APPENDIX I

ISOPERIMETRIC INEQUALITIES IN RIEMANNIAN MANIFOLDS

by M. GROMOV

1. GENERALITIES ON RIEMANNIAN MANIFOLDS

1.1. Local Riemannian structures. A *Riemannian C^r-structure* g on an open subset U in the Euclidean space $\mathbb{R}^n$ is, by definition, a C^r-map $u \to g_u$, $u \in U$, which assigns to each point $u \in U$ a positive definite quadratic form (i.e. an Euclidean metric) g_u on $\mathbb{R}^n$. If $X_1, \dots, X_n$ is a fixed basis in $\mathbb{R}^n$, then g is determined by the $\frac{n(n+1)}{2}$ scalar products $g_u(X_i, X_j)$ which are C^r-functions on U.

One defines next the $g-$*length* of each C^1-map $f : [0,1] \to U$ by

$$\int_0^1 (g_{f(t)}(f'(t), f'(t))^{\frac{1}{2}}\, dt \ ,$$

where f' stands for the derivative $\frac{df}{dt}$: $[0,1] \to \mathbb{R}^n$. For example, if g_u is constant in $u \in U = \mathbb{R}^n$, then the family g_u reduces to a single Euclidean structure on $\mathbb{R}^n$ and the above notion of the length agrees with the Euclidean one.

Finally, for *connected* domains U, one defines the distance $dist_g(u_0, u_1)$ between points u_0 and u_1 in U as the infimum of the lengths of *C^1-curves between u_0 and u_1*, which are C^1-maps $f : [0,1] \to U$ such that $f(0) = u_0$ and $f(1) = u_1$. This $dist_g$ is called *the Riemannian metric* associated to g and it obviously satisfies the ordinary metric axioms. Moreover, the topology associated to $dist_g$ agrees with the induced Euclidean topology in U. Observe that $dist_{g_1} = dist_{g_2}$ implies $g_1 = g_2$. Indeed, $(g_u(X,X))^{\frac{1}{2}} = \lim_{\varepsilon \to 0} \varepsilon^{-1} dist_g(u, u + \varepsilon X)$ for all $u \in U$ and $X \in \mathbb{R}^n$. However, only very special metrics on U are of the form $dist_g$ for some g. For example, no metric $dist(u_1, u_2) = \|u_1 - u_2\|$ for a *non-Euclidean* norm $\| \ \|$ on $\mathbb{R}^n$ is ever Riemannian.

EXAMPLES. (a) Let U be an open interval, say (a,b) in $\mathbb{R}^1$. Then each Riemannian structure is given by a single function g_u and the metric space (U, g_u) is isometric to the ordinary (i.e. with the metric $|u_1 - u_2|$) interval (a', b') in $\mathbb{R}^1$ for $a' = \int_a^c (g_u)^{\frac{1}{2}} du$ and $b' = \int_c^b (g_u)^{\frac{1}{2}} du$, where one may take an arbitrary fixed point in (a, b) for c.

(b) Take $\mathbb{R}^2$ with the standard Euclidean structure g_0 and let U be the unit disk in $\mathbb{R}^2$. Put $g_u = 4h(u)g_0$ for the function $h(u) = (1 - dist^2_{g_0}(u, 0))^{-2} = (1 - g_0(u, u))^{-2}$. This is called the *Poincare metric* and the metric space $(U, dist_g)$ is called the *hyperbolic plane*. Every map $f : [0, 1) \to U$ for which $g_0(f(t), f(t)) \to 1$ for $t \to 1$ clearly has *infinite* length. This implies (by an easy argument) the *completeness* of the hyperbolic plane. The remarkable (though easy to prove) property of this g is the invariance under conformal transformations of the disk U. In fact the isometry group of $(U, dist_g)$ is (easily seen to be) locally isomorphic to the multiplicative group of unimodular 2×2 matrices.

1.2. Riemannian manifolds. A metric space $(V, dist)$ is called an *n-dimensional Riemannian C^r-manifold* if it is locally isometric to some $(U, dist_g)$. Namely, for each point $v \in V$ there are some neighbourhoods U' and $U'' \subset U'$ of v in V and a homeomorphism H of U' on an open subset $U \subset \mathbb{R}^n$ with some Riemannian C^r-metric $dist_g$ such that $dist_g(H(u_1), H(u_2)) = dist(u_1, u_2)$ for all points u_1 and u_2 in U''.

EXAMPLES. (a) Let V be a C^{r+1}-smooth n-dimensional submanifold in some Euclidean space $\mathbb{R}^q$, $q \geq n$, with the standard Euclidean metric. Recall that a subset $V \subset \mathbb{R}^q$ is a C^{r+1}-submanifold if there exists, for each $v_0 \in V$, a linear n-dimensional subspace $\mathbb{R}^n \simeq L \subset \mathbb{R}^q$, such that the orthogonal projection, say $P : V \to L$, homeomorphically maps a small neighbourhood $U' \subset V$ of v_0 onto an open subset $U \subset L$ and such that the inverse map $P^{-1} : U \to U' \subset \mathbb{R}^q$ is a C^{r+1}-map of U into $\mathbb{R}^q$. Define $dist(v_1, v_2)$, for $v_1, v_2 \in V$, by taking the infimum of the lengths (measured by the Euclidean metric in $\mathbb{R}^q \supset V$) of C^1-curves in V between v_1 and v_2. Then the resulting metric in V is Riemannian. Indeed the orthogonal projection $\mathbb{R}^q \to L \approx \mathbb{R}^n$ sends the tangent n-planes $T_v(V) \subset \mathbb{R}^q$ of V at points $v \in U'$(which are close to v_0) onto L. This brings the Euclidean structure on $T_v(V)$ down to L, say to g_u on L for $u = P(v)$, and the projection $P : U' \to U$ clearly is an isometry near v_0 for the resulting Riemannian metric on U.

This metric on $V \subset \mathbb{R}^q$ is called the *induced Riemannian* metric. The corresponding Riemannian distance in V is typically greater than the distance in the ambient space $\mathbb{R}$, since straight intervals in $\mathbb{R}^q$ with the ends in V not always lie in V.

REMARKS. (a) A deep theorem of Nash claims that every Riemannian manifold V is isometric to some manifold in $\mathbb{R}^q$ (where q must be rather large compared to $n = dim\ V$). For instance, the hyperbolic plane can be realized by a C^1-submanifold in $\mathbb{R}^3$, but no such C^2-realization exists. However, every bounded ball (or rather disk) in the hyperbolic plane is isometric to a C^∞-submanifold in $\mathbb{R}^3$. It is unknown if there is a global realization of the hyperbolic plane in $\mathbb{R}^4$, but this is possible in $\mathbb{R}^6$ (see [Gr 3] for an account of the theory of isometric imbeddings of Riemannian manifolds). Unfortunately, the existence of isometric imbeddings $V \to \mathbb{R}^q$ has no application to the actual study of the Riemannian geometry of

V.

(a') The above example obviously generalizes to smooth submanifolds V in an arbitrary Riemannian manifold W. Namely, the restriction of the length functional on curves in W to those in V induces a Riemannian structure in V from W.

(b) Take the unit sphere $S^{n-1} \subset \mathbb{R}^n$ with the induced Riemannian metric (from the Euclidean structure in $\mathbb{R}^n$) and let P^{n-1} denote the projective space which is obtained from S^{n-1} by identifying the pairs of points s and $-s$ in S^{n-1}. Since the involution $s \to -s$ is an isometry on S^{n-1}, one obviously has a unique Riemannian metric on P^{n-1} for which the quotient map $S^{n-1} \to P^{n-1}$ is locally an isometry. Observe that P^{n-1} (unlike S^{n-1}) admits no isometric C^2-realization in $\mathbb{R}^n$. (In fact, the space P^{n-1} for $n \geq 3$ admits no isometric C^2-imbedding to $\mathbb{R}^{n+2}$, but P^2, for instance, can be isometrically C^∞-embedded into $\mathbb{R}^5$).

(b') Take linearly independent vectors $X_1, \ldots, X_n$ in $\mathbb{R}^n$ and let $\mathcal{L} \approx \mathbb{Z}^n$ be the free Abelian group (lattice) generated by these vectors. Then the quotient space, called a *flat torus*, $\mathbb{T}^n = \mathbb{R}^n/\mathcal{L}$ inherits a Riemannian metric from the Euclidean one in $\mathbb{R}^n$ for which the quotient map $\mathbb{R}^n \to \mathbb{T}^n$ is locally an isometry. The space of (isometry classes of) flat tori is (for a natural topology) a fairly complicated locally compact space of dimension $\frac{n(n+1)}{2}$. No flat torus admits an isometric imbedding into $\mathbb{R}^m$ for $m \leq 2n-1$ but many are embeddible into $\mathbb{R}^{2n}$ (e.g. 2-tori go into $\mathbb{R}^4$ and *split* n-tori which are *isometric* products of circles obviously go into $\mathbb{R}^{2n}$ for all n).

(c) One usually defines a Riemannian structure on a smooth manifold V as a family (or a field) of Euclidean structures in the tangent spaces $T_v(V)$ which is an assignment to each $v \in V$ of a positive quadratic form g_v on $T_v(V)$. For example, if V is a smooth submanifold in $\mathbb{R}^q$, then each tangent space $T_v(V)$ is realized by the n-plane which is the n-dimensional affine subspace in $\mathbb{R}^q$ (geometrically) tangent to V at v. Each $T_v(V)$ can be brought to the origin in $\mathbb{R}^q$ by the parallel translation $T_v(V) \to T_v(V) - v$. By restricting the Euclidean form in $\mathbb{R}^q$ to this (now linear) subspace $T_v(V) - v$ in $\mathbb{R}^q$ one gets a particular field of forms in $T_v(V)$ which gives the induced Riemannian structure in V described above in the language of length.

(c') Let $X_i = X_i(v)$, $i = 1, \ldots, n$, be linearly independent tangent vector fields on an n-dimensional manifold V. Then one may uniquely define a metric g on V by putting $g(X_i, X_j) = \delta_{ij}$, where $\delta_{ii} \equiv 1$ and $\delta_{ij} \equiv 0$ for $i \neq j$. This is, in fact, a very general procedure. For example, if V is homeomorphic to $\mathbb{R}^n$ then every Riemannian metric on V comes this way with some fields X_i. Indeed, the space $\mathbb{R}^n$ with any Riemannian metric admits, by elementary topology, a frame of n orthonormal vector fields X_i.

Another (by far more interesting) example arises from Lie groups. Namely, if V is endowed with a structure of a Lie group, then each tangent vector at the neutral element $e \in V$ uniquely extends by the left group translations to a left invariant field on V. Thus each basis of vectors $X_i \subset T_e(V)$ extends to a frame on all of V and we obtain with the above what is called a left *invariant Riemannian metric* on V. In other words, left translations extend each Euclidean structure in $T_e(V)$ to a Riemannian metric on V for which left translations are isometries. These metrics are easy to define, but not at all easy to understand. For example, if V is the multiplicative group of non-singular $(m \times m)$-matrices, then $T_e(V)$ equals the linear space $M^m = \mathbb{R}^n$, $n = m^2$ of all $(m \times m)$-matrices. The standard Euclidean structure in this $\mathbb{R}^n$ uniquely extends to a Riemannian metric in the space V of non-singular matrices. We suggest to the reader to evaluate the Riemannian distance between two matrices with given entries.

