L^2 HOLOMORPHIC FUNCTIONS ON PSEUDO-CONVEX COVERINGS.

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0. Preliminaries and main results.

1. Let M be a complex manifold with a non-empty smooth boundary which will be denoted by bM, $\dim_{\mathbf{C}} M = n$. Let us assume that a real-valued C^∞-function $\rho = \rho(z)$ is given in a complex neighbourhood $\tilde{M}$ of $\overline{M} = M \cup bM$, $\dim_{\mathbf{C}} \tilde{M} = n$, so that

$$(0.1) \qquad M = \{z |\ \rho(z) < 0\}, \ \ bM = \{z |\ \rho(z) = 0\}; \ \ \nabla\rho(z) \neq 0 \text{ for all } z \in bM .$$

For any $z \in bM$ denote by $T_z^c(bM)$ the complex tangent space to bM: the maximal complex subspace in the real tangent space $T_z(bM)$, $\dim_{\mathbf{C}} T_z^c(bM) = n-1$. If $z_1, \ldots, z_n$ are complex local coordinates in $\tilde{M}$ near $z \in bM$, then $T_z\tilde{M}$ is identified with $\mathbf{C}^n$ and

$$(0.2) \qquad T_z^c(bM) = \{w = (w_1, \ldots, w_n) |\ \sum_{j=1}^n \frac{\partial \rho}{\partial z_j}(z) w_j = 0\}.$$

The *Levi form* is an hermitian form on $T_z^c(bM)$ defined in the local coordinates as follows:

$$(0.3) \qquad L_z(w, \bar{w}) = \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(z) w_j \bar{w}_k .$$

The manifold M is called ***pseudoconvex*** if $L_z(w, \bar{w}) \geq 0$ for all $z \in bM$ and $w \in T_z^c(bM)$. It is called ***strongly pseudoconvex*** if $L_z(w, \bar{w}) > 0$ for all $z \in bM$ and all $w \neq 0$, $w \in T_z^c(bM)$. In this case replacing ρ by $e^{\lambda\rho} - 1$ with sufficiently large $\lambda > 0$ we can assume that $L_z(w, \bar{w}) > 0$ for all $w \neq 0$ (not only for w satisfying the condition in (0.2)).

Denote by $\mathcal{O}(M)$ the set of all holomorphic functions on M.

A point $z \in bM$ is called a *peak point* for $\mathcal{O}(M)$ if there exists a function $f \in \mathcal{O}(M)$ such that f is unbounded on M but bounded outside $U \cap M$ for any neighbourhood U of z in $\tilde{M}$. A point $z \in bM$ is called a *local peak point* for $\mathcal{O}(M)$ if there exists a function

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$f \in \mathcal{O}(M)$ such that f is unbounded in $U \cap M$ for any neighbourhood U of z in $\tilde{M}$ and there exists a neighbourhood U of z in $\tilde{M}$ such that for any neighbourhood V of z in $\tilde{M}$ the function f is bounded in $U - V$.

The Oka-Grauert theorem ([Gr], see also [F-K], [H]) states that if M is strongly pseudoconvex and $\overline{M}$ is compact, then every point $z \in bM$ is a peak point for $\mathcal{O}(M)$. (Moreover for every $z \in bM$ there exist functions $f_1, \ldots, f_n \in \mathcal{O}(M)$ which are local complex coordinates in $U \cap M$ for a neighbourhood U of z in $\tilde{M}$.) It follows in particular that the space $\mathcal{O}(M)$ is infinite-dimensional.

One of the goals of this paper is to extend this result to the case when $\overline{M}$ is not necessarily compact but admits a free holomorphic action of a discrete group Γ such that the orbit space $\overline{M}/\Gamma$ is compact (or in other words $\overline{M}$ is a regular covering of a compact complex manifold with a strongly pseudoconvex boundary). In this case we shall use the von Neumann Γ-dimension $\dim_\Gamma$ to measure Hilbert spaces of holomorphic functions (or some exterior forms) which are in L^2 with respect to a Γ-invariant smooth measure on $\overline{M}$. In case when the group Γ is trivial (i.e. has only one element) the Γ-dimension is just the usual dimension $\dim_{\mathbf{C}}$. We shall prove that in general case the space of L^2-holomorphic functions on a strongly pseudoconvex regular covering of a compact manifold has an infinite Γ-dimension and every point $z \in bM$ is a local peak point.

Let us choose a boundary point x for a strongly pseudoconvex manifold M and describe the classical E.Levi construction of a locally defined holomorphic function on $U \cap M$ (here U is a neighbourhood of x in $\tilde{M}$) with the peak point x. Let us consider the Taylor expansion of ρ at x:

$$\rho(z) = \rho(x) + 2\mathrm{Re}\, f(x,z) + L_x(z - x, \bar{z} - \bar{x}) + O(|z - x|^3), \tag{0.4}$$

where L_x is the Levi form at x and $f(x, z)$ is a complex quadratic polynomial with respect to z:

$$f(x,z) = \sum_{1 \le \nu \le n} \frac{\partial \rho}{\partial z_\nu}(x)(z_\nu - x_\nu) + \frac{1}{2} \sum_{1 \le \mu,\nu \le n} \frac{\partial^2 \rho}{\partial z_\mu \partial z_\nu}(x)(z_\mu - x_\mu)(z_\nu - x_\nu)\,.$$

The complex quadric hypersurface $S_x = \{z |\ f(x, z) = 0\}$ has $T_x^c(bM)$ as its tangent plane at x. Therefore the strict pseudoconvexity implies that $\rho(z) > 0$ if $f(x, z) = 0$ and $z \neq x$ is close to x. This means that near x the intersection of the hypersurface S_x with $\overline{M}$ consists of one point x. Hence the function $1/f(x, \cdot)$ is holomorphic in $U \cap M$ (where U is a neighbourhood of x in $\tilde{M}$) and x is its peak point.

