ESTIMATES OF BERNSTEIN WIDTHS OF SOBOLEV SPACES

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1. Introduction

We first recall some basic definitions on widths (cf. [P] for more details). Let X be a real complex normed linear space and A a closed, convex centrally symmetric subset of X.

The *n-Kolmogorov width* of A in X is given by

$$d_n(A;X) = \inf_{X_n} \sup_{x\in A} \inf_{y\in X_n} \|x-y\|$$

when the infimum is taken over all n-dimensional subspaces X_n of X.

The *n-Bernstein width* of A in X is defined as

$$\begin{aligned} b_n(A;X) &= \sup_{X_{n+1}} \sup\{\lambda \mid \lambda S(X_{n+1}) \subset A\} \\ &= \sup_{X_{n+1}} \inf_{x\in\partial(A\cap X_{n+1})} \|x\| \end{aligned}$$

where X_{n+1} in any $(n+1)$-dimensional subspace of X, $S(X_{n+1})$ is the unit ball of X_{n+1}.

The following inequality holds (see [P], Proof 1.6)

$$b_n(A;X) \le d_n(A;X)\ . \tag{1.1}$$

Clearly $d_n(A;X) \overset{n\to\infty}{\longrightarrow} 0$ implies compactness of A. The purpose of this paper is to develop a method to estimate certain Bernstein widths in situations where compactness is not available. More precisely, we will generalize to several variables the (semi)-classical fact that

$$b_n\big(S(W_1^1([0,1]))\ ,\ L_\infty([0,1])\big) \le \frac{1}{n} \tag{1.2}$$

where $L_p, \|\ \ \|_p$ refers to the usual Lebesgue spaces and $W_1^1([0,1])$ stands for the space of functions on $[0,1]$ with integrable derivative and norm

$$\|f\|_{W_1^1([0,1])} = \|f\|_1 + \|f'\|_1\ .$$

(Here and in the sequel, $S(\ \ )$ will refer to the unit ball.)

The proof of (1.2) is topological and rests on the Borsuk-Ulam antipodal mapping theorem. This method seems difficult to generalize to several variables. Our approach is based on Banach space theory of finite dimensional subspaces of L^p (in particular change-of-density and entropy-methods). There is the following generalization of (1.2)

Theorem 1.3. *Let $d \geq 1$ and Ω denote the unit ball in $\mathbb{R}^d$. Then*

$$b_n\big(S(W_1^1(\Omega))\ ,\ L_{d/d-1}(\Omega)\big) \leq c_d n^{-1/d} \tag{1.3}$$

where c_d is a constant only depending on d.

The relevant Sobolev inequality here is Gagliardo's inequality

$$\|u\|_{L_{d/d-1}(\mathbb{R}^d)} \leq c\|\operatorname{grad} u\|_{L_1(\mathbb{R}^d)} \quad \text{for} \quad u \in C_0^1(\mathbb{R}^d)\ . \tag{1.4}$$

More generally, let for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$, $|\alpha| = \sum\limits_i \alpha_i$

$$D^\alpha = D^{\alpha}_{x_1} \ldots D^{\alpha_d}_{x_d} \qquad\qquad D_{x_i} = \partial/\partial x_i$$

and denote

$$\|\nabla_\ell u\|_{L_p(\Omega)} = \left[\int_\Omega \Big(\sum_{|\alpha|=\ell} |D^\alpha u(x)|^2\Big)^{p/2} dx\right]^{1/p}$$

$$\|u\|_{W_p^\ell(\Omega)} = \|\nabla_\ell u\|_{L_p(\Omega)} + \|u\|_{L_p(\Omega)}\ .$$

Theorem 1.5. *Let $d \geq 1$ and Ω as in Theorem 1.3. Then*

$$b_n\big(S(W_1^d(\Omega))\ ,\ L_\infty(\Omega)\big) \leq c_d n^{-1}\ . \tag{1.6}$$

Notice that, up to the value of the constants, inequalities (1.3) and (1.4) are best possible. This fact is easily seen by considering polynomial spaces.