1.3. Equidistant translates of hypersurfaces in $\mathbb{R}^n$. Consider a C^2-smooth *hypersurface* V_0 (i.e. a submanifold of dimension $n-1$) in $\mathbb{R}^n$ with the Euclidean metric and let V_t, $t \in \mathbb{R}$, be a smooth one-parametric deformation of V_0, that is a C^2-map, say $F: V_0 \times \mathbb{R} \to \mathbb{R}^n$, such that the map F on $V_0 = V_0 \times 0$ equals the original embedding $V_0 \hookrightarrow \mathbb{R}^n$. The deformation is called *normal* to V_0 if the derivative $\frac{dF}{dt}(v,0) \in \mathbb{R}^n$ is normal to the tangent hyperplane $T_v(V) \subset \mathbb{R}^n$ for all $v \in V$. We want to study the deformation of the induced metric, now called g_t on the manifold $V_0 = V_0 \times t$ which is mapped into $\mathbb{R}^n$ by $v \to F(v,t)$. (The image $F(V_0 \times t)$ may not be a submanifold in $\mathbb{R}^n$, but this is rather irrelevant for our present purpose). The length $length_{g_t}(C)$ of each curve $C \subset V_0$ by definition of g_t is the Euclidean length of the curve $c \to F(c,t)$. This is equivalent to saying that $g_t(X,X)$ for each vector $X \in T_v(V_0)$ equals the squared Euclidean length of the image of X under the differential of the map F on $V_0 = V_0 \times t \subset V_0 \times \mathbb{R}$. Here each $T_v(V_0) \subset \mathbb{R}^q$ is viewed as a vector space where the point $v \in T_v(V_0)$ is taken for the origin.

PROPOSITION-DEFINITION (Gauss). *There exists a unique continuous map which assigns to each $v \in V_0$ and each vector $\nu = \nu_v$ normal to V_0 at v a symmetric linear operator $A_\nu : T_v(V_0) \to T_v(V_0)$ with the following property. If F is an arbitrary deformation for which $\frac{dF}{dt}(v,0) = \nu$, then $g'_{t=0}(X,Y) = 2g_0(A_\nu X, Y)$, where X and Y are arbitrary fixed tangent vectors in $T_v(V_0)$ and g'_t stands for the derivative $\frac{dg_t}{dt}$. Furthermore, $A_{\alpha\nu} = \alpha A_\nu$ for all $\alpha \in \mathbb{R}$.*

PROOF: Identify a small neighbourhood of a fixed point in V_0 with a domain $U \subset \mathbb{R}^{n-1}$, let $u_1, \ldots, u_{n-1}$ be the Euclidean coordinates in U and let $\partial_i = \frac{\partial}{\partial u_i}$ be the corresponding derivations (fields) in U. Then the scalar products $g_t(\partial_i, \partial_j)$ by the definition of the induced metric are

$$g_t(\partial_i, \partial_j) = < \partial_i F, \partial_j F > ,$$

where $< \, , \, >$ is the Euclidean scalar product in $\mathbb{R}^n$. Since $< \nu, \partial_i F > = 0$ for $t = 0$, we have (using $\frac{dF}{dt}(v,t) = \nu$ for $t = 0$ and operating with ∂_j)

$$< \frac{d}{dt}\partial_j F, \partial_i F > = - < \nu, \partial_i \partial_j F > \text{ at } t = 0$$

and so

$$\begin{aligned} g'_{t=0}(\partial_i, \partial_j) &= \frac{d}{dt} < \partial_i F, \partial_j F > \\ &= -2 < \nu, \partial_i \partial_j F > . \end{aligned}$$

Thus the operator A_ν is uniquely defined by $g_0(A_\nu\partial_i,\partial_j) = -<\nu,\partial_i\partial_j F>$ at $t=0$.

□

REMARK. Fix a point $v_0 \in V_0$, let ν_0 be a unit normal to V_0 at v_0 and let $\mathbb{R}^1 \subset \mathbb{R}^n$ be the 1-dimensional subspace parallel to ν_0. Take the orthogonal hyperplane $\mathbb{R}^{n-1} \approx L \subset \mathbb{R}^n$ which is parallel to $T_{v_0}(V_0) \subset \mathbb{R}^n$ and observe that the orthogonal projection $P: V_0 \to L$ *diffeomorphically* sends a small neighbourhood $U_0 \subset V_0$ of v_0 into a domain $U \subset L$. Denote by P_1 the orthogonal projection $\mathbb{R}^n \to \mathbb{R}^1$ and observe that $U_0 \subset \mathbb{R}^n$ equals the *graph* of the function $f = P_1P^{-1}: U \to \mathbb{R}^1$. Furthermore,

$$<\nu_0,\partial_i\partial_j F> = \partial_i\partial_j f\ ,$$

for the Euclidean coordinates u_i in this U and, hence, $-\frac{1}{2}g'_{v_0}$ equals the second differential $d^2 f$ of f at $u_0 = P(v_0) \in U$.

Next consider a hypersurface $\overline{V}_0 \subset \mathbb{R}^n$ which is an *inside tangent* (relative to ν_0) to V_0 at v_0. That is $\overline{V}_0$ contains v_0, the tangent hyperplane $T_{v_0}(\overline{V}_0) \subset \mathbb{R}^n$ equals $T_{v_0}(V_0)$ and the corresponding function $\overline{f}$ satisfies $\overline{f} \geq f$ near u_0. Then $d^2\overline{f} \geq d^2 f$ and we conclude to the following (simple but useful) inequality

$$\overline{g}'_{v_0} \leq g'_{v_0}\ .$$

See the first chapter in [Mi] for additional information.

Next we turn to the following *normal geodesic* deformation $F: V_0 \times \mathbb{R} \to \mathbb{R}^n$ which isometrically sends each line $R = v \times \mathbb{R} \subset V_0 \times \mathbb{R}$ onto the straight line in $\mathbb{R}^n$ normal to V_0 at v. Such a map exists if and only if V_0 is *normally orientable* in $\mathbb{R}^n$ (unlike the Möbius band in $\mathbb{R}^3$) which is always the case for small neighbourhoods of points in V_0. We want to study the second derivative g''_t for normal geodesic maps F. We denote by $A(t)$ the operators A_ν assigned to the hypersurface $F(V_0 \times t)$. Again, our consideration is local near a fixed point $v \in V_0$ and we only allow those values t for which the map F diffeomorphically sends $U' \times t$ for a small U' around v onto a smooth hypersurface in $\mathbb{R}^n$. This is the case (by the implicit function theorem) if and only if the differential of F is injective on the tangent space $T_{v,t}(V_0,t)$. The following theorem shows this property to fail exactly for those t which equal the reciprocals of the eigenvalues of the operator $A(0)$ at v and which are called the *principal curvatures* of V_0 at v relative to the unit normal $\nu = \frac{dF}{dt}(v,0)$.

THEOREM (Gauss-Weil). *The derivative of $A(t)$ in t satisfies*

$$A'(t) = -A^2(t)\ ,$$

where A^2 is the ordinary square of the operator A.

PROOF. Since $<\frac{dF}{dt},\partial_j F> = 0$, we have

$$<\frac{d}{dt}\partial_i F,\partial_j F> = <\partial_i F,\frac{d}{dt}\partial_j F> = \frac{1}{2}\frac{d}{dt}<\partial_i F,\partial_j F> = \frac{1}{2}g'_t(\partial_i,\partial_j) = g_t(A(t)\partial_i,\partial_j)\ .$$

Hence,

$$g_t(A^2(t)\partial_i,\partial_j) = g_t(A(t)\partial_i, A(t)\partial_j) =< \frac{d}{dt}\partial_i F, \frac{d}{dt}\partial_j F > \ .$$

Since $\frac{d^2F}{dt^2} = 0$, we obtain

$$2 < \frac{d}{dt}\partial_i F, \frac{d}{dt}\partial_j F >= \frac{d^2}{dt^2} < \partial_i F, \partial_j F >= 2\frac{d}{dt} g_t(A(t)\partial_i,\partial_j)$$
$$= 2g_t'(A(t)\partial_i,\partial_j) + 2g_t(A'(t)\partial_i,\partial_j) = 4g_t(A^2(t)\partial_i,\partial_j) + 2g_t(A'(t)\partial_i,\partial_j) \ ,$$

which equals, by the above, $2g_t(A^2(t)\partial_i,\partial_j)$.

□

EXAMPLES. (a) Let V_0 be the unit sphere $S^{n-1} \subset \mathbb{R}^n$ and let $F(s,t) = s(1-t)$ be the geodesic deformation corresponding to the interior normals. Then, clearly, $g_t' = -2(1-t)g_0$ and therefore $A(t) = -(1-t)^{-1}Id$. Thus all principal curvatures of the sphere of radius $1-t$ equal $-(1-t)^{-1}$ for the *interior* normal direction. They become infinite at $t = 1$ as the map F collapses S^{n-1} to a single point.

(b) Let V_0 be an arbitrary *closed* C^2-hypersurface in $\mathbb{R}^n$ which bounds a compact region $V_+ \subset \mathbb{R}^n$. Consider the interior normal geodesic deformation $F : V_0 \times \mathbb{R}_+ \to \mathbb{R}^n$ and let $r(v) \in R_+ = \mathbb{R}_+ \times v$ for $v \in V_0$ be the first singular point on $\mathbb{R}_+ \times v$ where the map F fails to be a local diffeomorphism. (We know by the above that $-(r(v))^{-1}$ equals the smallest eigenvalue of $A(v,0)$). Consider the open subset $W_+ = \{v,t|v \in V_0, t < r(v)\} \subset V_0 \times \mathbb{R}$.

PROPOSITION. *The n-dimensional measure of the difference* $V_+ \backslash F(W_+) \subset \mathbb{R}^n$ *equals zero.*

PROOF. Take a point $x \in V_+$ and let r_x be the radius of the greatest ball B_x in V_+ around x. The boundary ∂B_x is an inside tangent to V_0 at some point $v = v(x) \in V_0$. Hence, $x = F(v, r_x)$ and $r(v) \geq r_x$. The set $F\{v, r(v)\} \subset \mathbb{R}^n$ obviously has measure zero, and so almost all points $x \in V_+$ are contained in $F(W_+)$.

□

EXERCISE. Let V_0 be a closed *convex* hypersurface whose principle curvatures relative to the exterior normal are ≤ 1. Show the normal geodesic map $F(v,t)$, where $\frac{dF}{dt}$ is the *exterior* normal field, to be *injective* on $V_0 \times (-1, \infty) \subset V_0 \times \mathbb{R}$. Show that the convex region bounded by V_0 contains a unit ball. Next consider a convex subset V_+ with smooth convex boundary $V_0 = \partial V_+$ which has the principal curvatures > 1. Define $V_t \subset V_+$ as the set of those $v \in V_+$ for which $dist(v, V_0) \geq t$. Show that each set V_t is convex. Prove, moreover, that V_t is the intersection of convex subsets with *smooth* boundaries whose principal curvatures $> (1-t)^{-1}$. In particular, the minimal non-empty V_t is a single point, say, v in V_+. Show that V_+ is contained in some ball of radius 1 in $\mathbb{R}^n$.

1.4. Normal deformation of the Riemannian volume. Each Riemannian manifold of dimension n carries a canonical measure (volume) which is uniquely defined by the following two axioms:

(1) The volume of the unit cube in $\mathbb{R}^n$ equals 1.

(2) If $f: V_1 \to V_2$ is a distance decreasing map of V_1 *onto* V_2, then $Vol\ V_2 \leq Vol\ V_1$, where $dim\ V_1 = dim\ V_2 = n$.