The technique which allows to pass from locally defined holomorphic functions to global ones is $\bar{\partial}$-cohomologies on complex manifolds. For any integers p, q with $1 \le p, q \le n$ denote by $\Lambda^{p,q}(M)$ the space of all C^∞ forms on M which can be written in local complex coordinates as

$$\omega = \sum_{|I|=p,|J|=q} \omega_{I,J} dz^I \wedge d\bar{z}^J\,,$$

where $dz^I = dz^{i_1} \wedge \ldots \wedge dz^{i_p}$, $dz^J = d\bar{z}^{j_1} \wedge \ldots \wedge d\bar{z}^{j_q}$, $I = (i_1, \ldots, i_p)$, $J = (j_1, \ldots, j_q)$, $i_1 < \ldots < i_p$, $j_1 < \ldots < j_q$, and $\omega_{I,J}$ are C^∞ functions in local coordinates. For such a form ω its $\bar{\partial}$ differential is written as

$$\bar{\partial}\omega = \sum_{|I|=p,|J|=q} \sum_{k=1}^{n} \frac{\bar{\partial}\omega_{I,J}}{\partial \bar{z}^k} d\bar{z}^k \wedge dz^I \wedge d\bar{z}^J ,$$

so $\bar{\partial}$ defines a linear map $\bar{\partial} : \Lambda^{p,q}(M) \longrightarrow \Lambda^{p,q+1}(M)$. All these maps constitute a complex of vector spaces

$$\Lambda^{p,\bullet} : \quad 0 \longrightarrow \Lambda^{p,0} \longrightarrow \Lambda^{p,1} \longrightarrow \ldots \longrightarrow \Lambda^{p,n} \longrightarrow 0 .$$

Its cohomologies are denoted $H^{p,q}(M)$.

An important part of the Grauert theorem is the fact that $\dim_{\mathbf{C}} H^{p,q}(M) < \infty$ for all p, q with $q > 0$ provided M is strictly pseudoconvex and $\overline{M}$ is compact. This fact is used in constructing global holomorphic functions on M with a peak point $x \in bM$ as follows. We start with a locally defined function $g \in \mathcal{O}(U \cap M)$ (here U is a neighbourhood of x in $\tilde{M}$), multiply it by a cut-off function $\chi \in C_0^\infty(U)$ which equals 1 in a neighbourhood of x, then solve the equation $\bar{\partial} f = \bar{\partial}(\chi g)$ on M in appropriate function spaces consisting of bounded functions on M. If we can do this then the function $\chi g - f$ is holomorphic on M and x is its peak point. The solvability of the equation $\bar{\partial} f = \alpha \in \Lambda^{0,1}(M)$ for all forms α with $\bar{\partial}\alpha = 0$ is equivalent to the vanishing of $H^{0,1}(M)$. If we only know that the latter space has a finite dimension then we still can solve the equation $\bar{\partial} f = \alpha$ for all $\bar{\partial}$-closed forms α in the space of finite codimension in the space of all $\bar{\partial}$-closed forms. This is sufficient to construct holomorhpic functions on M with the peak point x because it is easy to provide an infinite-dimensional space of holomorphic functions in a neighbourhood of x having x as its peak point (e.g. we can take a linear space spanned by all powers of one function with the peak point x).

We should be also able to provide bounded solutions of the equation $\bar{\partial} f = \alpha$ provided α is $\bar{\partial}$-closed and sufficiently regular up to the boundary. Therefore we should consider cohomologies $H^{p,q}(M)$ with estimates.

2. Now we shall give a very brief description of the Γ-dimension. It will be used to measure Γ-invariant spaces (of functions and forms) which are infinite-dimensional in the usual sense. It is also convenient to use the Γ-trace. For more details we refer the reader to [A], [C] and textbooks on von Neumann algebras (e.g. [D], [N], [T]).

We shall denote the Γ-dimension by $\dim_\Gamma$. It is defined on the set of all (projective) Hilbert Γ-modules and takes values in $[0, \infty]$. The simplest Hilbert Γ-module is given by a left regular representation of Γ: it is the Hilbert space $L^2\Gamma$ consisting of all complex-valued L^2-functions on Γ. The group Γ acts unitarily on $L^2\Gamma$ by $\gamma \mapsto L_\gamma$ where L_γ is defined as follows:

$$L_\gamma f(x) = f(\gamma^{-1}x), \;\; x \in \Gamma; \quad f \in L^2\Gamma.$$

By definition $\dim_\Gamma L^2\Gamma = 1$.

For any (complex) Hilbert space $\mathcal{H}$ define a free Hilbert Γ-module $L^2\Gamma \bigotimes \mathcal{H}$. Its Γ-dimension equals $\dim_{\mathbf{C}} \mathcal{H}$. The action of Γ in $L^2\Gamma \bigotimes \mathcal{H}$ is defined by $\gamma \mapsto L_\gamma \otimes I$.

A general Hilbert Γ-module is a closed Γ-invariant subspace in a free Hilbert Γ-module. It would be natural to call such subspaces ***projective*** Hilbert modules, but the word " projective" is usually omitted, so only projective Hilbert modules are considered.

For any Hilbert space $\mathcal{H}$ denote by $\mathcal{A}_\Gamma$ a von Neumann algebra which consists of all bounded linear operators in $L^2\Gamma \bigotimes \mathcal{H}$ which commute with the action of Γ there. This algebra is in fact generated by the operators of the form $R_\gamma \otimes B,\ B \in \mathcal{B}(\mathcal{H}),\ \gamma \in \Gamma$, where $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators in $\mathcal{H}$, R_γ is the operator of the right translation in $L^2\Gamma$ i.e.

$$R_\gamma f(x) = f(x\gamma),\ x \in \Gamma;\ \ f \in L^2\Gamma.$$

This means that the algebra $\mathcal{A}_\Gamma$ is the weak closure of all finite linear combinations of the operators of the form $R_\gamma \otimes B$. So in fact $\mathcal{A}_\Gamma$ is a tensor product (in the sense of von Neumann algebras) of $\mathcal{R}_\Gamma$ and $\mathcal{B}(\mathcal{H})$ where $\mathcal{R}_\Gamma$ is the von Neumann algebra generated by the operators R_γ in $L^2\Gamma$ (it consists of all operators in $L^2\Gamma$ which commute with all operators $L_\gamma,\ \gamma \in \Gamma$).

