\,dt \ . \qquad \text{(Is)}$$

Corollaries. Observe that for $\eta \geq 0$ we have $\overline{J}^{\,+}_{\eta\delta}(t) \leq \overline{J}^{\,+}_{o\delta}(t)$ and in view of the explicit formula for $\overline{J}$ the inequality (Is) implies :

If $\mathrm{Vol}(V_o) \leq \mathrm{Vol}(V_1)$ then

$$\mathrm{Vol}(V_o) < \mathrm{Vol}(\partial V_o) \int_0^D \left(\cosh(|\delta|^{-\frac{1}{2}}t)\right)^n dt \qquad \text{(Is')}$$

Notice that this inequality is interesting only when $R(V) \leq 0$.

The following inequality is most interesting when $R(V) > 0$.

Set $\alpha = \dfrac{\mathrm{Vol}\ V_o}{\mathrm{Vol}\ V}$, $\beta = 1-\alpha$ and denote

$$\mu(\eta) = \min(\alpha^{-1} \int_0^D \overline{J}^{\,+}_{\eta\delta}(t)\,dt \ , \ \ \beta^{-1} \int_0^D \overline{J}^{\,+}_{-\eta\delta}(t)\,dt \ .$$

By combining the inequalities (Is) we immediately get

$$\frac{\mathrm{Vol}(H)}{\mathrm{Vol}(V)} \geq \inf_{-\infty<\eta<\infty} (\mu(\eta))^{-1} \qquad \text{(Is")}$$

When $\delta = 1$ and $D = \infty$ a straightforward calculation shows that

$$\inf_{-\infty<\eta<\infty} (\mu(\eta))^{-1} = Is_{n+1}(\alpha) = Is_{n+1}(\beta) \quad ,$$

and hence, (Is") implies the inequality (*) from §1.

Proof of (Is). Fix an α , $0 < \alpha < 1$, and consider all hypersurfaces in V which divide V into two parts V_o, V_1 with $\mathrm{Vol}(V_o) = \alpha$, $\mathrm{Vol}\ V_1 = 1-\alpha$. Among these hypersurfaces there is one with minimal n-dimensional volume (see [A]) . Suppose for a minute that this minimal hypersurface is non-singular. In such a case it has constant mean curvature. We denote by η the value of this curvature relative to the normal looking inward V_o . The curvature in the direction looking inward V_i is $-\eta$ and applying the Corollary of §3 to V_o and V_1 we get (Is).

About the singularity. When $\dim V \leq 7$ the minimal hypersurface H dividing V in the given volume proportion is known to be non-singular (see [A]) and the proof of (Is) is completed. In the general case H may have a singular locus but it does not affect our proof due to the following

Lemma. Take a point $v \in V$ and a geodesic segment $\gamma \in V$ with one end v and with the second end $h \in H$, such that $\mathrm{length}(\gamma) = \mathrm{dist}(v,H)$. Then the point $h \in H$ is non-singular.

Proof. Take the sphere (relative to the Riemannian metric in V) centered at the center of γ and of radius $\frac{1}{2}\,\mathrm{dist}(v,H)$. This sphere meets H only at h and it is smooth near this point. It follows that the tangent cone to H at h is contained in a half-space and hence, (see [A]) the point h is non-singular.

5. Estimates for λ_i, $i > 1$. Our method of estimating λ_1 can be extended to the other eigenvalues of the Laplacian on V but we shall establish these estimates by a more elementary method because for the higher λ_i the more refined Levy's method does not give much better constants.

Notations. Let V be a compact Riemannian manifold of dimension $n+1 \geq 2$. Denote by $N(\varepsilon)$, $\varepsilon > 0$, the minimal number N such that V can be covered by N balls of radius ε. Notice that

$$N(\varepsilon) \geq \frac{D}{2}$$

$$N(D) = 1, \quad D = \mathrm{Diam}(V).$$

Denote by $b_{n+1}(\varepsilon)$ the volume of the ε-ball in the $(n+1)$-dimensional hyperbolic space of curvature -1 . The Rauch comparison theorem (see [Ch-E]) implies that for a closed V with $R(V) \geq -n$ one has

$$N(\varepsilon) \geq (b_{n+1}(\varepsilon))^{-1} \operatorname{Vol}(V)$$

On the other hand we shall see below that

$$N(\varepsilon) \leq b_{n+1}(D)\,(b_{n+1}(\tfrac{\varepsilon}{2}))^{-1} ,$$

and, more generally, for $\varepsilon_1 \geq \varepsilon$ one has

$$N(\varepsilon_1) \leq N(\varepsilon) \leq b_{n+1}(2\varepsilon_1 + \tfrac{\varepsilon}{2})\,(b_{n+1}(\tfrac{\varepsilon}{2}))^{-1} N(\varepsilon_1)$$