This is obvious since each continuous Riemannian structure g_u in $U \subset \mathbb{R}^n$ can be approximated near each point $u_0 \in U$ by the Euclidean structure g_{u_0} (which is constant g_{u_0} in U). The following computational formula for this volume is equally obvious. Let μ_0 be the Euclidean Haar measure in $\mathbb{R}^n$ and let μ_u be the Haar measure associated to the form g_u on $\mathbb{R}^n$ (for which the measure of the g_u-unit cube is one). Then the ratio $r(u) = \mu_u/\mu_0$ is a continuous function in u and $Vol\ U = \int r(u)d\mu_0$. Moreover, one defines the Jacobian of a smooth map $f: V_1 \to V_2$ between Riemannian manifolds, say $J(v)$, $v \in V_1$, as the absolute value of the determinant of the differential $D_f: T_v(V_1) \to T_{v'}(V_2)$ for $v' = f(v)$, relative to the Euclidean structures g_v in $T_v(V_1)$ and $g'_{v'}$ in $T_{v'}(V_2)$, where g_v and $g'_{v'}$ are the Riemannian structures in V_1 and V_2 respectively (which by definition are Euclidean structures in the tangent spaces). If f is a bijective map, then $Vol\ V_2 = \int_{V_1} J(v)dv$ where dv is the Riemannian measure in V_1.

EXERCISES. (a) Show every *compact* manifold V to have *finite* total volume $Vol\ V$.

(b) Show the hyperbolic plane H^2 to have $Vol\ H^2 = \infty$.

(c) Let g_0 be a Euclidean structure in $\mathbb{R}^2$ and let a continuous function $h(u)$ on $\mathbb{R}^2$ equal $(\|u\| log\|u\|)^{-2}$ outside a compact subset in $\mathbb{R}^2$ for $\|u\| = dist_{g_0}(u, 0)$. Show that the Riemannian metric $g = hg_0$ on $\mathbb{R}^2$ is complete and $Vol(R^2, g) < \infty$.

(d) Show every non-compact Lie group with a left invariant metric to be complete of infinite volume.

(e) Let $SL_n\mathbb{R}$ be the group of unimodular $(n \times n)$-matrices with a left invariant metric and let $SL_n\mathbb{Z}$ be the subgroup of the matrices with integer entries. Show the quotient manifold $SL_n\mathbb{R}/SL_n\mathbb{Z}$ to be complete of *finite* volume (and noncompact for $n \geq 2$).

REMARK. The double coset space $S0_n \backslash SL_n\mathbb{R}/SL_n\mathbb{Z}$ is naturally homeomorphic to the space of flat tori $\mathbb{T}^n$ of unit volume.

Now, with the Gauss formula $g'_t(X, Y) = 2g_0(A_\nu X, Y)$ for normal deformations we immediately see the following formula for the derivative in t of the volume $Vol_t = Vol_{g_t}(U_0)$ for all domains $U_0 \subset V_0$

$$Vol'_{t=0} = \int_{U_0} trace\ A_\nu dv_0$$

for the volume element dv_0 of the metric g_0 (which is the usual volume of a submanifold in $\mathbb{R}^n$). The trace of A_ν relative to a unit normal is called the *mean curvature* of $V_0 \subset \mathbb{R}^n$ and it clearly equals the sum of principal curvatures. For example, the mean curvature of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ equals $n - 1$ for the exterior normal and it is $-(n-1)$ for the interior normal.

In order to apply this formula to $t \neq 0$ we express the volume element of g_t on $F(V_0 \times t) \subset \mathbb{R}^n$ by $dv_t = J(v_0,t)dv_0$ for the Jacobian of the map F. Then for all $t \in \mathbb{R}$,

$$Vol'_t = \frac{d}{dt}\int_{U_0} J(v_0,t)dv_0 = \int_{U_0} (trace\ A_\nu(v_0,t))J(v_0,t)dv_0 \ .$$

This is equivalent to the relation

$$\frac{dJ(v_0,t)}{dt} = J(v_0,t)Trace\ A_\nu(v_0,t)$$

which holds on the greatest interval $a(v_0) \leq t \leq b(v_0)$ around zero where the Jacobian does not vanish. Then one exresses the above by

$$\frac{d}{dt}log\ J(v_0,t) = Trace\ A_\nu(v_0,t)\ ,$$

and the Gauss-Weil formula implies

$$\frac{d^2}{dt^2}log\ J(v_0,t) = -trace\ A^2_\nu(v_0,t) \leq -\frac{1}{n-1}(trace\ A_\nu(v_0,t))^2 = -\frac{1}{n-1}(\frac{d}{dt}log\ J(v_0,t))^2\ .$$

Now, a straightforward computation shows that

$$J(v_0,t) \leq 1 + \left(\frac{t\ Trace\ A_\nu(v_0,0)}{n-1}\right)^{n-1}\ .$$

EXERCISE. Show that $J(v_0,t) = Det(1 + tA_\nu(v_0,0))$.

1.4.A. THEOREM. (Paul Levy [Lev]) *Let V_0 be a closed hypersurface in $\mathbb{R}^n$ whose mean curvature relative to the interior normal everywhere is $\leq \overline{\mu}$ for some number $\overline{\mu} < 0$. then the region V_+ bounded by V_0 has $Vol\ V_+ \leq -\frac{n-1}{n\overline{\mu}}Vol\ V_0$. (In fact the equality holds if and only if V_0 is a round sphere of radius $-\overline{\mu}/(n-1)$.*

PROOF. Let $F: V_0 \times \mathbb{R}_+ \to \mathbb{R}^n$ be the interior normal geodesic map and let $W_+ = \{v_0 \in V_0,\ 0 \leq t < r(v_0)\} \subset V_0 \times \mathbb{R}_+$ be the maximal open subset on which F is locally diffeomorphic. (This is equivalent to $J(v_0,t) > 0$ for $(v_0,t) \in W_+$). Then $J(v_0,t) \leq (1 + \frac{\overline{\mu}t}{n-1})^{n-1}$ which implies $r(v_0) \leq \overline{r} = -\frac{n-1}{\overline{\mu}}$ for all $(v_0,t) \in W_+$. Hence,

$$Vol\ F(W_+) \leq \int_{V_0}\int_0^{r(v_0)} J(V_0,t)dv_0dt \leq Vol\ V_0\int_0^{\overline{r}}(1 + \frac{\overline{\mu}t}{n-1})^{n-1} = -\frac{n-1}{n\overline{\mu}}Vol\ V_0\ .$$

Since $meas(V_+\backslash F(W_+)) = 0$, the proof follows.

□

1.5. Normal deformations in Riemannian manifolds. Let V be a C^∞-smooth Riemannian manifold of dimension n which is complete as a metric space. A C^1-map $f: \mathbb{R} \to V$ is called a *geodesic* if $dist(f(t_0), f(t)) = |t_0 - t|$ for all $t_0 \in \mathbb{R}$ and for all $t \in \mathbb{R}$ close to t_0. Let V_0 be a C^2-smooth normally oriented hypersurface in V. Then there exists a unique (with the given normal orientation) *normal geodesic map* $F: V_0 \times \mathbb{R} \to V$ for which the curve $F_{|v_0 \times \mathbb{R}}: R = v_0 \times \mathbb{R} \to V$ is a geodesic normal to V_0 at v_0 for all $v_0 \in V_0$ - (see [Mi], [G.K.M.], [C.E.]). We define as above the family g_t of the induced Riemannian metrics on $V_0 = V_0 \times t$ and we study g'_t and g''_t at $t = 0$ as earlier. First we define the operators $A_\nu = A(v_0, t)$, where the normal direction ν in V is understood as the image of the field $\frac{\partial}{\partial t}$ on $V_0 \times \mathbb{R}$ under the differential of the map F, by setting $g_t(A_\nu, X, Y) = \frac{1}{2} g'_t(X, Y)$ for all pairs of tangent vectors X and Y in V_0.

PROPOSITION-DEFINITION (Riemann). *There exists a smooth map $\nu \to K_\nu$ which for each $v_0 \in V_0 \subset V$ assigns to the unit vector $\nu \in T_{v_0}(V_0)$, normal to $T_{v_0}(V_0) \subset T_{v_0}(V)$ for all $v_0 \in V$, an operator $K_\nu: T_{v_0}(V_0) \to T_{v_0}(V_0)$, called normal curvature operators, such that $\frac{d}{dt} A_\nu = -A_\nu^2 - K_\nu$ at $t = 0$ for all normal vectors. Furthermore, the operator K_ν depends only on ν (and on the Riemannian metric in V, of course) but not on V_0. That is, if V_0 and V'_0 have a common tangent space at some point v_0, that is $v_0 \in V_0 \cap V'_0$ and $T = T_{v_0}(V_0) = T_{v_0}(V'_0) \subset T_{v_0}(V)$, then the operator $K_\nu: T \to T$ defined by $K_\nu = -(\frac{d}{dt} A_\nu + A_\nu^2)$ for $V_0 \cap V'_0$, automatically satisfy the same relation for V'_0. (Observe that the operator A_ν, unlike K_ν, does depend on V'_0.)*

The proof (by a straightforward computation) can be found in any textbook on Riemannian geometry (e.g. [G.K.M.], [C.E.], where an equivalent language of Jacoby fields is employed).

EXAMPLES. (a) Let V be the round sphere S^n in $\mathbb{R}^{n+1}$ of radius R and let V_0 be a round sphere S^{n-1} in S^n whose points are all within (geodesic in S^n) distance r_0 from a fixed point $v \in V = S^n$. then, clearly, the metric g_t on $V_0 = V_0 \times t$ satisfies $g_t = \rho\, sin^2[(t + r_0)/R] g_0$ for $\rho = sin^2(r_0/R)$. Then $g'_t = (\frac{2\rho}{R} sin \frac{t+r_0}{R} cos \frac{t+r_0}{R}) g_0$ and so $A_\nu = (R^{-1} ctg \frac{t+r_0}{R}) Id$. Next, $A'_\nu = (-R^{-2} - R^{-2} ctg (\frac{t+r_0}{R})^2) Id$ and so $K_\nu = R^{-2} Id$. Thus the sphere S^n of radius R has (constant) curvature R^{-2}.

(b) The exponential map. There exists a unique map $T_v(V) \to V$ for each point $v \in V$, called $exp: T_v(V) \to V$, which isometrically sends each straight line ℓ in $(T_v(V), g_v) \approx \mathbb{R}^n$ through the origin onto a geodesic γ in V through v which is tangent at v to ℓ. (See [Mi], [G.K.M.], [C.E.].) It follows that each ball $\tilde{B}_\varepsilon \subset T_v(V)$ around the origin of *small* radius $\varepsilon > 0$ is diffeomorphically sent onto the ε-ball $B_\varepsilon \subset V$ around v. Hence, the boundary S_ε of B_ε for small ε is a C^∞-smooth hypersurface in V, whose interior normal geodesic map $F: S_\varepsilon \times \mathbb{R}_+ \to V$ diffeomorphically sends $S_\varepsilon \times [0, \varepsilon)$ onto $B_\varepsilon \backslash \{v\}$, while the map $F(s, \varepsilon)$ collapses S_ε to v. This implies, like in the Euclidean case, the following property of the normal geodesic map F of an arbitrary hypersurface V_0 in V.

If for a fixed point $x_0 \in V$ the function $dist(x_0, v)$, $v \in V_0$, assumes a local minimum at some point $v_0 \in V_0$ then $F(v_0, r) = x_0$ for $r = dist(x_0, v_0)$ (and for the obvious choice of the normal ν_0 at v_0) and the map F is locally diffeomorphic at (v_0, t) for $0 \le t < r$.