There is a natural trace on $\mathcal{R}_\Gamma$. It is denoted by tr_Γ and defined as the diagonal matrix element (all of them are equal) in the δ-functions basis. For example we can define it by

$$\mathrm{tr}_\Gamma S = (S\delta_e, \delta_e),\quad S \in \mathcal{R}_\Gamma,$$

where e is the neutral element of Γ, $\delta_e \in L^2\Gamma$ is the "Dirac delta-function" at e, i.e. $\delta_e(x) = 1$ if $x = e$ and 0 otherwise. There is also a natural trace on $\mathcal{A}_\Gamma$ too: $\mathrm{Tr}_\Gamma = \mathrm{tr}_\Gamma \otimes \mathrm{Tr}$ where Tr is the usual trace on $\mathcal{B}(\mathcal{H})$.

Now for any Hilbert Γ-module which is a closed Γ-invariant subspace L in $L^2\Gamma \bigotimes \mathcal{H}$, its Γ-dimension is defined by the natural formula

$$\dim_\Gamma L = \mathrm{Tr}_\Gamma P_L,$$

where P_L is the orthogonal projection on L in $L^2\Gamma \bigotimes \mathcal{H}$.

3. Let us describe the spaces of reduced Dolbeault cohomologies on a complex (generally non-compact) manifold M with a given hermitian metric. Denote the Hilbert space of all (measurable) square-integrable (p,q)-forms on M by $L^2\Lambda^{p,q} = L^2\Lambda^{p,q}(M)$. The operator

$$\bar{\partial} : L^2\Lambda^{p,q}(M) \longrightarrow L^2\Lambda^{p,q+1}(M)$$

is defined as the maximal operator i.e. its domain $D^{p,q} = D^{p,q}(\bar{\partial}; M)$ is the set of all $\omega \in L^2\Lambda^{p,q}$ such that $\bar{\partial}\omega \in L^2\Lambda^{p,q+1}$ where $\bar{\partial}\omega$ is applied in the sense of distributions. Obviously $\bar{\partial}^2 = 0$ on $D^{p,q}$ and we can form a complex

$$L^2\Lambda^{p,\bullet} : \quad 0 \longrightarrow D^{p,0} \longrightarrow D^{p,1} \longrightarrow \ldots \longrightarrow D^{p,n} \longrightarrow 0\,.$$

Its cohomologies are denoted $L^2H^{p,q}(M)$ and called L^2 Dolbeault cohomologies of M:

$$L^2H^{p,q}(M) = \mathrm{Ker}\,(\bar{\partial} : D^{p,q} \to D^{p,q+1})/\mathrm{Im}\,(\bar{\partial} : D^{p,q-1} \to D^{p,q})\,.$$

We actually need reduced cohomologies

$$L^2\bar{H}^{p,q}(M) = \mathrm{Ker}\,(\bar{\partial} : D^{p,q} \to D^{p,q+1})/\overline{\mathrm{Im}\,(\bar{\partial} : D^{p,q-1} \to D^{p,q})}\,,$$

where the line over $\mathrm{Im}\,\bar{\partial}$ means its closure in the corresponding L^2 space. Since $\mathrm{Ker}\,\bar{\partial}$ is a closed subspace in L^2, the reduced cohomology space $L^2\bar{H}^{p,q}(M)$ is a Hilbert space.

Note that the space $L^2\bar{H}^{0,0}(M)$ coincides with the space $L^2\mathcal{O}(\mathcal{M})$ of all square-integrable holomorphic functions on M.

4. Let us assume now that M is a complex manifold (with boundary) with a free action of a discrete group Γ on $\overline{M}$ such that $\overline{M}/\Gamma$ is compact and the action is holomorphic on M. (Here $\overline{M} = M \cup bM$.) Let us assume that an hermitian Γ-invariant metric is given on $\overline{M}$. Then the reduced L^2 Dolbeault cohomologies become Hilbert Γ-modules. Hence they have a well defined Γ-dimension (possibly infinity).

Now we will formulate our main results. We will always assume that we are in the situation described above, M is strongly pseudoconvex and bM is nonempty.

Theorem 0.1. $\dim_\Gamma L^2\bar{H}^{p,q}(M) < \infty$ *for all* p, q *provided* $q > 0$.

Theorem 0.2. $\dim_\Gamma L^2\mathcal{O}(M) = \infty$ *and each point in* bM *is a local peak point for* $L^2\mathcal{O}(M)$.

Under the same conditions it is also possible to construct holomorphic functions which have stronger local singularities (not L^2) but are in L^2 in a generalized sense. For any $s \in \mathbf{R}$ denote by $W^s = W^s(M)$ the uniform (Γ-invariant) Sobolev space of distributions on M, based on the space $W^0 = L^2(M)$ constructed with the use of a smooth Γ-invariant measure on $\overline{M}$ (see e.g. [S1] for the details on the Sobolev spaces). The space W^{-s} for large $s > 0$ contains in particular holomorphic functions on M with power singularities at the boundary. For any $s \in \mathbf{R}$ the space W^s is a Hilbert Γ-module with respect to the natural action of Γ. Denote by $W^s\mathcal{O}(M)$ the space of all elements in W^s which are actually holomorphic functions on M. Now we can formulate another version of Theorem 0.2.

Theorem 0.3. *For any* $x \in bM$ *and any integer* $N > 0$ *there exists* $s > 0$ *and a closed* Γ*-invariant subspace* $L \subset W^{-s}\mathcal{O}(M)$ *such that*

(i) $\dim_\Gamma L = N$;

(ii) $L \cap L^2(M) = \{0\}$ *but for any* $f \in L$ *and any* Γ*-invariant neighbourhood* U *of* x *in* $\overline{M}$ *we have* $f \in L^2(M - U)$.