<u>Theorem</u>. <u>There are two positive constants</u> C_1 <u>and</u> C_2 <u>depending on</u> n , <u>such that for any closed Riemannian manifold</u> V <u>of dimension</u> $n+1 \geq 2$ <u>with</u> $R(V) \geq -n$ <u>and for each positive</u> $\varepsilon \leq D = \operatorname{Diam}(V)$ <u>one has the following estimates for the eigenvalue</u> λ_i <u>with</u> $i = N(\varepsilon)$

$$\varepsilon^{-2} C_1^{1+\varepsilon} \geq \lambda_i \geq \varepsilon^{-2} C_2^{1+\varepsilon}$$

<u>Corollary</u>. Applying these inequalities to $\varepsilon = 1$ and using the estimates for $N(\varepsilon)$ from above we get

(a) <u>if</u> $i \leq \frac{D}{2}$ then $\lambda_i \leq \text{const}_n = C_1^2$,

(a') <u>if</u> $i \leq \dfrac{\operatorname{Vol}(V)}{b_{n+1}(1)}$, then $\lambda_i \leq \text{const}_n = C_1^2$,

(b) <u>if</u> $i \geq b_{n+1}(D)\,(b_{n+1}(\tfrac{1}{2}))^{-1}$ then $\lambda_i \geq \text{const}_n = C_2^2$.

<u>Remark</u>. The inequality $\lambda_i \leq \varepsilon^{-2} C_1^{1+\varepsilon}$ and its corollaries (a) and (a') are not new. A more precise result is due to Chen (see [C]) , but we give the

proof here for the completeness sake.

Proof. We start with several simple Lemmas.

(A) (Rauch theorem). Take a point $v \in V$ and two balls B, B_1 centered at v of radii r and $r_1 > r$. Let $A \subset B_1$ be a star-connex set, i.e. a Borel set such that for each point $a \in A$ each geodesic segment which joins v with a and has length = dist(v,a) is contained in A. If $\mathrm{Ric}(V) \geq -n$ then

$$\frac{\mathrm{Vol}(A \cap B_1)}{\mathrm{Vol}(A \cap B)} \leq \frac{b_{n+1}(r_1)}{b_{n+1}(r)} \quad ,$$

where "Vol" denotes the (n+1)-dimensional mesure in V. In particular, one has

$$\frac{\mathrm{Vol}(B_1)}{\mathrm{Vol}(B)} \leq \frac{b_{n+1}(r_1)}{b_{n+1}(r)}$$

When $B' \subset V$ is a radius r ball centered at $v' \in V$ with $\mathrm{dist}(v,v') = d$ we get

$$\frac{\mathrm{Vol}(B_1)}{\mathrm{Vol}(B')} \leq \frac{b_{n+1}(r_1+d)}{b_{n+1}(r)} \quad .$$

It follows that each ε_1-ball in V can be covered by $p \leq b_{n+1}(2\varepsilon_1 + \frac{\varepsilon}{2})(b_{n+1}(\frac{\varepsilon}{2}))^{-1}$ balls of radius ε. It implies the relation between $N(\varepsilon)$ and $N(\varepsilon_1)$ which was stated above.

(A') Denote by $a_n(\varepsilon)$ the n-dimensional volume of the ε-sphere in the (n+1)-dimensional hyperbolic space of curvature -1. The n-dimensional volume of the set $A \cap \partial B$ and the (n+1)-dimensional volume of the part A' of A contained in the complement $B_1 \setminus B$ (i.e. $A' = A \cap (B_1 \setminus B)$) are related as follows

$$\frac{\mathrm{Vol}(A')}{\mathrm{Vol}(A \cap \partial B)} \leq \frac{b_{n+1}(r_1)-b_{n+1}(r)}{a_n(r)}$$

Proof. Both theorems (A) and (A') follow from the local Rauch theorem (see [Ch-E]) . Observe, that they can be also easily reduced to the Levy-Heintze-Karcher comparison theorem.