COROLLARY. *Let V_0 be a closed hypersurface in V which bounds a compact re-*

gion $V_+ \subset V$, *let* $F : V_0 \times I\!R_+ \to V$ *be the interior normal geodesic map and let* $W_+ = \{v_0 \in V_0,\ 0 \le t < r(t)\} \subset V_0 \times I\!R_+$ *be the maximal open subset on which F is locally diffeomorphic. Then* $meas_n(V_+ \backslash F(W_+)) = 0$.

REMARK. We shall need below a simple generalization of this fact to compact subsets $V_+ \subset V$ with possibly non-smooth boundary. Namely, let V_0 be the topological boundary of V_+, let $V_0' \subset V_0$ be the maximal open subset which is a C^∞-smooth hypersurface and let $\Sigma = V_0 \backslash V_0'$ be the complementary *singular part* of V_0. The singularity Σ is called *negligible* for V_+ if no compact subset $V_+' \subset V_+$ with C^∞*-smooth* $(n-1)$-dimensional boundary ever meets Σ.

EXAMPLE. Take a domain D in S^2 bounded by a smooth curve C in S^2 and let $V_+ \subset \mathbb{R}^3 \supset S^2$ be the cone over D from the origin. The boundary $V_0 = \partial V_+$ is singular at the origin unless C is the great circle and this singularity is negligible if and only if D contains no hemisphere.

Now, for V_0 with a negligible singularity we define the map F outside Σ, that is $F : V_0' \times \mathbb{R}_+ \to V$ and the image of the corresponding $W_+' \subset V_0' \times \mathbb{R}_+$ obviously covers almost all of V_+.

EXERCISES. (a) Show that the curvature of the hyperbolic plane everywhere is $-Id$.

(b) A hypersurface V_0 is called *convex* (*concave*) for a given normal orientation if the operators A_ν are positive (negative). Show that normal geodesic deformations preserve convexity if the (symmetric) operator K_ν is negative for all ν. Similarly, these deformations preserve the concavity if $K_\nu \ge 0$.

RICCI CURVATURE. Define *Ricci* ν for all unit vectors ν in $T(V)$ by*Ricci* ν = *Trace* K_ν. For example, the round n-dimensional sphere of radius R has *Ricci* $\nu = const = (n-1)R^{-2}$. The following (by now obvious but important) formula generalizes the volume deformation property from $\mathbb{R}^n$ to all V.

1.5.A. LEMMA. *The Jacobian of the map F satisfies*

$$\frac{d}{dt} log\ J(v_0,t) = Trace\ A_\nu(v_0,t)\ ,$$

and

$$\frac{d^2}{dt^2} log\ J(v_0,t) = -trace\ A_\nu^2(v_0,t) - Ricci\ \nu \le -\frac{1}{n-1}\left(\frac{d}{dt} log\ J(v_0,t)\right)^2 - Ricci\ \nu\ .$$

Observe that the equality here holds for spheres $V_0 = S^{n-1} \subset V = S^n$.

1.5.B. THEOREM. (Paul Levy). *Let a closed submanifold* V_0 *with a negligible singularity in* V *bounds a compact region* $V_+ \subset V$ *such that the mean curvature (that is Trace* A_ν *for the interior normal) of the non-singular locus* $V_0' \subseteq V_0$ *everywhere is* $\le \overline{\mu}$, *for a given* $\overline{\mu} \in I\!R$ *(of any sign now) and let the Ricci curvature of V everywhere* $\ge (n-1)R^{-2}$ *for some* $R > 0$. *Let* V_+^* *be the ball of the (geodesic) radius* r_0 *in the round sphere* $S^n \subset I\!R^{n+1}$ *of radius R, where*

$0 \le r_0 \le \pi R$ such that $R^{-1} ctg \frac{r_0}{R} = -\overline{\mu}/(n-1)$. (The boundary V_0^* of this ball has constant mean curvature $= \overline{\mu}$). Then, $Vol\ V_+ \le \frac{(Vol\ V_+^*) Vol\ V_0}{Vol\ V_0^*}$.

PROOF. The Jacobian $J(v_0, t)$ of the interior normal map on

$$W'_+ = \{v_0 \in V'_0,\ 0 \le t < r(t)\} \subset V'_0 \times \mathbb{R}_+$$

is majorized by the Jacobian of the corresponding map for V_0^*, that is $J(v_0, t) \le J^*(v_0^*, t)$, where the Jacobian J^* (obviously) depends only on t but not on v_0^*. This majorization is obtained by a direct computation with the above formulae. It follows that $r(v_0) \le r_0$ for all $v_0 \in V'_0$ and that

$$Vol\ V_+ \le Vol\ F(W'_+) \le \int_{V'_0} \int_0^{r(v_0)} J(v_0, t) dv_0 dt$$
$$\le \frac{Vol\ V'_0}{Vol\ V_0^*} \int_{V_0^*} \int_0^{r_0} J^*(t) dv_0^* dt = \frac{Vol\ V'_0\ Vol\ V_+^*}{Vol\ V_0^*}\ .$$

This is obvious for those who have confidence in their Riemannian geometry. A novice is invited to check all the details step by step by comparing with the Euclidean case.

EXERCISES. (a) Let V_+ be a compact region in V with possibly non-smooth boundary. Say that the mean curvature of the boundary is $\ge \overline{\mu}$ if V_+ is the intersection of a decreasing sequence of regions with smooth boundaries which have mean curvatures $\ge \overline{\mu}$. Take the ε-neighbourhood of the boundary, say $U_\varepsilon(\partial V) \subset V$ and define $Vol\ \partial V = liminf_{\varepsilon \to 0} \varepsilon^{-1} Vol(U_\varepsilon(\partial V) \cap V_+)$, where Vol on the right-hand- side denotes the n-dimensional Riemannian volume (or rather the measure) which is obviously defined for all Borel subsets in V. Generalize the above theorem to these regions V_+.

(b) Let V have $Ricci\ \nu \ge 0$ for all unit vectors $\nu \in T(V)$. Show that concentric balls B_1 and B_2 of radii R_1 and $R_2 \ge R_1$ have

$$\frac{Vol\ B_1}{Vol\ B_2} \ge \left(\frac{R_1}{R_2}\right)^n .$$

(b') Let $Ricci\ \nu \ge (n-1)R^{-1}$, for $R > 0$, and let B_1^* and B_2^* be the balls of radii R_1 and R_2 in the sphere S_R^n of radius R. Show that

$$\frac{Vol\ B_1}{Vol\ B_2} \le \frac{Vol\ B_1^*}{Vol\ B_2^*} \qquad (*)$$

and prove as a corollary that

$$Vol\ V \le Vol\ S_R^n \quad \text{and} \quad diam\ V \le diam\ S_R^n = \pi R\ .$$

(c) Generalize the above $(*)$ to the case $Ricci\ \nu \ge (n-1)R^{-2}$ for $R < 0$, by replacing the sphere S_R by a pertinent hyperbolic space (or, alternatively, by writing explicit formulae in place of $Vol\ B_1^*/Vol\ B_2^*$).

(d) Show all balls in a complete simply connected manifold of *negative curvature* (that is, the operators K_ν are negative) to be convex (i.e. to have the convex boundary for the exterior normal).

(e) Let a complete manifold V have $K_\nu \geq R^{-2} Id$ for some $R > 0$. Show that the complement to each ball in V of radius $\geq \pi R$ is a convex subset in V(i.e. it is an intersection of regions with smooth convex boundaries).

2. ISOPERIMETRIC INEQUALITIES

2.1. Inequalities for Banach spaces. The *classical isoperimetric inequality* in $\mathbb{R}^n$ claims that among all domains $V_+ \subset \mathbb{R}^n$ of a given volume the round ball B^n has the minimal $(n-1)$-dimensional volume of the boundary,

$$Vol\ V_+ \leq C_n (Vol\ \partial V_+)^{n/n-1} \tag{1}$$

where $C_n = Vol\ B^n/(Vol\ S^{n-1})^{n/n-1}$ for the unit sphere $S^{n-1} = \partial B^n$. This can be equally expressed with the characteristic function f_+ of V_+ (which is 1 on V_+ and 0 outside) by *Sobolev's inequality* which relates the L_p-norm of functions on $\mathbb{R}^n$ for $p = \frac{n}{n-1}$ to the L_1-norm of the differentials df,

$$\left(\int_{\mathbb{R}^n} |f|^{n/n-1} \right)^{(n-1)/n} \leq C_n^{(n-1)/n} \int_{R^n} \|df\| du \tag{2}$$

for all functions f on $\mathbb{R}^n$ with a compact support and where du is the (Euclidean) Haar measure in $\mathbb{R}^n$. If f is not differentiable then df is understood in the sense of distributions. For example,

$$\int_{\mathbb{R}^n} \|df_+\| du = Vol\ \partial V_+ \text{ for compact domains } V_+ \subset \mathbb{R}^n \text{ with } C^1 - \text{boundary} .$$

This can be seen with the approximation $f_\varepsilon \to f_+$, $\varepsilon \to 0$, where $f_\varepsilon(u) = f_+(u)$ for $u \in V_+$ and $f_\varepsilon(u) = max(0, 1 - dist(u, V_+))$ outside V_+. Hence the inequality (2) for f_+ reduces to (1). On the other hand, the inequality (2) can be obtained by applying (1) to the regions $V_t = \{u \in \mathbb{R}^n |\ |f(u)| \leq t\}$, $t \geq 0$, and by a (clever) integration in t (see [Maz], [Bu.Ma.]).

The inequality (2) is a member of a large family of inequalities between various L_p norms of f and df. For example, the *Poincare inequality* for functions f in a compact domain $U \subset \mathbb{R}^n$ with a smooth boundary claims

$$\int_U |f|^2 du \leq C(U) \int_U \|df\|^2 du , \tag{3}$$

where we assume $\int_U f\ du = 0$ and where $C(U) = (\lambda_1(U))^{-1}$ for the eigenvalue λ_1 of the Laplace operator $-\Delta$ on U (with Neumann's boundary condition). The non-trivial content of

(3) is the implied inequality $C(U) < \infty$ which amounts to $\lambda_1(U) > 0$. In fact, the inequality (3) can be obtained like (2) from an appropriate isoperimetric inequality for U. Namely, consider hypersurfaces $V_0 \subset U$ which have $\partial V_0 \subset \partial U$ and which divide U into two regions U_+ and U_- in U. Let

$$I(V_0) = min(Vol\ U_+, Vol\ U_-)/Vol V_0$$

and let $Is(U)$ be the supremum of $I(V_0)$ over all $V_0 \subset U$. Then *Cheeger's inequality* (see [Bus]) claims

$$\lambda_1(U) \geq (2\ Is(U))^{-2}\ . \tag{4}$$

which means

$$\int_U f^2(u)du \leq 2\ Is(U) \int_U \|df\|^2 du \tag{5}$$

for all functions f in U which have $\int_U f(u)du = 0$. Thus (5) is proven like (2) by applying the isoperimetric inequality to the levels of the function f (see [Ch]).

The inequalities (1) and (2) were generalized by Brunn back in 1888 to an arbitrary n-dimensional normed (Banach) space $X = (X, \|\ \|)$. Namely, let the Haar measure dx in X be normalized to have the unit ball $B = \{x \in X \mid \|x\| \leq 1\}$ of volume one (which disagrees with the Euclidean convention but has an advantage of simpler formulae) and let $\|\ \|^*$ denote the norm in the dual space X^*.