It is also possible to construct L^2-holomorphic functions on M which are in $C^\infty(\overline{M})$:

Theorem 0.4. *For any integer* $N > 0$ *there exists a* Γ*-invariant subspace* $L \subset L^2\mathcal{O}(M) \cap C^\infty(\overline{M})$ *such that* $\dim_\Gamma \bar{L} = N$ *where* $\bar{L}$ *is the closure of* L *in* $L^2(M)$.

Examples. 1) Let X be a compact real-analytic manifold with an infinite fundamental group $\Gamma = \pi_1(X)$. Assume that X is imbedded into its complexification Y and a Riemannian metric is chosen on Y. Let X_ε be a ε-neighbourhood of X in Y where ε is sufficiently small. It is known ([M1], [Gr]) that then X_ε is strongly pseudoconvex. Let M be the universal covering of X_ε. Theorems 0.1–0.4 can be applied to M and we conclude in particular that there are sufficiently many L^2 holomorphic functions on M.

A particular case: strip $\{z|\, |\mathrm{Im}\, z| < 1\}$ in $\mathbf{C}$ with the action of $\Gamma = \mathbf{Z}$ by translations along $\mathbf{R}$. Of course in this case L^2 holomorphic functions can be obtained by the Fourier transform or explicitly (e.g. take $1/(a^2 + z^2)$ where $a > 1$).

2) Let X be a compact complex manifold with a holomorphic positive vector bundle E on X. The positivity means that E is supplied with an hermitian metric and ε-neighbourhood X_ε of X in the total space of E is strongly pseudoconvex (for some $\varepsilon > 0$ or, equivalently, for any $\varepsilon > 0$).

Note that X_ε is not a Stein manifold because it has a non-trivial compact complex submanifold X (the zero section of E). But we are again in the situation of the Theorems 0.1–0.4 and these theorems give extensions of some results of Gromov [Gro] and Napier [Na]. Namely let M be the universal covering of X_ε. Theorems 0.2–0.4 garantee that there are many L^2 holomorphic functions on M. In particular, $\dim_\Gamma L^2\mathcal{O}(M) = \infty$. Spaces of L^2 holomorphic functions with a finite positive Γ-dimension were constructed in [Gro], [Na] from functions which are polynomial along the fibers.

Remarks.

1) If in the assumptions of the theorems above M/Γ is a Stein manifold then Stein [St] proved that M is also a Stein manifold. It follows from this result that there are sufficiently many holomorphic functions on M then, but it does not follow that there exist non-trivial L^2 holomorphic functions. On the other hand it can happen that M/Γ is not Stein (see Example 2 above). Then even the existence of any holomorphic function on M which is not constant along orbits of Γ is not obvious.

2) If $bM = \emptyset$ then it follows from the arguments of Atiyah [A] that $\dim_\Gamma L^2\bar{H}^{p,q}(M) < \infty$ for all p, q (including $q = 0$). In this case in fact $L^2\mathcal{O}(M) = \{0\}$. If E is a positive Γ-invariant holomorphic line bundle, then the Atiyah index theorem [A] implies that $\dim_\Gamma L^2\mathcal{O}(M, E^k) > 0$ for large k; in particular, the space of all holomorphic sections of E^k is infinite-dimensional in the usual sense. (See [Gro] and [Na] for further results.)

3) Theorems 0.2–0.4 remain valid if we replace holomorphic functions by holomorphic $(p, 0)$-forms. More generally all Theorems 0.1–0.4 are true for sections of arbitrary holomorphic vector Γ-bundles over M.

4) Theorems 0.1, 0.3, 0.4 can be extended to the case when M is strongly pseudoconvex but with possibly non-smooth boundary i.e. we can drop the requirement $\nabla\rho \neq 0$ on bM in (0.1) but require instead that the Levi form (0.3) is positive for all $z \in bM$ and all $w \neq 0$, $w \in \mathbf{C}^n$.

5) Let us assume that the Levi form (0.3) is non-degenerate on $T_z^c(bM)$ for all $z \in bM$ and the boundary bM is connected (this is authomatically true if bM is strongly pseudoconvex). Let r be the number of negative eigenvalues of the Levi form in $T_z^c(bM)$. Then

$$\dim_\Gamma L^2\bar{H}^{0,r} = \infty$$

and

$$\dim_\Gamma L^2 \bar{H}^{p,q} < \infty, \ q \neq r .$$

This is a generalization to the covering case of the classical theorems by Andreotti-Grauert and Andreotti-Vesentini (see [A-V], [F-K], [Hö]).

6) First applications of von Neumann algebras to constructions of non-trivial spaces of L^2-holomorphic functions or sections of holomorphic vector bundles are due to M.Atiyah [A] and A.Connes [Co]. J.Roe proved existence of an infinite-dimensional space of L^2 holomorphic sections of a power E^k for a uniformly positive holomorphic line bundle E over a complete Kähler simply connected manifold of non-positive curvature without any action of a discrete group (see [R] for further results and references).

1. $\bar\partial$-cohomologies of pseudoconvex coverings.

1. In this section we will prove Theorem 0.1. We will start by extending the Kohn-Morrey estimates ([F-K],[M2]) to our case. We will always assume that M is strictly pseudoconvex.

First we will consider a general Γ-invariant analytic situation. Namely let M be a C^∞-manifold (possibly with boundary) with a free action of a discrete group Γ such that $\overline{M}/\Gamma$ is compact. Let E be a (complex) vector Γ-bundle on $\overline{M}$ with a Γ-invariant hermitian metric in the fibers of E. We shall use Γ-invariant Sobolev spaces W^s of sections of E over M. The scale of the Hilbert spaces $W^s = W^s(M, E)$ is based on the Hilbert space $L^2(M, E)$ which is taken with respect to a smooth Γ-invariant measure on $\overline{M}$ and the given Γ-invariant hermitian metric on E over $\overline{M}$. Let $\tilde{M}$ be a Γ-invarant complex neighbourhood of $\overline{M}$. Assume that E and the measure on M are extended to $\tilde{M}$ in a smooth Γ-invariant way. For any $s \in \mathbf{R}$ the space $W^s = W^s(M, E)$ is a Hilbert space which consists of all restrictions to M of finite linear combinations of all sections Au where $u \in L^2(\tilde{M}, E)$ and A is a properly supported Γ-invariant pseudodifferential operator of order $-s$ on $\tilde{M}$ (see e.g. [A] or [S1]). The norm in W^s is denoted $\|\cdot\|_s$.