It will be clear from what follows that one may consider as well the more general type of domains Ω appearing in the theory of Sobolev spaces (see [M] for these matters). The aim of this exposition is to present the new ideas in the simplest cases. Essentially speaking, Bernstein width estimates may be derived from any localizable Sobolev inequality. In particular, application in the context of an inequality such as

$$\|\nabla u\|_2 \geq c\|u\|_p$$

is of potential interest for the spectrum of the Laplacian. Such applications will not be considered here.

This exposition is not self-contained. The reader will be referred to [M] for some basic facts about coverings and Sobolev inequalities. More importantly, essential use is made of methods and results of [BLM] on the geometry of finite dimensional subspaces of L^p and the truly interested reader is advised to consult this paper.

2. Entropy-Estimates in Subspaces of L^p

Details on what follows may be found in sections 4 and 9 of [BLM]. The next change-of-density principle is due to D. Lewis.

Lemma 2.1. *Let X be an n-dimensional subspace of $L_1(\mu)$, where μ is a probability measure. There is a density $\Delta \in L_1(\mu)$, $\Delta \geq \frac{1}{2}$, $\int \Delta d\mu = 1$ such that the space $\widetilde{X} = \Delta^{-1}X$ considered as subspace of $L_1(\widetilde{\mu})$, $\widetilde{\mu} = \Delta.\mu$ admits an orthonormal basis $\varphi_1, \ldots, \varphi_n$ satisfying*

$$\left\| \left(\sum_{i=1}^{n} |\varphi_i|^2 \right)^{1/2} \right\|_\infty \leq 2\sqrt{n}\ . \tag{2.2}$$

Definition 2.3 (entropy numbers). Denote for A, B convex centrally symmetric subsets of a linear space by $E(A, B)$ the minimum number (possibly infinity) of translates of B needed to cover A.

Definition 2.4. If X is a linear space of functions on some probability space (Ω, μ), denote for $1 \leq p \leq \infty$

$$S_p(X) = \left\{ f \in X \mid \|f\|_{L_p(\mu)} \leq 1 \right\}\ .$$

Denote also

$$S_{\psi_2}(X) = \left\{ f \in X \mid \|f\|_{L_{\psi_2}(\mu)} \leq 1 \right\}$$

where ψ_2 refers to the Orlicz-function $e^{x^2} - 1$. Recall also that $\|f\|_{L_{\psi_2}} \sim \sup\limits_{p \geq 1} \frac{\|f\|_p}{\sqrt{p}}$.

Lemmma 2.5. *Let X be an n-dimensional function space on a probability space (Ω, μ) admitting an orthonormal basis satisfying (2.2). Then for $t \geq 1$*

$$\log E\big(S_1(X), tS_{\psi_2}(X)\big) \leq c \frac{\sqrt{\log(1+t)}}{t} \cdot n\ . \tag{2.6}$$

Proof: By Proof 9.6 of [BLM]

$$\log E\big(S_1(X), tS_2(X)\big) \leq c \frac{\log(1+t)}{t^2} n \tag{2.7}$$

while, by Lemma 9.2 of [BLM]

$$\log E\big(S_2(X), tS_{\psi_2}(X)\big) \leq c \frac{n}{t^2}\ . \tag{2.8}$$

Since for all $1 \leq t_1 \leq t$

$$\log E\big(S_1(X), tS_{\psi_2}(X)\big) \leq \log E\big(S_1(X), t_1 S_2(X)\big) + \log\big(t_1 S_2(X), tS_{\psi_2}(X)\big)\ .$$

(2.6) is immediate from (2.7) and (2.8). It results from Lemmas 2.1, 2.5 that to any n-dimensional subspace of $L^1(\mu)$ there corresponds a change of density such that the space in its new position satisfies the natural (= as in the ℓ_n^p-scale) entropy-estimates. This fact will be exploited as follows