(B) Let (V,v) be as above and consider an arbitrary Borel set $W \subset V$. Let $H \subset V$ be a smooth hypersurface (possibly with a boundary) such that for each $w \in W$ each distance minimizing geodesic segment γ between w and v ("distance minimizing" means length(γ) = dist(v,w)) meets H at a point $h \in H \cap \gamma$. Let $d_1 = d_1(\gamma)$ denote the distance dist(v,h) and d_2 = dist(h,w) = length(γ) - d_1 . (When γ meets H at several points we take for h the nearest point to w). The (n+1)-dimensional volume of W and the n-dimensional volume of H are related by the following inequality.

$$\frac{\mathrm{Vol}(W)}{\mathrm{Vol}(H)} \leq \sup_{\gamma} \frac{b_{n+1}(d_1(\gamma)+d_2(\gamma))-b_{n+1}(d_1(\gamma))}{a_n(d_1(\gamma))}$$

Proof. This immediatly follows from (A'). Observe that (A') implies a more precise inequality. Denote by $H' \subset H$ the set of all intersection points $\gamma \cap H$ and for an $h \in H'$ denoted by $\alpha(h)$ the angle between H and the corresponding γ . Set $d_1(h)$ = dist(h,v) , $d_2(h)$ = dist(h,w) . Now one has

$$\mathrm{Vol}(W) \leq \int_{H'} \left(\frac{b_{n+1}(d_1(h)+d_2(h))-b_{n+1}(d_1(h))}{a_n(d_1(h))}\right) \sin(\alpha(h))d\mu$$

where $d\mu$ denotes the n-dimensional mesure in H' induced from H .

(C) Let V be an arbitrary closed Riemannian manifold and let H be a closed hypersurface dividing V into two parts V_0 and V_1 . Let

$W_o \subset V_o$, $W_1 \subset V_1$ be two Borel sets of positive measure. Then there is point w_o in one of the sets W_o, W_1 , say in W_o , and a subset W in another set, say in W_1 , such that each distance minimizing segment joining w_o with an arbitrary point $w_1 \in W$ meets H at a point h with $\mathrm{dist}(w_o,h) \geq \mathrm{dist}(w_1,h)$, and such that

$$\mathrm{Vol}(W) \geq \frac{1}{2} \mathrm{Vol}(W_1)$$

Proof. Consider the product $W_o \times W_1 \subset V \times V$ and the set $X \subset W_o \times W_1$ of the points (w_o,w_1) which can be joined (in V) by only one shortest segment γ . This set has full mesure in $W_o \times W_1$. Denote by $Y \subset X$ the set of the pairs (w_o,w_1) such that for an intersection point h of H with the corresponding γ we have $\mathrm{dist}(w_o,h) \geq \mathrm{dist}(w_1,h)$. Denote by $X' \subset W_1 \times W_o$ the set corresponding to X under the natural involution in $V \times V$. Denote by $\tilde{Y} \subset X'$ the set of pairs (w_1,w_o) such that for the corresponding h we have $\mathrm{dist}(w_1,h) \geq \mathrm{dist}(w_o,h)$. One of the sets Y , $\tilde{Y}$ must contain at least one half of the total measure of X (or X') and we can assume that this is the case with Y . By the Fubini Theorem there exists a point $w_o \in W_o$ such that the instersection $w_o \times W_1 \cap Y$ contains at least one half of the total mesure of W_1 . Q.E.D.

We are ready now to prove a version of the inequality (Is') of §4.

We apply (C) to the manifolds V_o,V_1 themselves and we find a point $w_o \subset V_o$ and a set $W \subset V_1$ as in (C). By using (B) we get

$$\mathrm{Vol}(V_1) \leq 2\ \mathrm{Vol}(W) \leq 2\ \mathrm{Vol}(H)\ \frac{b_{n+1}(D)-b_{n+1}(\frac{D}{2})}{a_n(\frac{D}{2})}$$

Of course, if W happens to be in V_o we get such estimate for $\mathrm{Vol}(V_o)$, but, in any case, we have an estimate for the smallest of the manifolds V_o and V_1 . According to the Cheeger theorem (see §2) this is sufficient for an estimate of λ_1 from below. In order to get an esimate for the rest of λ_i we need one more lemma.