In particular Γ-invariant Sobolev spaces $W^s\Lambda^{p,q}$ of (p,q)-differential forms on M are well defined.

Let us consider $\bar\partial$ as the maximal operator in L^2 and let $\bar\partial^*$ be the Hilbert space adjoint operator. We shall also use the corresponding Laplacian

$$\Box_{p,q} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial \quad \text{on} \quad L^2\Lambda^{p,q}(M) .$$

We shall denote the domain of any operator A by $D(A)$. Let $\Lambda^\bullet_c(\overline{M})$ denotes the set of all C^∞ forms with compact support on $\overline{M}$.

For any complex 1-form α denote by $i(\alpha)$ the substitution operator $i(v)$ of the (complex) vector field v corresponding to the form α with the use of the given hermitian metric, so $i(\alpha) : \Lambda^p(M) \to \Lambda^{p-1}(M)$. In particular we shall use $i(\partial\rho)$ where $\partial : \Lambda^{p,q} \to \Lambda^{p+1,q}$ appears in the standard decomposition $d = \partial + \bar\partial$ for the de Rham differential d.

The following Lemma gives a description of the operators $\bar\partial^*$, $\Box$ (as well as their domains $D(\bar\partial^*)$, $D(\Box)$).

Lemma 1.1. (i) *The operator* $\bar\partial^*$ *can be obtained as the closure from the initial domain*

$$D_0(\bar\partial^*) = \{\omega|\ \omega \in \Lambda^\bullet_c(\overline{M}),\ i(\partial\rho)\omega = 0 \text{ on } bM\}\ . \tag{1.1}$$

(ii) *The operator* $\Box = \Box_{p,q}$ *can be obtained as the closure from the initial domain*

$$D_0(\Box) = \{\omega|\ \omega \in \Lambda^\bullet_c(\overline{M}),\ i(\partial\rho)\omega = 0 \text{ and } i(\partial\rho)\bar\partial\omega = 0 \text{ on } bM\}\ . \tag{1.2}$$

Proof. The proof is a simple combination of the one given by Hörmander [Hö] in the compact case and Gaffney's cut-off trick (see [G], [Gro]). □

Remark. The conditions on ω in (1.2) are called the *$\bar\partial$-Neumann conditions.*

Proposition 1.2. *The domain* $D(\Box_{p,q})$, $q > 0$, *is included into* $W^{1/2}$ *and there exists a constant* $C > 0$ *such that the following estimate is true*

$$\|\omega\|^2_{1/2} \le C((\Box_{p,q}\omega, \omega) + \|\omega\|^2_0),\quad \omega \in D(\Box_{p,q}),\ q > 0\ .$$

Proof. Summing up the corresponding local Morrey estimates from [M1], [M2] with the use of a Γ-invariant partition of unity we obtain the following fundamental inequality:

$$\|\omega\|^2_{L^2(bM)} \le C((\Box_{p,q}\omega, \omega) + \|\omega\|^2_0),\quad \omega \in \Lambda^{p,q}_c(\overline{M}),\ q > 0\ .$$

This inequality, the Kohn-type estimate (see [F-K])

$$\|u\|^2_{1/2} \le C(\|u\|^2_{L^2(bM)} + \|\bar\partial u\|^2_{L^2(M)}),\ u \in C^\infty(\overline{M}),$$

and Lemma 1.1 imply the necessary statement. □

Corollary 1.3. *There exists* $C > 0$ *such that*

$$\|\omega\|_{1/2} \le C\|\omega\|_0\ ,\ \omega \in \operatorname{Ker} \Box_{p,q},\ q > 0\ ,$$

where $\operatorname{Ker} \Box = \{\omega|\ \omega \in D(\Box),\ \Box\omega = 0\}$.

2. Let us formulate the necessary version of the Hodge-Kodaira decomposition (see e.g. [F-K]):

Proposition 1.4. *The following orthogonal decompositions hold:*

$$L^2\Lambda^\bullet(M) = \overline{\operatorname{Im}\bar\partial} \oplus \operatorname{Ker} \Box \oplus \overline{\operatorname{Im}\bar\partial^*} \tag{1.3}$$

and

$$\operatorname{Ker}\bar\partial = \overline{\operatorname{Im}\bar\partial} \oplus \operatorname{Ker} \Box\ .$$

In particular we have an isomorphism of Hilbert Γ*-modules*

$$L^2\bar H^{p,q}(M) = \operatorname{Ker} \Box_{p,q}\ .$$

We shall use the following rather general

Lemma 1.5. *Let L be a closed Γ-invariant subspace in $L^2(M,E)$, $L \subset W^\varepsilon$ for some $\varepsilon > 0$ and there exists $C > 0$ such that*

$$\|u\|_\varepsilon \leq C\|u\|_0, \ u \in L. \tag{1.4}$$

Then $\dim_\Gamma L < \infty$.

To prove this Lemma we need the following simple Lemma about estimates of Sobolev norms on compact manifolds with boundary.