Lemma 2.9. *Given an n-dimensional subspace of X of $L^1(\mu)$, there exists a density-function Δ (as in Lemma 2.1) such that the following holds: Let Y be a subspace of X of dimension $\dim Y \geq \frac{n}{2}$. Then there is $f \in Y$ fulfilling the conditions*

$$\|f\|_{L^1(\mu)} = 1 \tag{2.10}$$

$$\int \left[\exp\left(\frac{f}{\Delta}\right)^2\right] \cdot \Delta \, d\mu \leq C \tag{2.11}$$

where C is an absolute constant.

Proof: In virtue of Lemma 2.1, there is clearly no restriction to assume (2.2) valid, in which case, by Lemma 2.5, (2.6) holds. (The density Δ appearing in Lemma 2.9 is given by Lemma 2.1.) From volumetric considerations,

$$\log E\Big(S_1(Y), \frac{1}{4}S_1(Y)\Big) \geq n \; . \tag{2.12}$$

From (2.6), for sufficiently large (numerical) value of t

$$\log E\big(S_1(Y), tS_{\psi_2}(X)\big) \leq \frac{n}{2} \; . \tag{2.13}$$

Since the centers of the covering balls may always be taken in Y, provided the radius t is doubled, also

$$\log E\big(S_1(Y), 2tS_{\psi_2}(Y)\big) \leq \frac{n}{2} \tag{2.14}$$

which together with (2.12) implies that

$$\tfrac{1}{4}S_1(Y) \not\subset 2tS_{\psi_2}(Y) \; .$$

Thus there is $\varphi \in Y$ satisfying $\|\varphi\|_{\psi_2} \leq 2t$ and $\|\varphi\|_1 > \frac{1}{4}$, completing the proof.

3. Proof of Theorem 1.3

Recall first following covering property due to Besicovitch (see [M], Theorem 1.2.1 for a proof).

Lemma 3.1. *Let Ω be a bounded set in $\mathbb{R}^d$. Associate to each point $x \in \Omega$ a ball $B_{r(x)}(x)$, $r(x) > 0$ and denote $\mathcal{B}$ the collection of these balls. Then one may choose a sequence $\{B_m\}$ of balls in $\mathcal{B}$ such that*

$$\Omega \subset \bigcup_m B_m \tag{3.2}$$

and

$$\{B_m\} \text{ has multiplicity bounded by } M = M(d) \tag{3.3}$$

(= a constant only depending on the dimension d). Thus each point is in at most M balls B_m.

Lemma 3.2. *Let Ω be the unit ball in $\mathbb{R}^d$ and Ω_1 the intersection of Ω and a ball centered at a point in Ω. Then, letting $p = \frac{d}{d-1}$, on a subspace of codimension 1, the inequality*

$$\|u\|_{L_p(\Omega_1)} \leq C\|\nabla u\|_{L_1(\Omega_1)} \tag{3.3}$$

holds. Here $c = c_d$ only depends on the dimension.

Proof: Notice homothetic invariance in (3.3) which permits the assumption Ω_1 is 1-bounded and satisfies the cone-property in a uniform way. Thus Ω_1 admits a Stein extension operator $\xi = W_1^1(\Omega_1) \longrightarrow W_1^1(\mathbb{R}^d)$ with a uniform bound (cf. [M]). Also, by Lemma 1.1.11 of [M], there is a function φ_0 on Ω_1 such that

$$\|u - \langle u, \varphi_0\rangle\|_{L_1(\Omega_1)} \leq C\|\nabla u\|_{L_1(\Omega_1)} \tag{3.4}$$