(D) Let V be an arbitrary compact Riemannian manifold and let $B_1, B_2, \dots, B_i$ be arbitrary Borel sets in V of positive measure. Let $f_1, f_2, \dots, f_i$ be linearly independent continuous functions orthogonal to the constant function. Then there are some constants $c_o, c_1, \dots, c_i$ such that the function $f = c_o + \sum_1^i c_j f_j$ is not constant and it has the following property :

For each $j = 1,\dots,i$ both intersections $B_j \cap f^{-1}(-\infty,0]$ and $B_j \cap f^{-1}[0,\infty)$ have volume not less than $\frac{1}{2}\,\mathrm{Vol}(B_j)$.

Remark. When f_j are nice smooth functions the level $f^{-1}(0)$ has measure zero and we get the equalities :

$$\mathrm{Vol}(B_j \cap f^{-1}(-\infty,0]) = \mathrm{Vol}(B_j \cap f^{-1}[0,\infty)) = \frac{1}{2}\,\mathrm{Vol}\ \mathrm{Vol}(B_j) \ .$$

Proof. The functions f_j map V into $\mathbb{R}^i$. By the Ulam-Borsuk theorem there is a hyperplane which divides each B_j into the pieces of measure $\geq \frac{1}{2}\,\mathrm{Vol}(B_j)$. This hyperplane is defined by a combination $c_o + \sum_1^i c_j f_j$. Q.E.D.

Proof of the inequality $\lambda_i \geq \varepsilon^{-2}\, C_2^{1+\varepsilon}$. Cover V by $i = N(\varepsilon)$ balls of radius ε . Take the eigenfunctions $f_1,\dots,f_i$ and form $f = c_o + \sum_{j=1}^i c_j f_j$ as above. For each ball B_j take the concentric ball $\tilde{B}_j$ of radius 2ε and set $g = f^2$. Then for each $j = 1,2,\dots i$, we have :

$$\int_{B_j} g\,dv \leq c(n,\varepsilon) \int_{\tilde{B}_j} \|\mathrm{grad}\ g\|\,dv \quad , \tag{i}$$

where $c(n,\varepsilon) = 2\,\dfrac{b_{n+1}(2\varepsilon)-b_{n+1}(\varepsilon)}{a_n(\varepsilon)}$.

Proof. Set $B_j(t) = B_j \cap g^{-1}[t,\infty)$ and $\tilde{B}_j(t) = \tilde{B}_j \cap g^{-1}[t,\infty)$. One obviously has (see [B-G-M] , for example)

$$\int_{B_j} g\,dv = \int_o^\infty \mathrm{Vol}(B_j(t))\,dt \ ,$$

$$\int_{\widetilde{B}_j} \|\mathrm{grad}\ g\|\,dv = \int_o^\infty \mathrm{Vol}(\partial\widetilde{B}_j(t))\,dt \ ,$$

where in the first case "Vol" denotes the (n+1)-dimensional volume in V and in the second formula we use the n-dimensional volume of the hypersurface $\partial\widetilde{B}_j(t)$. In order to prove (i) we must only show, that for each non-critical value $t \in (0,\infty)$ one has

$$\mathrm{Vol}(B_j(t)) \leq c(n,\varepsilon)\ \mathrm{Vol}(\ \widetilde{B}_j(t)) \ .$$

Denote by $B^+ = B_j^+(t) \subset B_j(t)$ the set where $f \geq 0$ and denote by $B^- \subset B_j(t)$ the set where $f \leq 0$. When $t > 0$ these sets are disjoint and their volumes do not exceed $\frac{1}{2}\mathrm{Vol}(B_j(t))$.

The hypersurface $\partial\widetilde{B}_j(t)$ is also divided according to the sign of f into two parts which we denote by H^+ and H^- correspondingly.

Let us show that

$$\mathrm{Vol}(B^+) \leq c(n,\varepsilon)\mathrm{Vol}(H^+) \ .$$

Set $W_o = B^+$ and $W_1 = B_j(t) \setminus B^+$. For any two point $w_o \in W_o$, $w_1 \in W_1$ the shortest geodesic segment between these points has length at most 2ε and it meets H^+ at some point.

By applying (C) to W_o, W_1 and to the hypersurface $f^{-1}(\sqrt{t}) \supset H^+$ and by applying (B) to H^+ we get our inequality.