Lemma 1.6. *Let X be a compact Riemannian manifold, possibly with a boundary. Let E be a (complex) vector bundle with an hermitian metric over $\overline{X}$. Denote by $(\cdot,\cdot)$ the induced hermitian inner product in the Hilbert space $L^2(X,E)$ of square-integrable sections of E over X. Denote by $W^s = W^s(X,E)$ the corresponding Sobolev space of sections of E over X, $\|\cdot\|_s$ the norm in this space. Let us choose a complete orthonormal system $\{\psi_j;\ j=1,2,\dots\}$ in $L^2(X,E)$. Then for all $\varepsilon > 0$ and $\delta > 0$ there exists an integer $N > 0$ such that*

$$\|u\|_0 \leq \delta\|u\|_\varepsilon \text{ provided } u \in W^\varepsilon \text{ and } (u,\psi_j) = 0,\ j = 1,\dots,N.$$

Proof. Assuming the opposite we conclude that there exist $\varepsilon > 0$ and $\delta > 0$ such that for every $N > 0$ there exists $u_N \in W^\varepsilon$ with $(u_N,\psi_j) = 0,\ j = 1,\dots,N$ satisfying the estimate $\|u_N\|_\varepsilon \leq \delta^{-1}\|u_N\|_0$. Normalizing u_N we can assume that $\|u_N\|_0 = 1$, so the previous estimate gives $\|u_N\|_\varepsilon \leq \delta^{-1}$ for all N. It follows from the Sobolev compactness theorem that the set $\{u_N|\ N = 1,2,\dots\}$ is compact in $L^2 = L^2(X,E)$. On the other hand obviously $u_N \to 0$ weakly in L^2 as $N \to \infty$. Therefore $\|u_N\|_0 \to 0$ as $N \to \infty$ which contradicts to the chosen normalization. □

Proof of Lemma 1.5. Let us choose a Γ-invariant covering of M by balls γB_k, $k = 1,\dots,m$, $\gamma \in \Gamma$, so that all the balls have smooth boundary (e.g. have sufficiently small radii). Let us choose a complete orthonormal system $\{\psi_j^{(k)}\ j = 1,2,\dots\}$ in $L^2(B_k,E)$ for every $k = 1,\dots N$. Then $\{(\gamma^{-1})^*\psi_j^{(k)},\ j = 1,2,\dots\}$ will be an orthonormal system in γB_k (here we identify the element γ with the corresponding transformation of M).

Given the subspace L satisfying the conditions in the Lemma let us define a map

$$P_N : L \longrightarrow L^2\Gamma \otimes \mathbf{C}^{mN}$$

$$u \mapsto \{(u,(\gamma^{-1})^*\psi_j^{(k)}),\ j = 1,2,\dots,N;\ k = 1,\dots,m;\ \gamma \in \Gamma\}.$$

Since $\dim_\Gamma L^2\Gamma \otimes \mathbf{C}^{mN} = mN < \infty$ the desired result will follow if we prove that P_N is injective for large N. Assume that $u \in L$ and $P_N u = 0$. Using Lemma 1.6 we get then

$$\|u\|_{0,\gamma B_k}^2 \leq \delta_N^2\|u\|_{\varepsilon,\gamma B_k}^2, \quad k = 1,\dots,m;\ \gamma \in \Gamma,$$

where $\delta_N \to 0$ as $N \to \infty$ and $\|u\|_{s,\gamma B_k}$ means the norm in the Sobolev space W^s over the ball γB_k. Summing over all k and γ we get

$$\|u\|_0^2 \leq C_1^2\delta_N^2\|u\|_\varepsilon^2,$$

where $C_1 > 0$ does not depend on N. This clearly contradicts (1.4) unless $u = 0$. □

Remark 1.7. It is not necessary to require that L is closed in L^2 in Lemma 1.5. For any L satisfying (1.4) we can consider its closure $\bar{L}$ in L^2. Then obviously $\bar{L} \subset W^\varepsilon$ and Lemma 1.5 implies that $\dim_\Gamma \bar{L} < \infty$.

Proof of Theorem 0.1. Propositions 1.2, 1.4 and Lemma 1.5 immediately imply Theorem 0.1. □

2. L^2 holomorphic functions.

1. We shall use some simple linear algebra and Γ-Fredholm operators in Hilbert Γ-modules. Necessary background and similar arguments can be found in [B] and [S2].

Lemma 2.1. *Let L be a Hilbert Γ-module, L_1, L_2 its Hilbert Γ-submodules such that $\dim_\Gamma L_1 >$ $codim_\Gamma L_2$ where $codim_\Gamma L_2$ means the Γ-dimension of the orthogonal complement of L_2 in L. Then $L_1 \cap L_2 \neq \{0\}$. Moreover*

$$\dim_\Gamma L_1 \cap L_2 \geq \dim_\Gamma L_1 - \operatorname{codim}_\Gamma L_2 \ . \tag{2.1}$$

Proof. Denote by $L_1 \ominus L_2$ the orthogonal complement of $L_1 \cap L_2$ in L_1. Clearly $\dim_\Gamma L_1 \ominus L_2 \leq \operatorname{codim}_\Gamma L_2$. Therefore if (2.1) is not true, then we get

$$\dim_\Gamma L_1 = \dim_\Gamma L_1 \cap L_2 + \dim_\Gamma L_1 \ominus L_2 \leq \dim_\Gamma L_1 \cap L_2 + \operatorname{codim}_\Gamma L_2 < \dim_\Gamma L_1$$

which is a contradiction. □

We will use unbounded Γ-Fredholm operators. The corresponding definition slightly extends the corresponding definition for bounded operators given by M.Breuer [B] (see also [S2]).

Definition 2.2. Let L_1, L_2 be Hilbert Γ-modules, $A : L_1 \to L_2$ a closed densely defined linear operator (with the domain $D(A)$) which commutes with the action of Γ in L_1 and L_2. The operator A is called *Γ-Fredholm* if the following conditions are satisfied:

(i) $\dim_\Gamma \operatorname{Ker} A < \infty$;

(ii) there exists a closed Γ-invariant subspace $Q \subset L_2$ such that $Q \subset \operatorname{Im} A$ and $\operatorname{codim}_\Gamma Q (= \dim_\Gamma (L_2 \ominus Q)) < \infty$.

Let us also recall the following definition from [S2]:

Definition 2.3. Let L be a Hilbert Γ-module, $Q \subset L$ is a Γ-invariant subspace (not necessarily closed). Then

(i) Q is called *Γ-dense* in L if for every $\varepsilon > 0$ there exists a Γ-invariant subspace $Q_\varepsilon \subset Q$ such that Q_ε is closed in L and $\operatorname{codim}_\Gamma Q_\varepsilon < \varepsilon$ in L.

