Société Mathématique de France
Astérisque, hors série, 1985, p. 129-133

ISOMETRIC IMMERSIONS OF RIEMANNIAN MANIFOLDS

BY

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Denote by $\mathcal{G}_r$ the space of C^r-smooth quadratic differential forms g on a smooth manifold V (that are C^r-sections of the symmetric square of the cotangent bundle of V) and let $\mathcal{F}_r^q$ be the space of C^r-maps $f : V \to \mathbf{R}^q$. Denote by $\mathcal{D} : \mathcal{F}_r^q \to \mathcal{G}_{r-1}$, for $r \geq 1$, the (first order non-linear differential) operator which assigns to each f the *induced* quadratic form g on V, that is

$$g(\partial, \partial') = \langle D_f \partial, D_f \partial' \rangle, \tag{1}$$

for all pairs of tangent vectors ∂ and ∂' in $T_v(V)$, $v \in V$, where

$$D_f : T(V) \to T(\mathbf{R}^q)$$

stands for the differential of f and where $\langle\ ,\ \rangle$ denotes the Euclidean scalar product in $T_w(\mathbf{R}^q) = \mathbf{R}^q$ for all $w = f(v) \in \mathbf{R}^q$. The relation (1) can be expressed in local coordinates $u_1, \ldots, u_n$ on V, for $n = \dim V$, by $p = n(n+1)/2$ equations in the partial derivatives

$$\partial_i f = \frac{\partial}{\partial u_i} f, \quad i = 1, \ldots, n,$$

$$g(\partial_i, \partial_j) = \langle \partial_i f, \partial_j f \rangle, \qquad 1 \leq i \leq j \leq n. \tag{2}$$

Observe that the induced form g is always positive semi-definite : it is positive definite if and only if f is an *immersion*, (i.e. the differential D_f is injective on $T_v(V)$ for all $v \in V$). Thus $\mathcal{D}$ restricts to an operator from immersions to positive forms, called

$$\mathcal{D}_+ : \mathrm{Im}_r \to \mathcal{G}^+_{r-1}.$$

The study of $\mathcal{D}_+$ was originally motivated by the *isometric immersion problem* asking for a solution f to the equation $\mathcal{D}_+ f = g$ for a given Riemannian metric $g \in \mathcal{G}^+$. This question was raised by SCHLÄFLI in 1873 and one probably believed at that time that the existence of an isometric immersion $f : (V, g) \to \mathbf{R}^q$ might be helpful in the study of the intrinsic geometry of (V, g). Although this belief has not materialized so far, the operator $\mathcal{D}_+$ has turned out to be an amusing non-linear specimen worth of a study in its own right. An essential feature of $\mathcal{D}$ is the abundance of *characteristic directions.* In fact, *all directions* are characteristic for $\mathcal{D}$ if $\dim V \geq 2$. Namely, for no hypersurface $V_0 \subset V$ the initial value problem

$$\mathcal{D} f = g\,, \qquad f \mid V_0 = f_0 \tag{3}$$

can be solved unless the initial map $f_0 : V_0 \to \mathbf{R}^q$ satisfies certain differential equations of its own. Indeed, if the map $f : (V, g) \to \mathbf{R}^q$ is isometric, then f_0 is isometric for the restricted metric $g_0 = g \mid V_0$ on V_0. Now if f_0 *is* isometric, then the system (3) can be locally solved in the real analytic case by applying the Cauchy-Kovalevska theorem to an auxiliary second order system obtained by differentiating (1) *two* times and then by eliminating the third derivatives with an appropriate anti-symmetrization. To see how it works, we assume, for the sake of simplicity, the manifold V to be the metric product,

$$(V, g) = (V_0 \times \mathbf{R}, g_0 + dt^2)$$

and we write the equations (2) as follows :

$$\begin{aligned} &\langle \partial_i f, \partial j f \rangle = g_{ij} = g(\partial_i, \partial_j)\,, \quad 1 \leq i, j \leq n-1, \\ &\langle \partial_i f, \partial_t f \rangle = 0\,, \qquad \langle \partial_t f, \partial_t f \rangle = 1, \end{aligned} \tag{4}$$

where ∂_t stands for $\partial / \partial t$, and where the functions g_{ij} on $V = V_0 \times \mathbf{R}$ are constant in t. Hence

$$0 = \partial_t \langle \partial_i f, \partial_j f \rangle = \langle \partial_{it} f, \partial_j f \rangle + \langle \partial_i f, \partial_{jt} f \rangle.$$