THEOREM. (Brunn [Br]). *An arbitrary C^1-function f on X with a compact support satisfies*

$$\left(\int_X |f(x)|^{n/(n-1)}\right)^{(n-1)/n} dx \leq n^{-1} \int_X \|df(x)\|^* dx\ . \tag{6}$$

PROOF. (Knothe [Kn]) Fix a linear coordinate system $x_1, \ldots, x_n$ in X such that $dx_1, dx_2, \ldots, dx_n = dx$ and let $\mu(x)$ be a continuous function on X with a compact support $S \subset X$, whose interior is denoted by $S^0 \subset S$.

LEMMA. *There exists a C^1-map Y of S^0 into the cube $C = \{0 < x_i < 1\} \subset X$ with the following two properties:*

(1) The map Y is triangular: the i-th coordinate function of Y depends on $x_1, \ldots, x_i$ only. That is

$$Y = (y_1(x_1), y_2(x_1, x_2), \ldots, y_n(x_1, \ldots, x_n))\ .$$

(2) The partial derivatives $\frac{\partial y_i}{\partial x_i}$ are non-negative on S^0 and the Jacobian

$$J(x) = Det(\frac{\partial y_i}{\partial x_j}) = \prod_{i=1}^{n} \frac{\partial y_i}{\partial x_i}\ \ satisfies\ \ J(x) = \mu(x)/\int_S \mu(x)dx,\ \ for\ all\ \ x \in S^0\ .$$

PROOF. For $s = (x_1(s), \ldots, x_n(s)) \in S^0$ set

$$A_i(s) = \{x = (x_1, \ldots, x_n) \mid x_j = x_j(s)\ \text{ for }\ j < i\ \text{ and }\ x_i \leq x_i(s)\}$$

and

$$B_i(s) = \{x = (x_1, \ldots, x_n) \mid x_j = x_j(s) \text{ for } j < i\} .$$

Then the map $Y(s) = \{y_i(s)\}$ with

$$y_i(s) = \int_{A_i(s)} \mu(s) dx_i, \ldots, dx_n / \int_{B_i(s)} \mu(s) dx_i, \ldots, dx_n \ , \ i = 1, \ldots, n$$

clearly satisfies (1) and (2).

□

REMARK. The above formula equally applies to the characteristic function of an arbitrary open convex subset U in X which gives a *triangular C^1-diffeomorphism of U onto C* whose Jacobian identically equals $(Vol\ U)^{-1}$. We are especially interested in the inverse of this diffeomorphism in case U is the open unit ball $B = \{\|x\| < 1\} \subset X$. This inverse map, called $Y^* : C \to B$, clearly is triangular with the Jacobian $\equiv 1$.

We now compose the above map $Y = Y_\mu$ with Y^* and thus obtain a C^1-map

$$(z_1, \ldots, z_n) = Z = Y^* \circ Y : S^0 \to B$$

which satisfies the above properties (1) and (2) as well as Y. This Z obviously has $\|Z(x)\| < 1$ for all $x \in S^0$ and the divergence $div\ Z(x) = \Sigma_{i=1}^n \frac{\partial Z_i}{\partial x_i}(x)$ satisfies by the geometric-arithmetic mean inequality,

$$(div\ Z(x))^n \geq n^n \mu(x) / \int_S \mu(x) dx, \quad \text{for all} x \in S^0 .$$

Finally, we take $\mu(x) = |f(x)|^{n/n-1}$ and get

$$0 = \int_S div(|f(x)| Z(x)) dx = \int_S |f(x)| div\ Z(x) dx + \int_S < d|f(x)|,\ Z(x) > dx \ ,$$

for the canonical bilinear pairing $< , >$ between X^* and X. Therefore,

$$\int |f(x)|^{n/n-1} dx = \int |f(x)| \mu(x)^{1/n} dx \leq n^{-1} (\int |f(x)| div\ Z(x) dx) (\int \mu(x) dx)^{1/n}$$
$$\leq n^{-1} (\int \|df(x)\| dx) (\int \mu(x) dx)^{1/n} .$$

Since the inequality (6) is homogeneous, we may normalize to $\int \mu(x) dx = 1$, and then (6) follows from the above.

□

REMARKS. (a) The inequality (6) generalizes with an obvious approximation argument to all functions f with (generalized) derivatives in L_1 which, for $X = \mathbb{R}^n$, amounts to Sobolev's inequality (2).

(b) The above proof shows that the equality in (6) holds if and only if f is a scalar multiple of the characteristic function of a metric ball in X. In fact, the proof gives an integral formula for the *isoperimetric deficiency*

$$\|f(x)\|_{L_{n/n-1}} - n^{-1}\|df\|^*_{L_1} .$$

(c) The inequality (6) (and its proof as well) obviously generalizes to the spaces X with *non-symmetric* norms. This generalization (in a slightly different but equivalent form) is called the *Brunn-Minkowski inequality* (see [Had]).

(d) Brunn's inequality generalizes, up to a certain extent, to *differential forms* f on X of degree > 1, as was discovered (in the dual isoperimetric language by Federer and Fleming (see [F.F], [Gr2]).

2.2. Levy's Inequality. Let V be a closed n-dimensional Riemannian C^∞-manifold whose Ricci curvature is everywhere $\geq (n-1)R^{-2}$, $R > 0$, (which is the Ricci curvature of the round sphere $S^n \subset \mathbb{R}^{n+1}$ of radius R). Let $V_+ \subset V$ be a compact region with smooth boundary $V_0 = \partial V_+$ and let V_+^* be a round ball in the sphere S^n of radius R in $\mathbb{R}^{n+1}$ such that $Vol\ V_+^*/Vol\ S^n = Vol\ V_+/Vol\ V$.

THEOREM. (Levy [Lev]). *The $(n-1)$-dimensional volume of V_0 is related to that of the sphere $V_0^* = \partial V_+^*$ by the inequality*

$$\frac{Vol\ V_0}{Vol\ V} \geq \frac{Vol\ V_0^*}{Vol\ S^n} .$$

PROOF. Consider the functional $Vol\ \partial\Omega_+$ on all domains $\Omega_+ \subset V$ with a fixed n-dimensional volume $Vol\ \Omega_+ = Vol\ V_+$. Then, by the global calculus of variations, there exists an *extremal* domain, say $\overline{\Omega}_+$ in V for which the $(n-1)$-volume $\partial\overline{\Omega}_+$ is the least possible. Unfortunately, the boundary $\partial\overline{\Omega}_+$ is not necessarily smooth. However, a deep theorem of F. Almgren [Al] claims the singularity to be negligible both from inside and outside $\overline{\Omega}_+$. (Notice that $\partial\overline{\Omega}_+$ is smooth for $n \leq 7$, see [Law].) Furthermore, the non-singular locus $\overline{\Omega}_0' \subset \overline{\Omega}_0 = \partial\overline{\Omega}_+$ of this Ω has *constant* mean curvature (this is obvious since the mean curvature equals the normal derivative of the $(n-1)$-dimensional volume of $\overline{\Omega}_0'$). If this curvature, called $\overline{\mu}$, does not exceed that of V_0^*, then the proof follows from 1.5.B. Otherwise, we go to the complement $\overline{\Omega}_- = V\backslash\overline{\Omega}_+$, in which (interior) direction the mean curvature of $\overline{\Omega}_0'$ equals $-\overline{\mu}$ which is necessarily less than the mean curvature of V_0^* in the direction of the ball $V_-^* = S^n\backslash V_+^*$. Then 1.5.B applies to $\overline{\Omega}_-^*$ and the proof is concluded.

□

REMARKS AND COROLLARIES. (a) If $V = S^n$, then Levy's inequality amounts to the classical isoperimetric inequality on S^n:

Among all domains in S^n with a fixed volume the minimal volume of the boundary is assumed by a round ball.

A similar inequality holds true in the n-dimensional hyperbolic space H^n (see [Schm]), but for no space except $\mathbb{R}^n$, S^n and H^n one knows the exact solution of the isoperimetric problem.

(b) Let $V_\varepsilon \subset V$ denote the ε-neighbourhood of V_0 in V. Then an obvious integration in ε shows that

$$\frac{Vol\ V_\varepsilon}{Vol\ V} \geq \frac{Vol\ V_\varepsilon^*}{Vol\ S^n} .$$

(c) Levy's inequality (and his proof as well) generalizes to all Riemannian manifold with a given (possibly negative) lower bound on the Ricci curvature (see [Gr1]). This leads to sharp estimates on the eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$ of the Laplace operator on V (see [B.G.M.], [Ber]).

(d) In order to apply Levy's inequality to a specific manifold V one needs some information on the Ricci curvature. In fact, the Ricci curvature is an easily computable invariant.

EXAMPLES. (1) Let $V^n \subset \mathbb{R}^{n+1}$ be a smooth hypersurface whose principal curvatures $x_1, \ldots, x_n$ at some point v_0 satisfy

$$x_1(x_2 + \ldots + x_n) \geq \alpha, \quad x_2(x_1 + x_3 + \ldots x_n) \geq \alpha, \ldots, x_n(x_1 + \ldots + x_{n-1}) \geq \alpha .$$

Then $Ricci(\nu, \nu) \geq \alpha$ for all $\nu \in T_{v_0}(V)$.

This follows from the famous *theorema egregium* of Gauss which expresses the intrinsic curvature of V in terms of the principal curvatures (see [G.K.M] or prove it yourself).

(2) Let Riemannian manifolds $V_1, \ldots, V_k$ have $Ricci(V_j) \geq \alpha$, $j = 1, \ldots, k$. Then the *Riemannian product* $V = V_1 \times \ldots \times V_k$ also have $Ricci(V) \geq \alpha$. This is immediate from the definition of the Riemannian structure g in V, which is $g = g_1 \oplus \ldots \oplus g_k$.

(3) Let V be the orthogonal group $0(n)$ and let g be the (natural) left invariant metric on $0(n)$ which is invariant under conjugations (which is equivalent to being right invariant as well as left invariant) and such that the circle consisting of the rotations around a fixed subspace $\mathbb{R}^{n-2} \subset \mathbb{R}^n$ has length 2π. (With these conditions g obviously is unique)then $Ricci(0(n), g) \geq \frac{n}{4}$ everywhere (see [C.E.] for an explicit computation of the curvature of Lie groups).

APPENDIX II

GAUSSIAN AND RADEMACHER AVERAGES

II.1. THEOREM: *For all $C < \infty$ and $2 \le q < \infty$ there exists a constant $K = K(C, q)$ such that if $\beta_q(X) \le C$ then for all n and $x_1, \ldots, x_n \in X$*

$$\left\| \sum_{i=1}^{n} g_i x_i \right\|_{L_q(X)} \le K \left\| \sum_{i=1}^{n} r_i x_i \right\|_{L_q(X)} .$$

In particular, $C_q(X) \le K\beta_q(X)$ ($(g_i)_{i=1}^n$ are independent symmetric gaussian variables normalized in L_2, $(r_i)_{i=1}^n$ are the Rademacher functions).

Given a 1-unconditional basis $z_1, z_2, \ldots$ in some Banach space X and a function $f : \mathbb{R} \to \mathbb{R}$, we may view f as acting on the finitely supported elements in X by applying f to each of the coefficients

$$f(\Sigma a_n z_n) = \Sigma f(a_n) z_n .$$

II.2. DEFINITION: *The p-convexity constant A_p (resp. q-concavity constant B_q)* of the basis (z_i) is the smallest A (resp. B) satisfying

$$\|(\Sigma |x_n|^p)^{1/p}\| \le A(\Sigma \|x_n\|^p)^{1/p}$$
$$(\text{resp.} \quad (\Sigma \|x_n\|^q)^{1/q} \le B\|(\Sigma |x_n|^q)^{1/q}\|)$$

for all finite sequences (x_n) of finitely supported elements in X.