(ii) Q is called *almost closed* if Q is Γ-dense in its closure $\bar{Q}$.

If Q is Γ-dense in L then it is also dense in L in the usual sense i.e. $\bar{Q} = L$ (see Lemma 1.8 in [S2]). Note also that if Γ is trivial (or finite) then Q is Γ-dense in L if and only if $Q = L$ (in particular in this case Q is almost closed if and only if it is closed).

Lemma 2.4. *If $A : L_1 \to L_2$ is a Γ-Fredholm operator then $\operatorname{Im} A$ is almost closed.*

Proof. This statement can be reduced to the case when A is bounded by replacing L_1 by $D(A)$ with the graph norm. Then the statement is due to M.Breuer [B] (see also Lemma 1.15 in [S2]). $\square$

Lemma 2.5. ([S2]) *Let L be a Hilbert Γ-module, $L_1 \subset L$ and $Q \subset L$ its Γ-invariant subspaces in L such that L_1 is closed and Q is Γ-dense in L. Then $Q \cap L_1$ is Γ-dense in L_1. More generally, if Q is almost closed then $Q \cap L_1$ is almost closed and its closure equals $\bar{Q} \cap L_1$.*

Corollary 2.6. *Let $A : L_1 \to L_2$ be a Γ-Fredholm operator, $L_3 \subset L_2$ is a closed Γ-invariant subspace such that $L_3 \subset \overline{\operatorname{Im} A}$. Then $L_3 \cap \operatorname{Im} A$ is Γ-dense in L_3.*

Now let us return to the analytic situation described above.

Proposition 2.7. *Let A be a self-adjoint linear operator in $L^2(M, E)$ such that A commutes with the action of Γ, $D(A) \subset W^\varepsilon$ where $\varepsilon > 0$ and*

$$\|u\|_\varepsilon^2 \leq C(\|Au\|^2 + \|u\|_0^2), \quad u \in D(A) . \tag{2.2}$$

Then A is Γ-Fredholm.

Proof. It follows from (2.2) that the estimate (1.4) is satisfied on $L = \operatorname{Ker} A$. Therefore Lemma 1.5 implies that $\dim_\Gamma \operatorname{Ker} A < \infty$.

Let $\tilde{E}_\delta$ be the spectral projection of A corresponding to the interval $(-\delta, \delta)$. Then again $\dim_\Gamma \operatorname{Im} \tilde{E}_\delta < \infty$ by Lemma 1.5. On the other hand

$$\operatorname{Im}(I - \tilde{E}_\delta) = (\operatorname{Im} \tilde{E}_\delta)^\perp \subset \operatorname{Im} A ,$$

which immediately implies that A is Γ-Fredholm. $\square$

2. Now using Theorem 0.1 we will be able to provide the complete proof of Theorem 0.2. We shall start with the following elementary

Lemma 2.8. *Let U be an arbitrary set, $g : U \to \mathbf{C}$ an unbounded function. Then for any integer $N > 0$ the functions $g, g^2, \ldots, g^N$ are linearly independent modulo bounded functions i.e. if $B(U)$ is the space of all bounded functions on U and*

$$c_1 g + c_2 g^2 + \ldots + c_N g^N \in B(U) , \tag{2.3}$$

then $c_1 = \ldots c_N = 0$.

Proof. Assuming that (2.3) is fulfilled consider the polynomial

$$p(t) = c_1 t + c_2 t^2 + \ldots c_N t^N, \ t \in \mathbf{C}.$$

Then (2.3) implies that this polynomial is bounded along an unbounded sequence of complex values of t. Clearly this is only possible if the polynomial p is identically 0. $\square$

Proof of Theorem 0.2. We shall use the notations from the introduction to this paper.

Let us choose a defining function ρ of the manifold M (see (0.1)) so that the Levi form (0.3) is positive for all $w \in \mathbf{C}^n - \{0\}$ (and not only for $w \in T_z^c(bM) - \{0\}$) at all points $z \in bM$. Using (0.4) we see that $\operatorname{Re} f(x,z) < 0$ if $x \in bM$ and $z \in M$ is sufficiently close to x. It follows that we can choose a branch of $\log f(x,z)$ so that $g_x(z) = \log f(x,z)$ is a holomorphic function in $z \in M \cap U_x$ where U_x is a sufficiently small neighbourhood of x in $\overline{M}$. Note that we can (and will) choose $U_{\gamma x} = \gamma U_x$.

Let us fix an arbitrary point $x \in bM$. Clearly $g_x^m \in L^2(\overline{M} \cap U)$ for all $m = 1, 2, \ldots$ and all functions g_x^m have a peak point at x. Besides all these functions are linearly independent modulo bounded functions by Lemma 2.8.

Let us choose a cut-off function $\chi \in C_c^\infty(U)$ where U is a sufficiently small neighbourhood of x, so that $\chi = 1$ in a neighbourhood of x. We shall identify χ with its extension by 0 to $\overline{M}$, so it becomes a function from $C_c^\infty(\overline{M})$. The translation of χ by $\gamma \in \Gamma$ is a function $\gamma^*\chi$ which is supported in a small neighbourhood of γx: $\gamma^*\chi(z) = \chi(\gamma^{-1}z)$.

Denote by L the closed Γ-invariant subspace in $L^2(M)$ generated by all functions χg_x^m; $m = 1, \ldots, N$. Clearly

$$L = \{f |\, f = \sum_{\gamma\in\Gamma} \sum_{m=1}^{N} c_{m,\gamma} \gamma^*(\chi g_x^m);\ \sum_{m,\gamma} |c_{m,\gamma}|^2 < \infty\}\ , \tag{2.4}$$

where $c_{m,\gamma}$ are complex constants. It follows that L has the form $L^2\Gamma \otimes \mathbf{C}^N$, hence $\dim_\Gamma L = N$.