Next,

$$0 = \partial_j \langle \partial_i f, \partial_t f \rangle = \langle \partial_{ij} f, \partial_t f \rangle + \langle \partial_i f, \partial_{jt} f \rangle.$$

Now, by alternating i and j we obtain with the above,

$$\langle \partial_{ij} f, \partial_t f \rangle = 0. \tag{5}$$

which implies by differentiating in t,

$$\langle \partial_{ij} f, \partial_{tt} f \rangle + \langle \partial_t f, \partial_{ijt} f \rangle = 0. \tag{5'}$$

On the other hand

$$0 = \partial_{ij}\langle \partial_t f, \partial_t f\rangle = 2(\langle \partial_t f, \partial_{ijt} f\rangle + \langle \partial_{it} f, \partial_{jt} f\rangle).$$

Therefore,

$$\langle \partial_{tt} f, \partial_{ij} f\rangle = \langle \partial_{ij} f, \partial_{jt} f\rangle. \tag{6}$$

Finally, we differentiate in t the last two equations in (4) and obtain

$$\begin{aligned} \langle \partial_{tt} f, \partial_i f\rangle &= -\langle \partial_t f, \partial_{it} f\rangle, \\ \langle \partial_{tt} f, \partial_t f\rangle &= 0. \end{aligned} \tag{7}$$

LEMMA. — *A C^3-map $f : V_0 \times \mathbf{R} \to \mathbf{R}^q$ satisfies the equation $\mathcal{D}f = g = g_0 + dt^2$ if and only if it satisfies the system $(6)+(7)$ (which consists of $p = n(n+1)/2$ equations of the second order) as well as the following initial value conditions on $V_0 = V_0 \times 0$:*

$$\begin{aligned} &\langle \partial_i f, \partial_j f\rangle = g_0(\partial_i, \partial_j), \qquad \langle \partial_t f, \partial_t f\rangle = 1, \\ &\langle \partial_t f, \partial_i f\rangle = 0, \qquad \langle \partial_t f, \partial_{ij} f\rangle = 0. \end{aligned} \tag{8}$$

Proof. — The "only if" claim (i.e. the implication (2) $\Rightarrow$ (6) + (7) + (8)) has been already established. The "if" part follows by reversing the above computation.

COROLLARY. — *Let $f_0 : (V_0, g_0) \to \mathbf{R}^q$ be a real analytic isometric immersion whose derivatives $\partial_i f_0$, $\partial_{ij} f_0$, $1 \leq i, j \leq n-1$, are linearly independent at every point $v_0 \in V$. Then, if V_0 is a contractible manifold and if $q \geq p = n(n+1)/2$, there is a neighborhood $U \subset V_0 \times \mathbf{R}$ of $V_0 = V_0 \times 0$ which admits a real analytic isometric immersion $f : (U, g) \to \mathbf{R}^q$, such that $f \mid V_0 = f_0$.*

Proof. — The immersed manifold $f(V_0) \subset \mathbf{R}^q$ admits, under our assumptions, a real analytic unit vector field $X_0 : V_0 \to T(\mathbf{R}^q) \mid V_0$ which is normal to the vectors $\partial_i f_0$ and $\partial_{ij} f_0$, $1 \leq i, j \leq n-1$ at every point $v_0 \in V_0$. Then the initial data $f \mid V_0 = f_0$ and $\partial_t f \mid V_0 = X_0$ satisfy the assumption of the lemma. Furthermore, the independence of $\partial_i f_0$, $\partial_{ij} f_0$ and X_0 at all points $v_0 \in V$, allows one to resolve the system (7) in $\partial_{tt} f$, and then to apply the Cauchy-Kovaleskaya theorem (see 3.1.2. in [G] for details).

These considerations are due to JANET (see [J], [Bu]) who applied them to an arbitrary metric g and thus proved by induction in n the following.

THEOREM [JANET, 1926]. — *If a metric g on V is real analytic, then a small neighborhood $U \subset V$ of any given point $v_0 \in V$ admits a real analytic isometric immersion $(U, g) \to \mathbf{R}^p$ for $p = n(n+1)/2$.*

(This result was also proven by É. CARTAN by a somewhat different method; see [C] and [B]).