where C is a uniform constant. Hence, applying Gagliardo's inequality

$$\|v\|_{L_p(\mathbb{R}^d)} \leq C\|\nabla v\|_{L_1(\mathbb{R}^d)} \tag{3.5}$$

(cf. [M]), it follows

$$\begin{aligned}\|u - \langle u, \varphi_0\rangle\|_{L_p(\Omega_1)} \leq &\|\xi(u - \langle u, \varphi_0\rangle)\|_{L_p(\mathbb{R}^d)} \leq \\ &C\big\|\nabla\big(\xi(u - \langle u, \varphi_0\rangle)\big)\big\|_{L_1(\mathbb{R}^d)} \leq \\ &C\|\xi\|\|u - \langle u, \varphi_0\rangle\|_{W_1^1(\Omega_1)} \leq \\ &C'\|\nabla u\|_{L_1(\Omega_1)}\end{aligned}$$

This implies Lemma 3.2.

Remark. By Lemma 1.1.11 of [M], the generalization of previous localization argument, when there are higher order derivatives involved, is clear.

Proof of Theorem 1.3. Let X be an n-dimensional subspace of $W_1^1(\Omega)$ and consider its image $\nabla(X)$ under the linear map

$$\nabla = (\partial_{x_1}, \ldots, \partial_{x_d}) : W_1^1(\Omega) \longrightarrow L_1(\Omega) \oplus \cdots \oplus L_1(\Omega) \simeq L_1(\Omega \oplus \cdots \oplus \Omega) .$$

Application of Lemma 2.5 to the space $\nabla(X)$ gives a density Δ on Ω such that any $\frac{n}{2}$-dimensional subspace Y of X contains an element f fulfilling the properties

$$\|\nabla f\|_1 = 1 \quad \text{and} \quad \int_\Omega \Delta \exp\left(\frac{|\nabla f|}{\Delta}\right)^2 \leq C\ . \tag{3.6}$$

(Notice that in the preceding the linearity of ∇ is important.)

Define $\kappa = \frac{2M}{n}$, where M is the constant of Lemma 3.1. To each $x \in \Omega$, assign $r(x) > 0$ so that

$$\int_{B_{r(x)}(x)} \Delta(y)dy = \kappa\ . \tag{3.7}$$

Let $\{B_m\}$ be obtained applying Lemma 3.1. Notice that by (3.3),(3.7), $n_1 = \#\{B_m\}$ has to satisfy

$$\kappa n_1 = \sum_m \int_{B_m} \Delta \leq \Big\| \sum_m \chi_{B_m} \Big\|_\infty \left(\int_\Omega \Delta\right) \leq M$$

hence

$$n_1 \leq \frac{n}{2}\ .$$

Denote $\Omega_m = \Omega \cap B_m$ for $1 \leq m \leq n_1$ to which Lemma 3.2 is applied. Thus

$$\|u\|_{L_p(\Omega_m)} \leq C\|\nabla u\|_{L_1(\Omega_m)} \qquad (1 \leq m \leq n_1) \tag{3.8}$$

holds, for all u in a space of codimension $\leq n_1$. In particular, (3.8) holds for all f in a half-dimensional subspace Y of X. Take f satisfying (3.6) in addition. Since in particular

$$\int_\Omega |\nabla f|^p \Delta^{1-p} \leq C \tag{3.9}$$

it follows from (3.2),(3.8),(3.7),(3.3),(3.9) and Hölder's inequality

$$\begin{aligned}
\|f\|^p_{L_p(\Omega)} &\leq \sum_m \|f\|^p_{L_p(\Omega_m)} \leq C \sum_m \|\nabla f\|^p_{L_1(\Omega_m)} \\
&\leq C \sum_m \left\| \frac{|\nabla f|}{\Delta^{1/p'}} \right\|^p_{L_p(\Omega_m)} \cdot \|\Delta^{1/p'}\|^p_{L_{p'}(\Omega_m)} \\
&= C \sum_m \left(\int_{\Omega_m} |\nabla f|^p \Delta^{1-p}\right)\left(\int_{\Omega_m} \Delta\right)^{p-1} \\
&\leq C\kappa^{p-1} M
\end{aligned}$$

Hence, by definition of p and κ, $\|f\|_{L_p(\Omega)} \leq Cn^{-1/d}$, completing the proof.