Let us consider the set of (0,1)-forms (which are smooth on $\overline{M}$ and have compact support):

$$\bar\partial(\chi g_x^m);\ m = 1, 2, \ldots, N\ . \tag{2.5}$$

They are linearly independent for any integer $N > 0$. Indeed, assuming that

$$c_1\bar\partial(\chi g_x) + c_2\bar\partial(\chi g_x^2) + c_N\bar\partial(\chi g_x^N) = 0$$

with some complex constants $c_1, \ldots, c_N$, we see that

$$c_1\chi g_x + c_2\chi g_x^2 + \ldots + c_N\chi g_x^N$$

is holomorphic on M and has a compact support, hence it is identically 0, which implies that $c_1 = \ldots = c_N = 0$ due to Lemma 2.8.

Let L_1 be a closed Γ-invariant subspace in $L^2\Lambda^{0,1}(M)$ generated by the set of forms (2.5). Then again

$$L_1 = \{\omega |\, \omega = \sum_{\gamma\in\Gamma} \sum_{m=1}^{N} c_{m,\gamma}\bar\partial(\gamma^*(\chi g_x^m)),\ \ \sum_{m,\gamma} |c_{m,\gamma}|^2 < \infty\}\ ,$$

where $c_{m,\gamma}$ are complex constants, and $\dim_\Gamma L_1 = N$. Clearly $L_1 \subset C^\infty\Lambda^{0,1}(\overline{M})$ i.e. all elements of L_1 are C^∞ forms of type (0,1) on $\overline{M}$. Also $L_1 \subset \operatorname{Im}\bar\partial$, hence $L_1 \subset \overline{\operatorname{Im}\,\square}$ due to the orthogonal decomposition (1.3).

Now we can apply Corollary 2.6 to conclude that $\operatorname{Im}\,\square \cap L_1$ is Γ-dense in L_1. Hence for any $\delta > 0$ there exists a closed Γ-invariant subspace $Q_1 \subset L_1$ such that $Q_1 \subset \operatorname{Im}\,\square$ and $\dim_\Gamma Q_1 > N - \delta$. Solving the equation $\square\omega = \alpha$ with $\alpha \in Q_1$ we can assume that $\omega \perp \operatorname{Ker}\,\square$ and in this case the solution ω will be unique. Denote the space of all such solutions by K. Then $\dim_\Gamma K = \dim_\Gamma Q_1 > N - \delta$.

Applying $\bar\partial$ to both sides of the equation $\square\omega = \alpha$ we see that $\bar\partial\bar\partial^*\bar\partial\omega = 0$, hence $\bar\partial^*\bar\partial\omega = 0$ and $\bar\partial\omega = 0$. Therefore $\bar\partial\bar\partial^*\omega = \alpha$. Also $\omega \in \Lambda^{0,1}(\overline{M})$ (i.e. $\omega \in C^\infty$ on $\overline{M}$) for any such solution ω due to the local regularity theorem for the $\bar\partial$ Neumann problem (see [F-K]).

Now denote

$$Q = \{f \mid f \in L, \bar\partial f = \alpha \in Q_1\}\,.$$

As we have seen earlier $\bar\partial$ is injective on L, hence $\dim_\Gamma Q = \dim_\Gamma Q_1 > N - \delta$. If $f \in Q$ then we can find a (unique) solution $\omega \in K$ of the equation $\square\omega = \alpha = \bar\partial f$ and then $h = f - \bar\partial^*\omega \in L^2\mathcal{O}(M)$. All these functions h form a closed Γ-invariant subspace $H \subset L^2\mathcal{O}(M)$ with $\dim_\Gamma H > N - \delta$. Hence $\dim_\Gamma L^2\mathcal{O}(M) = \infty$. Besides using the Γ-invariance of H we see that we can always find a function $h \in H$ such that one of the coefficients $c_{m,e}; m = 1, \ldots, N$, in the expansion (2.4) (for the corresponding function f) does not vanish. The point x will be a local peak point for this function. This completes the proof of Theorem 0.2. $\square$

Proof of Theorem 0.3. We should modify the proof of Theorem 0.2 by another choice of locally given holomorphic functions with singularities at a point $x \in bM$. Namely, if f is a holomorphic polynomial from (0.4), then we should use $\{f^{-k}, f^{-2k}, \ldots, f^{-kN}\}$ with sufficiently large integer $k > 0$ instead of $\{\log f, \ldots, (\log f)^N\}$ as we did in the proof of Theorem 0.2. It is easy to check that all functions χf^{-k} are in appropriate Sobolev spaces. Then we should apply Lemma 2.1 to evaluate the Γ-dimension of the intersection $L_1 \cap L_2$ where L_1 is the Γ-invariant subspace generated by all forms $\bar\partial(\chi f^{-km})$, $m = 1, \ldots, N$, and $L_2 = \overline{\operatorname{Im}\bar\partial}$ in $L^2\Lambda^{0,1}(M)$.

All other arguments are similar to the ones used in the proof of Theorem 0.2. $\square$

Remark. An interesting feature of Theorem 0.3 is that its proof does not use the regularity results for the $\bar\partial$-Neumann problem and so this theorem can be extended to a number of less regular situations.

Proof of Theorem 0.4. We should apply the arguments given in the proof of Theorem 0.3 to a strongly pseudoconvex Γ-invariant neighbourhood $\hat{M}$ of $\overline{M}$, find a sufficiently large space H of holomorphic functions on $\hat{M}$ with singularities on the boundary of $\hat{M}$ and then take the space L of restrictions of all functions from H to M. Since the restriction operator is injective the closure of L will have the same Γ-dimension as H. $\square$

3. Open questions.

Here we give a list of open questions of various difficulty. It is assumed in all questions that we are in the situation of Theorems 0.1–0.4.

1. Does there exist a finite number of functions in $L^2\mathcal{O}(M) \cap C(\overline{M})$ which separate all points in bM?

2. Does there exist $f \in L^2(M) \cap C(\overline{M})$ such that $f(x) \neq 0$ for all $x \in bM$?

3. Is it true that for every CR-function $f \in L^2(bM) \cap C(bM)$ ($\bar{\partial}_b f = 0$) there exists $F \in L^2\mathcal{O}(M) \cap C(\overline{M})$ such that $F|_{bM} = f$?

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