The same argument shows that

$$\mathrm{Vol}(B^-) \leq c(n,\varepsilon)\ \mathrm{Vol}(H^-) \ ,$$

and by adding the two inequalities we obtain our estimate for $\mathrm{Vol}(B_j(t))$, and hence, for $\int_{B_j} g dv$.

Denote by M the maximal multiplicity of the covering $\{\tilde{B}_j\}_{j=1,2,\dots,i}$. This number M satisfies

$$M \leq M_n(\varepsilon) = b_{n+1}(\frac{13}{2}\varepsilon)\,(b_{n+1}(\frac{\varepsilon}{2}))^{-1} .$$

Really, take a point $v \in V$ which is contained in M balls $\tilde{B}_j$. The ball $B_{2\varepsilon}$ of radius 2ε around v contains M centers of the balls B_j . But the ball $B_{3\varepsilon}$ around v can be covered by $p \leq M_n(\varepsilon)$ balls of radius ε and the inequality $M > p$ would contradict to the minimality of the covering $\{B_j\}$. (The definition of $N(\varepsilon)$ implies that V cannot be covered by $N(\varepsilon)-1$ balls of radius ε).

Now we have

$$\int_V g dv \leq \sum_{j=1}^{i} \int_{B_j} g dv \leq$$

$$\leq c(n,\varepsilon) \sum_{j=1}^{i} \int_{\tilde{B}_j} \|\mathrm{grad}\, g\| dv \leq$$

$$\leq M_n(\varepsilon)\, c(n,\varepsilon) \int_V \|\mathrm{grad}\, g\| dv .$$

We recall now that $g = f^2$, and hence,

$$\int_V \|\mathrm{grad}\, g\| dv \leq 2\left(\left(\int \|\mathrm{grad}\, f\|^2 dv\right)\left(\int f^2 dv\right)\right)^{\frac{1}{2}}$$

So we have

$$\int_V f^2 dv \leq E \int_V \|\mathrm{grad}\, f\|^2 dv ,$$

where $E = 4(M_n(\varepsilon)\, c(n,\varepsilon))^2$.

The function f was constructed as a combination $f = c_o + \sum_{j=1}^{i} c_j f_j$, where one of c_j, $j > 0$, is different from zero and where the eigenfunctions f_i can be assumed orthonormal. So we have

$$\int_V f^2 dv = \sum_{o}^{i} c_j^2 \quad ,$$

$$\int_V \|\mathrm{grad}\, f\|^2 \, dv = \sum_{j=1}^{i} \lambda_j c_j^2 \quad ,$$

and so

$$\sum_{j=o}^{i} c_j^2 \leq E \sum_{j=1}^{i} \lambda_j c_j^2 \quad .$$

Because one of c_j, $j > 0$, is non-zero one of the numbers λ_j must be at least E^{-1}, and hence, the largest eigenvalue λ_i satisfies

$$\lambda_i \geq E^{-1} \quad .$$

We recall that $E = 4(M_n(\varepsilon)\ c(n,\varepsilon))^2$, where

$$M_n(\varepsilon) = b_{n+1}(\tfrac{13}{2}\varepsilon)\ (b_{n+1}(\tfrac{\varepsilon}{2}))^{-1} \quad ,$$

$$c(n,\varepsilon) = 2\,\frac{b_{n+1}(2\varepsilon) - b_{n+1}(\varepsilon)}{a_n(\varepsilon)}$$

and $b_{n+1}(\varepsilon)$ denotes the volume of the $(n+1)$-dimensional hyperbolic ball and $a_n(\varepsilon)$ denotes the volume of its boundary. So, for a sufficiently small $C_2 = C_2(n)$ we have

$$\lambda_i \geq \varepsilon^{-2}\ C_2^{1+\varepsilon} \quad .$$

<u>Proof of the inequality</u> $\lambda_i \leq \varepsilon^{-2}\ C_1^{1+\varepsilon}$. Let V be any manifold. In order to show that $\lambda_i \leq \lambda$ it is sufficient, according to the variational principle,

to find $2i$ non-zero functions $\varphi_1,\ldots,\varphi_{2i}$ on V with the following properties

(a) The supports of φ_i are pairwise disjoint,

(b) $\int_V \varphi_j^2 \, dv \geq \lambda \int \|\operatorname{grad} \varphi_j\|^2$, $j = 1,\ldots,2i$.