Remark. The pevious argument breaks down for $d = 1$. This case or, more generally (1.4), will be considered in the next section.

4. Proof of Theorem 1.4

Since the ball has the Stein extension property, one may as well take for Ω the cube $]0,1[^d$ (this simplifies the construction of certain partitions introduced later). We first draw the following corollary from Lemma 2.5

Lemma 4.1. *Let X be an n-dimensional subspace of $L^1(\Omega,\mu)$ with an orthonormal basis satisfying (2.2). Let $(\mathcal{P}_i)_{i=1,2,\dots}$ be a sequence of finite partitions of Ω such that*

$$\#\mathcal{P}_i = N_i \quad \text{and} \quad \mu(A) = \frac{1}{N_i} \quad \text{for} \quad A \in P_i . \tag{4.2}$$

Let (λ_i) be a sequence of positive numbers fulfilling the conditions

$$\lambda_i < N_i \quad \text{and} \quad \sum N_i \exp\left(-\frac{1}{2}\frac{N_i^2}{\lambda_i^2}\right) < n . \tag{4.3}$$

If Y is a subspace of X, $\dim Y > \frac{n}{2}$, there exists $f \in Y$ satisfying

$$\left.\begin{aligned} &\|f\|_{L^1(\mu)} = 1 \\ &\sup_i \sup_{A\in\mathcal{P}_i} \lambda_i |\langle f,\chi_A\rangle| \le C . \end{aligned}\right\} \tag{4.4}$$

Proof: For each i, consider the operator $T_i f = (\langle f,\chi_A\rangle)_{A\in\mathcal{P}_i}$ ranging in $\ell^\infty_{N_i}$. Again from entropy considerations, it will suffice to fulfill an inequality

$$\log E\big(S_1(X), \{f \in X \mid \|T_i f\| < \lambda_i^{-1} \quad \text{for each} \quad i\}\big) < Cn . \tag{4.5}$$

Since, by Lemma 1.5, (2.6) holds, (4.5) may be replaced by

$$\log E\big(S_{\psi_2}(X)\ ,\ \{f \in X \mid \|T_i f\| < \lambda_i^{-1} \quad \text{for each} \quad i\}\big) < Cn$$

or equivalently

$$\log E\big(\{(\lambda_i T_i f) \mid f \in S_{\psi_2}(X)\}\ ,\ S(\oplus_\infty \ell^\infty_{N_i})\big) < Cn . \tag{4.6}$$

Estimate the left number of (4.6) by

$$\sum_i \log E\big(\{\lambda_i T_i f \mid f \in S_{\psi_2}(X)\}\ ,\ S(\ell^\infty_{N_i})\big) . \tag{4.7}$$

By (4.2), taking conditional expectation with respect to $\mathcal{P}_i$, for $\|f\|_{\psi_2} \le 1$

$$\frac{1}{N_i} \sum_{A \in \mathcal{P}_i} \exp(N_i \langle f, \chi_A \rangle)^2 \le 1$$

hence

$$\|T_i f\|_{L^{\psi_2}_{N_i}} \le \frac{1}{N_i}$$

where $L^{\psi_2}_N$ stands for the L^{ψ_2} space on $\{1, \ldots, N\}$ with normalized counting measure. Thus (4.7) is bounded by

$$\sum_i \log E\Big(S(L^{\psi_2}_{N_i}) \ , \ \frac{N_i}{\lambda_i} S(\ell^\infty_{N_i})\Big) \ . \tag{4.8}$$

It is easily verified that $(t > 1)$

$$\log E\big(S(L^{\psi_2}_n) \ , \ tS(\ell^\infty_N)\big) \le C e^{-\frac{t^2}{2}} N \tag{4.9}$$