II.3. DEFINITION: *The upper p-estimate constant a_p* (resp. *lower q-estimate constant b_q*) of the basis (z_i) is the smallest a (resp. b) satisfying

$$\|\Sigma x_n\| \le a(\Sigma \|x_n\|^p)^{1/p}$$
$$(\text{resp.} \quad (\Sigma \|x_n\|^q)^{1/q} \le b\|\Sigma x_n\|)$$

for all finite sequences (x_n) of finitely and *disjointly* supported (with respect to (z_i)) elements in X.

Clearly, $a_p \le A_p$, $b_p \le B_p$. Also for $1 \le p \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$A_p(z_i) = B_q(z_i^*) \quad , \quad A_p(z_i^*) = B_q(z_i) ,$$

$$a_p(z_i) = b_q(z_i^*) \quad , \quad a_p(z_i^*) = b_q(z_i)$$

where (z_i^*) are the biorthogonal functionals to (z_i) (see [L.T2]). Note also that $a_p \le \alpha_p(X)$, $b_q \le \beta_q(X)$. (α_p, β_q are the gaussian type and cotype constants).

II.4. PROPOSITION: *For all $1 < r < q < \infty$ there exists a constant $K(r,q)$ such that*

$$B_q \le K(r,q) b_r \quad for \quad 1 < r < q < \infty$$

and

$$A_p \le K(s,p) a_s \quad for \quad 1 < p < s < \infty \; .$$

In particular

$$B_q \le K(r,q)\beta_r(X) \quad for \quad 1 < r < q < \infty$$

and

$$A_p \le K(s,p)\alpha_s(X) \quad for \quad 1 < p < s < \infty \; .$$

PROOF: Let $1 < p < s < \infty$. We shall show $A_p \le K(s,p)a_s$, the result for B_q, b_r follows by duality.

We need the following lemma, the proof of which we delay until after the proof of the proposition.

II.5. LEMMA: *Let $1 < p < s < \infty$. There exist a sequence of independent random variables (f_i) on some probability space $(\Omega, \mathcal{F}, \mu)$ and a constant $K = K(s,p)$ such that*

$$K^{-1} \int_\Omega \left(\sum_{i=1}^n |a_i f_i|^s \right)^{1/s} d\mu \le \left(\sum_{i=1}^n |a_i|^p \right)^{1/p} = \int \max_{1 \le i \le n} |a_i f_i| d\mu$$

for all finite sequences $a_1, \dots, a_n$.

Assume the lemma and let $x_1, \dots, x_n$ be finitely supported elements in X

$$x_i = \sum_j a_{ij} z_j \; .$$

For each $\omega \in \Omega$ we define the function $\max_{1 \le i \le n} |x_i f_i(\omega)|$ coordinatewise:

$$\max_{1 \le i \le n} |x_i f_i(\omega)| = \sum_j \max_i |a_{ij} f_i(\omega)| z_j \; .$$

Note that for each ω

$$\max_{1 \le i \le n} |x_i f_i(\omega)| = \sum_{i=1}^n y_i(\omega)$$

where the $y_i(\omega)$-s are disjointly supported elements of X and $|y_i(\omega)| \le |x_i f_i(\omega)|$ in the order induced by (z_i). It follows from the lemma that

$$(\Sigma |x_i|^p)^{1/p} \le \int \max_{1 \le i \le n} |x_i f_i(\omega)| = \int \sum_{i=1}^n y_i(\omega) \; .$$

By the 1-unconditionality of (z_i)

$$\|(\Sigma|x_i|^p)^{1/p}\| \le \|\int \sum_{i=1}^{n} y_i(\omega)\| \le \int \|\sum_{i=1}^{n} y_i(\omega)\| \le a_s \int (\Sigma\|y_i(\omega)\|^s)^{1/s}$$
$$\le a_s \int (\Sigma\|x_i\|^s|f_i(\omega)|^s)^{1/s} \le K\ a_s(\Sigma\|x_i\|^p)^{1/p}$$

and

$$A_p \le K\ a_s\ .$$

□

II.6. PROOF OF LEMMA II.5: Let $(f_i(\omega))_{i=1}^{\infty}$ be independent random variables with

$$\mu(f_i(\omega) > \lambda) = 1 - e^{-c/\lambda^p} \quad \textit{for all} \quad \lambda > 0$$

where c is such that $\int f_i d\mu = 1$. Then

$$\mu(\max_{1\le i\le n} |a_i f_i| \le \lambda) = \prod_{i=1}^{n} \mu(a_i f_i \le \lambda) = exp\frac{-c\Sigma|a_i|^p}{\lambda}\ .$$

We conclude that, if $\Sigma_{i=1}^n|a_i|^p = 1$, then $\max_{1\le i\le n}|a_i f_i|$ has the same distribution as f_1 and in particular $\int \max_{1\le i\le n}|a_i f_i| d\mu = 1$. This proves the right hand side equality. To prove the left hand side inequality let $p < s < \infty$ and let $(g_i)_{i=1}^{\infty}$ be independent random variables such that

$$\mu(g_i > \lambda) = 1 - e^{-d/\lambda^s} \quad \text{for all} \quad \lambda > 0$$

where d is such that $\int g_i d\mu = 1$. Then, by the first part of the proof

$$\int \left(\sum_{i=1}^{n} |a_i f_i|^s\right)^{1/s} = \int\int \max_{1\le i\le n} |a_i f_i g_i|$$
$$= \int \left(\sum_{i=1}^{n} |a_i g_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{n} |a_i|^p \int |g_i|^p\right)^{1/p}$$
$$= K\left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p}$$

with

$$K = \left(\int |g_i|^p\right)^{1/p} < \infty\ .$$

□

II.7. Before proceeding to the proof of the Theorem we need another inequality which is a generalization due to Maurey of Khinchine's inequality.

THEOREM: *for all C and q, $1 \le C, q < \infty$, there exists a constant $M = M(C,q)$ such that, if (z_i) is a 1-unconditional basis in some Banach space X with q concavity constant $\le C$, then*

$$\frac{1}{\sqrt{2}}\|(\sum_i |x_i|^2)^{1/2}\| \le \int_0^1 \|\sum_i r_i(t)x_i\| \le (\int_0^1 \|\sum r_i(t)x_i\|^q)^{1/q} \le M\|(\sum_i |x_i|^2)^{1/2}\|$$

for all (x_i) in X.

PROOF: The left hand side inequality follows immediately from the triangle inequality and Khinchine's inequality in L_1, the concavity assumption is not needed here. For the right hand side inequality:

$$\begin{aligned}(\int \|\sum r_i(t)x_i\|^q)^{1/q} &\le C\|(\int |\sum r_i(t)x_i|^q)^{1/q}\| \\ &\le CK_q\|(\sum |x_i|^2)^{1/2}\|\end{aligned}$$

where K_q is Khinchine's constant in L_q. □

We turn now to the

II.8. PROOF OF THEOREM II.1: Let $\tilde{x}_i = r_i(t)x_i$ in $L_q(X)$. Since $\|\Sigma r_i(s)x_i\|_{L_q(X)} = \|\Sigma r_i(s)\tilde{x}_i\|_{L_q(L_q(X))}$ and $\|\Sigma g_i(s)x_i\|_{L_q(X)} = \|\Sigma g_i(s)\tilde{x}_i\|_{L_q(L_q(X))}$ and since $L_q(X)$ has the same cotype q constant as X, we may assume we are dealing with $(\tilde{x}_i)$ instead of (x_i). In particular, without loss of generality, (x_i) is 1-unconditional. Fix $r > q$, then, by Proposition II.4, (x_i) is r-concave with r-concavity constant depending on q, r and C only.

Let $(r_{ij})_{i=1}^{n}\,{}_{j=1}^{\infty}$ be a sequence of independent Rademacher functions. By Theorem II.7, for all $m \in \mathbb{N}$,

$$\begin{aligned}(\int_0^1 \|\sum_{i=1}^n\sum_{j=1}^m r_{ij}(t)\frac{1}{\sqrt{m}}x_i\|^q)^{1/q} &\le (\int_0^1 \|\sum_{i=1}^n\sum_{j=1}^m r_{ij}(t)\frac{1}{\sqrt{m}}x_i\|^r)^{1/r} \\ \le M\|(\sum_{i=1}^n |x_i|^r)^{1/r}\| &\le M\sqrt{2}(\int \|\sum_{i=1}^n r_i(t)x_i\|^q)^{1/q}\ .\end{aligned}$$

By the central limit theorem, the sequence

$$\left(\frac{1}{\sqrt{m}}\sum_{j=1}^m r_{ij}(t)\right)_{i=1}^n$$

tends in distribution to $(g_i)_{i=1}^n$, a sequence of independent, mean zero, gaussian random variables normalized in L_2. Thus we get

$$\left(\int_0^1 \|\sum_{i=1}^n g_i(t)x_i\|^q\right)^{1/q} \le M\sqrt{2}\left(\int \|\sum_{i=1}^n r_i(t)x_i\|^q\right)^{1/q} .$$

□

APPENDIX III
KAHANE'S INEQUALITY

We bring here a proof, essentially due to C. Borell, of Kahane's inequality 9.2. We begin with the Brunn-Minkowsky inequality (see also Appendix I).

III.1. THEOREM: *Let A, B be two compact sets in $I\!R^n$. Then*

$$Vol(A+B)^{1/n} \geq Vol(A)^{1/n} + Vol(B)^{1/n} .$$

PROOF: By a simple approximation procedure, we may assume that each of A and B is a union of finitely many disjoint sets, each of which is a product of intervals. The proof is by induction on the total number k of such rectangular boxed in A and B. If $k = 2$, i.e. if A and B are rectangular boxes with sides of lengths $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ respectively, then

$$Vol(A+B)^{1/n} = \prod_{i=1}^n (a_i + b_i)^{1/n} , \quad Vol(A) = \prod_{i=1}^n a_i^{1/n} , \quad Vol(B) = \prod_{i=1}^n b_i^{1/n} .$$

By the inequality between the geometrical and arithmetical means,

$$\prod_{i=1}^n \left(\frac{a_i}{a_i+b_i}\right)^{1/n} + \prod_{i=1}^n \left(\frac{b_i}{a_i+b_i}\right)^{1/n} \leq \frac{1}{n}\sum_{i=1}^n \frac{a_i}{a_i+b_i} + \frac{1}{n}\sum_{i=1}^n \frac{b_i}{a_i+b_i} = 1 .$$

Assume now that A and B are composed of a total of $k > 2$ rectangular boxes and that the inequality holds for all sets A', B' such that A', B' are composed of a total of at most $k-1$ boxes. We may and shall assume that the number of rectangular boxes in A is at least 2. Note that parallel shifts of A and B do not change the volume of A, B or $A + B$. We find such a shift of A with the property that one of the coordinate hyperplanes divides A in such a manner that least one rectangular box in A is in each side of the plane. A is now divided into two sets A', A'' each of which is a disjoint union of finite number of rectangular boxes and the number of boxes in each of A', A'' is strictly smaller than the number in A. Now shift B parallel to the axes in such a manner that the same hyperplane divides B into B', B'' with $\frac{Vol\ B'}{Vol\ B} = \frac{Vol\ A'}{Vol\ A} = \lambda$.