Observe that a *generic* real analytic manifold (V, g) admits no analytic (not even C^∞) immersion into $\mathbf{R}^q$ for $q < p = n(n+1)/2$. This is seen by viewing the (infinite dimensional!) space $\mathcal{F}^q$ of maps $V \to \mathbf{R}^q$ as a q-dimensional variety while the space g of metrics on V is assigned dimension $p = n(n+1)/2$ (as metrics are sections of a p-dimensional bundle over V. Infact a simple application of Sard's theorem to finite dimensional jet spaces shows (see [G-R]) the image $\mathcal{D}(\mathcal{F}^q_\infty) \subset \mathcal{G}_\infty$ to be a meager subset in $\mathcal{G}_\infty$.

A similar consideration suggests the following *rigidity* of generic immersions $V \to \mathbf{R}^q$ for $q < p$. To state this we divide $\mathcal{F}^q$ by the group Is of isometries of $\mathbf{R}^q$ and observe $\mathcal{D}$ to admit a factorization to an operator

$$\overline{\mathcal{D}} : \mathcal{F}^q_\infty / \operatorname{Is} \to \mathcal{G}_\infty.$$

CONJECTURE. — *If $q < p = n(n+1)/2$ then the operator $\overline{\mathcal{D}}$ is one-to-one on an open dense subset in $\mathcal{F}^q_\infty/\operatorname{Is}$. (See [B] for the recent progress in the rigidity problem).*

The above conjecture claims the double points of the map (operator) $\overline{\mathcal{D}}$ to be nowhere dense in the C^∞-topology.

Yet the subset of double points is expected to be quite substantial for $p < 2q$. The following result (see § 3.3.4. in [G]) shows this subset to be C^0-dense for $p \ll 2q$.

THEOREM. — *If $q \geq p/2 + 2n + 2$ then arbitrary continuous maps f_1 and f_2 of V into $\mathbf{R}^q$ admit C^0-approximations by real analytic immersions, say by f_1', and by f_2' respectively, such that $\mathcal{D}f_1' = \mathcal{D}f_2'$.*

Now we turn to the global isometric immersion problem for $q \geq p$. If $q = p$, the structure of $\mathcal{D}^+$ (and, in particular, of the image $\mathcal{D}^+(\mathrm{Im}_\infty) \subset \mathcal{G}^+_\infty$) appears formidably complicated. Yet, for $q > p$, one expects, the operator $\mathcal{D}^+$ to behave like a reasonable smooth map of a q-dimensional variety to a p-dimensional one. In particular, one expects the following dual to the rigidity conjecture.

CONJECTURE. — *If $q > p$, then there is an open dense subset Ω in Im_∞ on which the operator $\mathcal{D}^+$ is a submersion (in particular an open map) with infinite dimensional fibers.*

The truth of this conjecture for $q \geq p + 2n$ was established by J. NASH (see [N_2]) in 1956 in the course of his solution of the isometric immersion problem. Namely, NASH considers *free* maps $f : V \to \mathbf{R}^q$ whose osculating spaces (generated by the derivatives $\partial_i f$ and $\partial_i \partial_j f$) have dimension $n+p$ at all points $v \in V$. Then he proves the above conjecture for $\Omega =$ the subspace of free maps in Im_∞. Moreover, the techniques in § 3.1. of [G] show *the map* $\mathcal{D}^+ : \Omega \to \mathcal{G}^+_\infty$ *to be a Serre fibration* for $q \geq p + 2n + 3$. This immediately implies the following

ISOMETRIC IMMERSION THEOREM (see § 3.1.7. in [G]). — *Every C^∞-smooth riemannian manifold V admits an isometric C^∞-immersion into $\mathbf{R}^q$ for $q = p+2n+3$.* (One does not know what happens for $p \leq q < p+2n+3$).

The above immersion theorem remains valid for real analytic manifolds and maps but the image

$$D_+(\mathrm{Im}_r) \subset \mathcal{G}^+_{r-1}$$

is poorly understood for $1 \leq r < \infty$. However, the isometric immersion problem for $r = 0$ admits the following solution (see $[N_1]$, [K]).

THEOREM [NASH-KUIPER]. — *If V admits some immersion into $\mathbf{R}^q$, for $q > n \approx \sqrt{2}p$, then there also exists an isometric C^1-immersion $(V, g) \to \mathbf{R}^q$ for an arbitrary C^0-metric g on V. Moreover, the operator $\mathcal{D}^+ : \mathrm{Im}_1 \to G^+_0$ is a Serre fibration.*

See § 2.4.9. in [G] for a conceptual proof of this remarkable theorem.

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