(See [BLM], section 9 for a proof.) Substitution of (4.9) in (4.8) gives the condition (4.3)

$$\sum_i N_i \exp\left(-\frac{1}{2}\left(\frac{N_i}{\lambda_i}\right)^2\right) < n \ .$$

Remark. It is easily verified that Lemma 4.1 remains valid if the hypothesis of the $\mathcal{P}_i$ being partitions is replaced by a hypothesis of bounded multiplicity. Let now X be an n-dimensional subspace of $W_1^d(\Omega)$ and consider its image $\nabla_j X$ under the mapping

$$\nabla_d : W_1^d(\Omega) \longrightarrow \underbrace{L^1(\Omega) \oplus \cdots \oplus L^1(\Omega)}_{d_1 \text{ components}} = L^1(\Omega \oplus \cdots \oplus \Omega) : f \longrightarrow (D^\alpha f)_{|\alpha|=d}$$

where $d_1 = \frac{(2d-1)!}{d!(d-1)!}$.

Application of Lemma 2.1 to $\nabla_d X$ gives a density $(\Delta_\alpha)_{|\alpha|=d}$ on $\Omega \oplus \cdots \oplus \Omega$ and we let $\Delta = \frac{1}{d_1} \sum_{|\alpha|=d_1} \Delta_\alpha$. Then $(\Delta \oplus \cdots \oplus \Delta)^{-1} \cdot \nabla_d X$ is a subspace of $L^1(\oplus\Omega, \Delta\, dx)$ satisfying the hypothesis of Lemma 4.1.

Next apply as in the previous section Lemma 3.1 in order to obtain a sequence $B^1, B^2, \ldots, B^{n'}$ of intersections of Ω and some cube centered at a point of Ω, such that

$$\int_{B^m} \Delta(x) dx \sim \frac{1}{n} \qquad (1 \le m \le n') \tag{4.10}$$

$$\{B^1, \ldots, B^{n'}\} \quad \text{has bounded multiplicity} \tag{4.11}$$

(Lemma 3.1 remains valid if balls are replaced by cubes.) Here $n' \sim n$ and will be specified according to later needs. Fix $1 \leq m \leq n'$. Construct partitions $\mathcal{P}^m_{i_1,\dots,i_d}$ of $B^m = J_1 \times \cdots \times J_d$ in rectangles

$$A = I_{s_1} \times I_{s_1,s_2} \times \cdots \times I_{s_1,\dots,s_d} \qquad (s_j \leq 2^{i_j})$$

where for $s_1, \dots, s_{j-1}$ fixed, $I_{s_1,\dots,s_{j-1},s}$ $(1 \leq s \leq 2^{i_j})$ are consecutive intervals chosen such that

$$\int_Q \Delta = 2^{-i_1-\dots-i_{j-1}-i_j} \int_{B^m} \Delta \sim \frac{1}{n} 2^{-i_1-\dots-i_j} \ . \tag{4.12}$$

where $Q = I_{s_1} \times \cdots \times I_{s_1,\dots,s_{j-1}} \times I_{s_1,\dots,s_{j-1},s} \times J_{j+1} \times \cdots J_d$. Thus letting $\bar{\imath} = (i_1, \dots, i_d)$, $|\bar{\imath}| = i_1 + \cdots + i_j$, it follows that

$$N_{\bar{\imath}} \equiv \# \mathcal{P}^m_{\bar{\imath}} = 2^{|\bar{\imath}|} \tag{4.13}$$

and

$$\int_A \Delta \sim 2^{-|\bar{\imath}|} \frac{1}{n} \quad \text{for} \quad A \in \mathcal{P}^m_{\bar{\imath}} \ . \tag{4.14}$$

Let $\mathcal{P}_{\bar{\imath}}$ be the collection of rectangles obtained as $\bigcup_{1 \leq m \leq n'} \mathcal{P}^m_{\bar{\imath}}$, of bounded multiplicity, by (4.11). Defining $\lambda_{\bar{\imath}} = n' 2^{\frac{1}{2}|\bar{\imath}|}$, condition (4.3) becomes

$$n > \sum_{i_1,\dots,i_d} |\mathcal{P}_{\bar{\imath}}| \exp\big(-\tfrac{1}{2}\lambda_{\bar{\imath}}^{-2} |\mathcal{P}_{\bar{\imath}}|^2\big) = n' \sum_{\bar{\imath}} 2^{|\bar{\imath}|} \exp\big(-\tfrac{1}{2} 2^{|\bar{\imath}|}\big) \tag{4.15}$$

which may be satisfied for $n' \sim n$.