Each of B', B'' has at most the same number of rectangular boxes as B has.

By the induction hypothesis

$$\begin{aligned} Vol(A+B) &\ge Vol(A'+B') + Vol(A''+B'') \\ &\ge [(Vol\ A')^{1/n} + (Vol\ B')^{1/n}]^n + [(Vol\ A'')^{1/n} + (Vol\ B'')^{1/n}]^n \\ &= \lambda[(Vol\ A)^{1/n} + (Vol\ B)^{1/n}]^n + (1-\lambda)[(Vol\ A)^{1/n} + (Vol\ B)^{1/n}]^n \\ &= [(Vol\ A)^{1/n} + (Vol\ B)^{1/n}]^n \end{aligned}$$

□

III.2. It follows from Theorem III.1 that

$$Vol(\lambda A + (1-\lambda)B) \ge (Vol\ A)^\lambda (Vol\ B)^{1-\lambda}$$

for all compact sets A, B in $\mathbb{R}^n$ and all $0 < \lambda < 1$. Indeed

$$Vol(\lambda A + (1-\lambda)B)^{1/n} \ge \lambda (Vol\ A)^{1/n} + (1-\lambda)(Vol\ B)^{1/n} \ge \{(Vol\ A)^\lambda (Vol\ B)^{1-\lambda}\}^{1/n} .$$

If K is any convex body of finite volume in $\mathbb{R}^n$, put $\mu_K(A) = \frac{Vol(A\cap K)}{Vol\ K}$. Then μ_K clearly satisfies the same kind of inequality

$$\mu_K(\lambda A + (1-\lambda)B) \ge \mu_K(A)^\lambda \mu_K(B)^{1-\lambda} .$$

III.3. THEOREM (C. Borell). *Let μ be any Borel probability measure on $\mathbb{R}^n$ satisfying $\mu(\lambda A + (1-\lambda)B) \ge \mu(A)^\lambda \mu(B)^{1-\lambda}$. Then for all symmetric convex sets $A \subseteq \mathbb{R}^n$ with $\mu(A) = \theta > \frac{1}{2}$*

$$\mu((tA)^c) \le \theta\left(\frac{1-\theta}{\theta}\right)^{(1+t)/2} \quad for\ all\ \ t > 1 .$$

PROOF. We have the inclusion

$$\mathbb{R}^n \backslash A \supseteq \frac{2}{t+1}(\mathbb{R}^n \backslash tA) + \frac{t-1}{t+1} A$$

(check!). Consequently

$$1 - \theta = \mu(A^c) \ge \mu((tA)^c)^{2/(t+1)} \theta^{(t-1)/(t+1)} .$$

□

III.4. Let now $(X, \|\ \ \|)$ be any normed space and let $x_1, \dots, x_n \in X$. Assume

$$\int_K \|\sum_{i=1}^n a_i x_i\| d\mu(a) = 1$$

where μ is the Lebesgue measure on the cube $K = [-1,1]^n$ normalized as to give $\mu(K) = 1$. Let

$$A = \{a \in K;\ \|\sum_{i=1}^n a_i x_i\| \le 3\} .$$

Then A is convex symmetric and $\mu(A) \geq \frac{2}{3}$, (since $3\mu(A^c) \leq \int_K \|\Sigma a_i x_i\| d\mu(a) = 1$). Applying Theorem III.3 we get, for all $t > 1$,

III.4.1.
$$\mu\{\|\Sigma_{i=1}^n a_i x_i\| > 3t\} \leq \frac{2}{3}(\frac{1}{2})^{(1+t)/2}$$

and consequently, for all $p > 1$,

$$\int \|\sum_{i=1}^n a_i x_i\|^p d\mu(a) = p\int_0^\infty t^{p-1}\mu(\|\sum_{i=1}^n a_i x_i\| > t)dt$$
$$\leq p\ 3^{p-1} + \frac{2p}{3}\int_3^\infty t^{p-1}(\frac{1}{2})^{(3+t)/6} dt \leq K_p^p$$

for some constant K_p depending only on p. It follows by homogeneity that

III.4.2.
$$(\int_K \|\Sigma_{i=1}^n a_i x_i\|^p d\mu(a))^{1/p} \leq K_p \int_K \|\Sigma_{i=1}^n a_i x_i\| d\mu(a) \ .$$

Kahane's inequality now easily follows from III.4.2. Indeed, by the 1-unconditionality of the Rademacher functions,

$$(\int_K \|\sum_{i=1}^n a_i x_i\|^p d\mu(a))^{1/p} = (\int_0^1 \int_K \|\sum_{i=1}^n a_i r_i(t) x_i\|^p d\mu(a) dt)^{1/p}$$
$$\leq (\int_0^1 \|\sum_{i=1}^n r_i(t) x_i\|^p dt)^{1/p} \ .$$

By the fact that $(a_i)_{i=1}^n$ and $(|a_i| r_i(t))_{i=1}^n$ have the same distribution and by the triangle inequality

$$\int_K \|\sum_{i=1}^n a_i x_i\|^p d\mu(a))^{1/p} = (\int_0^1 \int_K \|\sum_{i=1}^n |a_i| r_i(t) x_i\|^p d\mu(a) dt)^{1/p}$$
$$\geq (\int_0^1 \|\int_K (\sum_{i=1}^n |a_i| r_i(t) x_i) d\mu(a)\|^p dt)^{1/p} = \frac{1}{2}(\int_0^1 \|\sum_{i=1}^n r_i(t) x_i\|^p)^{1/p} \ .$$

We can now conclude from III.4.2 that, for $p > 1$,

$$(\int_0^1 \|\sum_{i=1}^n r_i(t) x_i\|^p dt)^{1/p} \leq 2K_p \int_0^1 \|\sum_{i=1}^n r_i(t) x_i\|$$

which proves Kahane's inequality. (The other side inequality is trivial.)

We remark that this proof does not give the right order of magnitude for the growth rate of the constant. It only gives $K_p \leq Kp$ with $K < \infty$ absolute. See [L.T2] for a proof which gives the right order of magnitude for the constant, $K_p \approx \sqrt{p}$.

APPENDIX IV
PROOF OF THE BEURLING-KATO THEOREM 14.4

Define

$$Ax = \lim_{t \to 0} \frac{S_t x - x}{t}$$

whenever the limit exists. Then A is a linear (usually unbounded) closed operator with dense domain $D(A)$. For any $x \in D(A)$

$$\frac{dS_t x}{dt} = \lim_{s \to 0} \frac{S_{t+s} x - S_t x}{s}$$

exists and

$$S_t x = AS_t x = S_t A x \ .$$

Let $\sup_t \|I - S_t\| = \rho_0$ and let $\rho_0 < \rho < 2$, say $\rho = \frac{2+\rho_0}{2}$.

STEP 1: For all $\beta \geq \rho$, $((\beta - 1)I + S_t)^{-1}$ exists and $\|((\beta - 1)I + S_t)^{-1}\| \leq \frac{M}{\beta}$ with $M = M(\rho_0)$.

Indeed, since $\|\frac{I - S_t}{\beta}\| \leq \frac{\rho_0}{\rho}$, $T = \Sigma_{n=0}^{\infty}(\frac{I-S_t}{\beta})^n$ converges and its norm is at most $\Sigma_{n=0}^{\infty}(\frac{\rho_0}{\rho})^n = M$. Also,

$$T = (I - \frac{I - S_t}{\beta})^{-1} = \beta((\beta - 1)I + S_t)^{-1} \ .$$

Thus,

$$\|((\beta - 1)I + S_t)^{-1}\| \leq \frac{M}{\beta} \ .$$

STEP 2: There exists $0 < \theta = \theta(\rho_0) \leq \frac{\pi}{2}$ and $M' = M'(\rho_0)$ such that for $z \in \mathbb{C}$ with $|arg\ z| \geq \frac{\pi}{2} - \theta$, $zI + A$ has a bounded inverse and

$$\|(zI + A)^{-1}\| \leq \frac{M'}{|z|} \ .$$

PROOF: Note that for all $t \geq 0$ and $x \in D(A)$,

$$(e^{zt} S_t - I)x = \int_0^t \frac{d}{ds}(e^{zs} S_s) x ds = \int_0^t e^{zs} S_s (z + A) x ds \ .$$

Fix $z = a + ib$. If one can choose t such that $e^{-zt} = 1 - \beta$ with β real and $\beta \geq \rho$, then $(e^{zt}S_t - I)^{-1}$ exists and

$$(zI + A)^{-1} = (e^{zt}S_t - I)^{-1} \int_0^t e^{zs} S_s ds \ .$$

Choose $t = \frac{\pi}{|b|}$. Then,

$$e^{-zt} = e^{-a\pi/|b|} e^{i\pi sign\ b} = -e^{-a\pi/|b|} \ .$$

If $\beta = 1 + e^{-a\pi/|b|}$, then $\beta \geq \rho$ as long as $\frac{|a|}{|b|}$ is small enough, i.e., as long as $|arg\ z| > \frac{\pi}{2} - \theta$ with θ small enough.

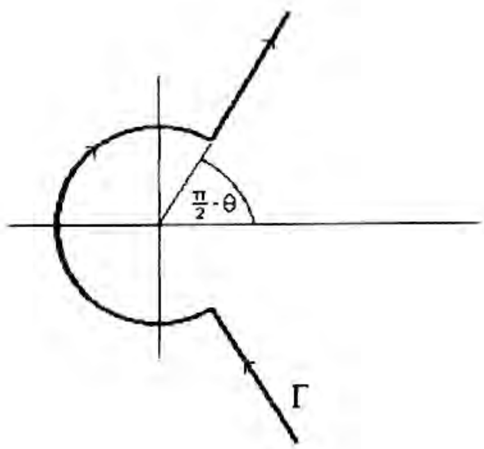
For such z and t we get

$$\begin{aligned}
\|(zI + A)^{-1}\| &\leq |e^{-zt}|\ \|(S_t - e^{-zt}I)^{-1}\| \int_0^t |e^{zs}| ds \\
&\leq e^{-at}\frac{M}{\beta}\int_0^t e^{as} ds \\
&= e^{-at}\frac{M}{\beta}\frac{e^{at}-1}{a} = \frac{M}{\beta}\frac{1-e^{-at}}{a} \\
&\leq \begin{cases} \frac{M'}{\beta} inf(t, \frac{1}{a}) & a \geq 0 \\ M'\ inf(t, \frac{1}{|a|}) & a \leq 0 \ , \end{cases}
\end{aligned}$$

so that,

$$\|(zI + A)^{-1}\| \leq M'\ inf(\frac{1}{|a|}, \frac{1}{|b|}) \leq \frac{M''}{|z|} \ .$$

STEP 3: For Γ in the illustration



define, for ξ with $|arg\ \xi| < \frac{\theta}{2}$,

$$S_\xi = -\frac{1}{2\pi i}\int_\Gamma e^{-z\xi}(zI+A)^{-1}dz\ .$$

Since $|arg\ z\xi| \leq \frac{\pi}{2} - \theta + \frac{\theta}{2} = \frac{\pi}{2} - \frac{\theta}{2}$ on the infinite rays, $|e^{-z\xi}| \leq e^{-\alpha|z|}$ for some positive α so that the integral converges. One checks that it defines a holomorphic semigroup which extends S_t. To show that it extends S_t compute its derivative,

$$\begin{aligned}\frac{d}{dt}(-\frac{1}{2\pi i}\int_\Gamma e^{-zt}(zI+A)^{-1}dz) &= -\frac{1}{2\pi i}\int_\Gamma e^{-zt}(-z)(zI+A)^{-1}dz\\ &= \frac{1}{2\pi i}\int_\Gamma e^{-zt}(zI+A-A)(zI+A)^{-1}dz\\ &= \frac{1}{2\pi i}\int_\Gamma e^{-zt}(I-A(zI+A)^{-1}dz\\ &= \frac{1}{2\pi i}\int_\Gamma e^{-zt}dt - \frac{A}{2\pi i}\int_\Gamma e^{-zt}(zI+A)^{-1}dz\\ &= A(-\frac{1}{2\pi i}\int_\Gamma e^{-zt}(zI+A)^{-1}dz)\end{aligned}$$

and we get that the two semigroups have the same generator. This of course means that they are equal.