When $i = N(\varepsilon)$ one can always find $2i$ disjoint balls B_j in V of radius $\frac{\varepsilon}{5}$. With each ball B_j with center $v_j \in B_j$ we associate a function φ_i defined as follows

$$\varphi_j = \varphi_j(v) = \begin{cases} \operatorname{dist}(v_j,v) - \frac{\varepsilon}{5} & , \quad v \in B_j , \\ 0 , & v \notin B_j . \end{cases}$$

One obviously has

$$\int_V \|\operatorname{grad} \varphi_j\|^2 dv = \operatorname{Vol}(B_j) .$$

Denote by $\tilde{B}_j \subset B_j$ the concentric ball of radius $\frac{\varepsilon}{10}$. One has

$$\int_V \varphi_j^2 \, dv \geq \frac{\varepsilon^2}{100} \operatorname{Vol} \tilde{B}_j .$$

When $R(V) \geq -n$ we know that $\dfrac{\operatorname{Vol} B_j}{\operatorname{Vol} \tilde{B}_j} \leq \dfrac{b_{n+1}(\frac{\varepsilon}{5})}{b_{n+1}(\frac{\varepsilon}{10})}$ and so we get

$$\lambda_i \leq 100 \, \varepsilon^{-2} \frac{b_{n+1}(\frac{\varepsilon}{5})}{b_{n+1}(\frac{\varepsilon}{10})} \leq \varepsilon^{-2} C_1^{1+\varepsilon} .$$

Notice, that the argument we used above is close to the considerations of Cheeger (see [Ch]) and Yau (see [Y]).

Final remarks. Our lower estimate for λ_i can also be stated as follows.

<u>There exists a constant</u> $C = C_n$ <u>such that for any closed</u> $(n+1)$-<u>dimensional manifold</u> V <u>with</u> $R(V) \geq -n\delta$, $\delta \geq 0$, <u>and</u> $\operatorname{Diam} V \leq D$ <u>the</u> i-th <u>eigenvalue</u> λ_i <u>satisfies</u>

$$\lambda_i \geq D^{-2}\, C^{1+D\sqrt{\delta}}\, i^{\frac{2}{n+1}}$$

On the other hand the asymptotics of λ_i when $i \to \infty$, depends only on the volume $\operatorname{Vol}(V)$. It would be interesting to find a theorem interpolating between these two facts. The following remark indicates one possibility.

Let V be as above and suppose that the fundamental group of V is infinite and has no torsion. Denote by ℓ the length of the shortest non-contractable loop. Then one has

$$\underline{\text{If}}\quad i \geq \frac{D^n}{\ell^n}\exp(2nD+2n)\ ,\ \underline{\text{then}}\quad \lambda_i \geq (D^n \ell)^{-\frac{2}{n+1}}\, C_n^{1+D\sqrt{\delta}}\, i^{\frac{2}{n+1}}$$

<u>Proof</u>. We must estimate from above the number $N(\varepsilon)$, $\varepsilon \leq \ell$. Take the universal covering $\pi : \tilde{V} \to V$ and a point $\tilde{v} \in \tilde{V}$ such that the shortest loop in V passes through $\pi(\tilde{v}) \in V$. Consider the ball $\tilde{B} \subset \tilde{V}$ of radius $2D$ centered at $\tilde{v}$. One can easily see that for each $v \in V$ the set $f^{-1}(v) \cap \tilde{B}$ contains at least $q \geq D\ell^{-1}$ points and for any two distinct points $\tilde{v}_1, \tilde{v}_2 \in f^{-1}(v)$ one has $\operatorname{dist}(\tilde{v}_1, \tilde{v}_2) \geq \ell$.

When $R(V) \geq -n$ each ε-neighbourhood of $\tilde{B}$ contains at most $N \leq \frac{b_{n+1}(2D+\varepsilon)}{b_{n+1}(\varepsilon)}$ disjoint balls of radius ε , and when $\varepsilon < \ell$, we conclude that V itself has at most $\frac{N}{\ell}$ such balls. It follows that V can be covered by $N_1 \leq \frac{N}{\ell}$ balls of radius 2ε . This is exactly the estimate we need for the case $\delta = 1$ and by scaling the metric in V by $\sqrt{\delta}$ we reduce the general to $\delta = 1$.

Notice also, that for the manifolds with pinched negative curvature

P. Buser (see [B]) obtained much sharper results by dividing V into the pieces which are not necessarily balls, but still have a sufficiently simple geometry.

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