Assume Y a subspace of X, $\dim Y \geq \frac{n}{2}$ and apply the conclusion of Lemma 4.1 to the subspace $(\Delta \oplus \cdots \oplus \Delta)^{-1} \nabla_d Y$ of $(\Delta \oplus \cdots \oplus \Delta)^{-1} \nabla_d X$ (the systems $\mathcal{P}_{\bar{\imath}}$ are reproduced here on each of the components of the direct sum space $\Omega \oplus \cdots \oplus \Omega$). One gets then some $f \in Y$ satisfying

$$\sup_{|\alpha|=d} \|\Delta^{-1} D^\alpha f\|_{L^1(\Delta dx)} = \sup_{|\alpha|=d} \|D^\alpha f\|_{L^1(\Omega)} = 1 \tag{4.16}$$

and

$$|\langle D^\alpha f, \chi_A \rangle| = \left| \int_A (\Delta^{-1} D^\alpha f)(\Delta \ dx) \right| \leq C 2^{-\frac{1}{2}|\bar{\imath}|} n^{-1}$$

$$\text{for} \quad |\alpha| = d \ , \ 1 \leq m \leq n' \ , \ A \in \mathcal{P}^m_{\bar{\imath}} \ . \tag{4.17}$$

Fixing $m = 1, \dots, n'$ and considering the domain B_m, one has a uniform inequality

$$\|f\|_{L^\infty(B_m)} \leq C \sup_{|\alpha|=d} \sup_R |\langle D^\alpha f, \chi_R \rangle| \tag{4.18}$$

for all f taken in a space of codimension $\leq C_d$. The supremum in R relates to all parallelopipeda contained in B_m (the validity of this claim is easily seen by rescaling B_m to the unit cube and applying the fundamental theorem of calculus or Theorem 1.1.10.1 of [M]). If $n'C_d < \frac{n}{2}$ and the space $Y \hookrightarrow X$ considered above is defined appropriately, one gets some $f \in X$ satisfying (4.18) in addition to (4.16),(4.17). It remains to evaluate $\|f\|_{L^\infty(\Omega)} = \max_{1\le m\le n'} \|f\|_{L^\infty(B_m)}$. By (4.18), this may be done by estimating $\langle D^\alpha f, \chi_R\rangle$, where R is a parallelopipedum in B_m with same origin as B_m. It easily follows from the construction of the partitions $\mathcal{P}_{\bar{\imath}}^m$ ($\bar{\imath} = (i,\dots,i_d)$) that R admits a representation as disjoint union

$$R = \bigcup_{\bar{\imath}} R_{\bar{\imath}} \tag{4.19}$$

where $R_{\bar{\imath}} = \emptyset$ or $R_{\bar{\imath}} \in \mathcal{P}_{\bar{\imath}}^m$. The reader will easily verify this fact by considering $d = 1, 2$ etc. From (4.19) and (4.17)

$$|\langle D^\alpha f, \chi_R\rangle| \le \sum_{\bar{\imath}} |\langle D^\alpha f, \chi_{R_{\bar{\imath}}}\rangle| \le Cn^{-1} \sum_{i_1,\dots,i_d} 2^{-\frac{1}{2}(i_1+\cdots+i_d)} = Cn^{-1} .$$

Hence $\|f\|_\infty < Cn^{-1}$, concluding the proof of Theorem 1.4.

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