To check that S_ξ is a semigroup, notice that $S_\xi = \frac{-1}{2\pi i}\int_{\Gamma'} e^{-z\xi}(zI+A)^{-1}dz$ for Γ' being, say, $\Gamma - 2$. Then,

$$S_\xi S_{\xi'} = \frac{1}{(2\pi i)^2}\int_\Gamma\int_{\Gamma'} e^{-z\xi - z'\xi'}(zI+A)^{-1}(z'I+A)^{-1}dz'dz\ .$$

By the resolvent formula,

$$S_\xi S_{\xi'} = \frac{1}{(2\pi i)^2}\int_\Gamma\int_{\Gamma'} e^{-z\xi - z'\xi'}(z'-z)^{-1}((zI+A)^{-1} - (z'I+A)^{-1})dz'dz\ ,$$

and by the Cauchy formula,

$$\begin{aligned}S_\xi S_{\xi'} &= \frac{-1}{2\pi i}\int_\Gamma e^{-z\xi - z\xi'}(zI+A)^{-1}dz\\ &= S_{\xi+\xi'}\ .\end{aligned}$$

Finally, we have to check that $\|S_\xi\|$ is bounded in the sector $\{|arg\ \xi| < \frac{\theta}{2}\}$. We leave this to the reader (replace the unit circle with a larger one).

APPENDIX V

THE CONCENTRATION OF MEASURE PHENOMENON FOR GAUSSIAN VARIABLES

We bring here a simple proof, due to Maurey and Pisier, of Theorem V.1 below which in turn is equivalent to Levy's Lemma 2.3 (up to modification of the constants). We also prove a proposition (V.4) relating the deviation from the mean to the deviation from the median.

V.1. THEOREM: *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function with constant σ ($\mathbb{R}^n$ is endowed with the euclidean metric). Let $g_1, \ldots, g_n$ be independent, mean zero, normalized in L_2, gaussian variables. Then,*

$$P(|F(g_1, \ldots, g_n) - EF(g_1, \ldots, g_n)| > C) \le 2\ exp(\frac{-2C^2}{\pi^2\sigma^2}) .$$

PROOF: We may and shall assume that F is continuously differentiable. Let $H = (h_1, \ldots, h_n)$ be another sequence with the same distribution as $G = (g_1, \ldots, g_n)$ and independent of it. For each $0 \le \theta \le \frac{\pi}{2}$ put $G_\theta = G\ sin\ \theta + H\ cos\ \theta$. Then, by the invariance of the gaussian distribution under orthogonal transformations, G_θ and $\frac{d}{d\theta}G_\theta = G\ cos\ \theta - H\ sin\ \theta$ have the same joint distribution as G and H. Consequently, for any convex function φ

$$\begin{aligned} E\varphi(F(G) - EF(\cdot)) \le E\varphi(F(G) - F(H)) &= E\varphi(\int_0^{\pi/2} \frac{d}{d\theta}F(G_\theta)d\theta) \\ &= E\varphi(\int_0^{\pi/2} (grad\ F(G_\theta),\ \frac{d}{d\theta}G_\theta)d\theta) \\ &\le \frac{2}{\pi}\int_0^{\pi/2} E\varphi(\frac{\pi}{2}(grad\ F(G_\theta),\ \frac{d}{d\theta}G_\theta))d\theta \\ &= E\varphi(\frac{\pi}{2}(grad\ F(G), H)) . \end{aligned}$$

For any $\lambda \in \mathbb{R}$ we get

$$E\ exp(\lambda(F(G) - EF(\cdot))) \le E\ exp(\lambda\frac{\pi}{2}\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}(G) \cdot h_i) .$$

Integrating first with respect to the h_i and using the fact that

$$E\ e^{t\Sigma a_i h_i} = E\ e^{t(\Sigma a_i^2)^{1/2} h_1} = e^{t^2\Sigma a_i^2/2}$$

we get

$$E\ exp(\lambda(F(G) - EF(\cdot))) \leq E\ exp\left(\frac{\lambda^2\pi^2}{8}\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}(G)^2\right) .$$

Now, $\Sigma_{i=1}^n(\frac{\partial F}{\partial x_i}(x))^2 \leq \sigma^2$ and we get

$$E\ exp(\lambda(F(G) - EF(\cdot))) \leq exp(\lambda^2\pi^2\sigma^2/8) .$$

The rest of the proof is now standard.

□

V.2. Next we show how to deduce from Theorem V.1 a similar theorem for S^{n-1}.

COROLLARY: *Let $f : S^{n-1} \to \mathbb{R}$ be a function with Lipschitz constant σ. Then*

$$\mu(|f - \int f d\mu| > C) \leq 4e^{-\delta C^2 n/\sigma^2}$$

where μ is the Haar measure on S^{n-1} and $\delta > 0$ an absolute constant.

PROOF: First notice that, by the invariance of the canonical gaussian measure on $\mathbb{R}^n$ under orthogonal transformations,

$$\frac{1}{(\Sigma_{i=1}^n g_i^2)^{1/2}}(g_1, g_2, \ldots, g_n)$$

has the same distribution as $a = (a_1, \ldots, a_n)$ does with respect to $d\mu(a)$.

Given a function $f : S^{n-1} \to \mathbb{R}$ with Lipschitz constant σ and $\int f d\mu = 0$, define $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ by $\tilde{f}(x) = \|x\| f(\frac{x}{\|x\|})$, where $\|x\| = (\Sigma\ x_i^2)^{\frac{1}{2}}$. Then it is easily checked that the Lipschitz constant of $\tilde{f}$ is at most 3σ.

Now, for any $\delta > 0$,

$$\begin{aligned}
\mu(|f| > C) &= P(f(\frac{g}{\|g\|}) > C) \\
&= P(\|g\|^{-1}|\tilde{f}(g)| > C) = P(|\tilde{f}(g)| > C\|g\|) \\
&\leq P(|\tilde{f}(g)| > \delta C\sqrt{n}) + P(\|g\| < \delta\sqrt{n}) \\
&= P(|\tilde{f}(g)| > \delta C\sqrt{n}) + P(|\|g\| - E\|g\|| > E\|g\| - \delta\sqrt{n}) .
\end{aligned}$$

A simple computation shows that $E\|g\| > 2\delta\sqrt{n}$ for some $\delta > 0$ absolute. For that δ we get from Theorem V.1, applied to $\tilde{f}$ and to $\|\cdot\|$, that

$$\mu(|f| > C) \leq 2e^{-\frac{2\delta^2 C^2 n}{\pi^2 9\sigma^2}} + 2e^{-\frac{2\delta^2 n}{\pi^2}} .$$

Since we may assume $C \leq \sigma$, we get that the first term is dominating and the result follows.

□

V.3. REMARK: The last corollary evaluates the probability of a large deviation of f from its expectation while the results in Section 2 deals with deviation from the median. This is another advantage of the method here. It saves us the trouble of passing from the median to the average (Lemma 5.1). Also, as we shall see below, it is not hard to get, using Theorem V.1, a similar estimate for the deviation from the median.

V.4. We conclude this appendix with a proposition showing that estimation of large deviation from the median is equivalent to the estimation of large deviation from the expectation.

PROPOSITION: *Each of the following four conditions implies the other three, where the constants (K_i, δ_i) depend linearly each on the other.*

1. $\exists A$ *s.t.* $\forall C$

$$P(|f - A| > C) \le K_1 e^{-\delta_1 C^2}$$

2. $\forall C$

$$P(|f - \tilde{f}| > C) \le K_2 e^{-\delta_2 C^2}$$

where f and $\tilde{f}$ are independent and identically distributed.

3. $\forall C$

$$P(|f - Ef| > C) \le K_3 e^{-\delta_3 C^2}$$

4. $\forall C$

$$P(|f - Mf| > C) \le K_4 e^{-\delta_4 C^2}$$

(Mf is the median of f).

More precisely,

$$K_1 \le K_4 \le 2K_3 \le 2K_2 \le 4K_1$$

and

$$\delta_1 \ge \delta_4 \ge \frac{\delta_3}{4} \ge \frac{\delta_2}{8} \ge \frac{\delta_1}{32} .$$

PROOF: $1 \Longrightarrow 2$

$$\begin{aligned} P(|f - \tilde{f}| > C) &\le P(|f - A| > \frac{C}{2}) + P(|\tilde{f} - A| > \frac{C}{2}) \\ &\le 2K_1 e^{-\delta_1 C^2/4} \end{aligned}$$

$2 \Longrightarrow 3$

$$\begin{aligned} E\ exp(\lambda^2 |f - \tilde{f}|^2) &= \int_0^\infty 2\lambda^2 C e^{\lambda^2 C^2} P(|f - \tilde{f}| > C) dC \\ &\le 2K_2 \lambda^2 \int_0^\infty C e^{\lambda^2 C^2 - \delta_2 C^2} dC . \end{aligned}$$

For $\lambda = \sqrt{\frac{\delta_2}{2}}$, we get

$$E\ exp\left(\frac{\delta_2|f-\bar{f}|^2}{2}\right) \le K_2\ .$$

Now, $\varphi(t) = e^{\delta_2 t^2/2}$ is convex, so (as in the proof of V.1),

$$E\ exp(\frac{\delta_2}{2}|f-Ef|^2) \le K_2$$

and

$$\begin{aligned} P(|f-Ef|>C) &\le E\ exp(\frac{\delta_2}{2}|f-Ef|^2 - \frac{\delta_2}{2}C^2) \\ &\le K_2 e^{-\frac{\delta_2}{2}C^2}\ . \end{aligned}$$

$3 \Longrightarrow 4$ is similar to the end of the proof 7.5: Let $C_0 = \sqrt{\frac{log\ 2K_3}{\delta_3}}$. Then

$$P(|f-Ef|>C_0) \le \frac{1}{2}\ .$$

In particular,

$$P(f > C_0 + Ef),\ P(f < Ef - C_0) \le \frac{1}{2}$$

so that

$$Ef - C_0 < Mf < C_0 + Ef\ .$$

Now, for $C \ge 2C_0$,

$$\begin{aligned} P(|f-Mf|>C) &\le P(|f-Ef|>C-C_0) \\ &\le K_3 e^{-\delta_3(C-C_0)^2} \\ &\le K_3 e^{-\frac{\delta_3}{4}C^2}\ . \end{aligned}$$

For $C < 2C_0$,

$$e^{-\frac{\delta_3}{4}C^2} \ge e^{-\delta_3 C_0^2} = (2K_3)^{-1}$$

and

$$P(|f-Mf|>C) \le 1 \le 2K_3 e^{-\frac{\delta_3 C^2}{4}}\ ,$$

so that 4 holds with $\delta_4 = \frac{\delta_3}{4}$ and $K_4 = 2K_3$.

$4 \Longrightarrow 1$ is trivial.

□