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Cuts in Kähler Groups

Thomas Delzant and Misha Gromov

Abstract. We study fundamental groups of Kähler manifolds via their cuts or relative ends.

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1. A group G is called Kähler if it serves as the fundamental group $\pi_1(V)$ of a compact Kähler manifold V. Equivalently, such a group G appears as a discrete free co-compact isometry group of a complete simply connected Kähler manifold X – the Galois group acting on the universal covering of V denoted X. To keep the perspective (compare [De-Gr]) we indicate possible generalization of this setting.

(a) Dropping "free", i.e. allowing discrete actions with fixed points (having finite stabilizers).

(b) Replacing "co-compact" by a weaker smallness condition on the quotient X/G, e.g. by requiring X/G to have finite volume (or slow volume growth) combined with a sufficiently simple geometry at infinity in the spirit of the following two examples.

(b') Complete (and natural non-complete) Kähler metrics on quasi-projective varieties V and the corresponding metrics on coverings of such V.

(b") Complete Kähler metrics on X with bounded geometry i.e. with curvature bounded from above and the injectivity radius from below.

(c) Admitting (closed) non-discrete isometry groups G of X.

(d) Allowing singular spaces V and X.

(e) Replacing Kähler groups by Kähler groupoïds that are leaf-wise Kähler foliations with transversal measures ([Gr]$_{FPP}$).

1.1. Central Problem. Identify the constraints imposed by the Kähler nature of the space X on its asymptotic metric invariants and then express these constraints in terms of some algebraic properties of G.

If V is projective algebraic then the structure of the profinite completion of its fundamental group G is accounted for by finite, and hence algebraic, coverings of V. Here we are primely concerned with infinite coverings and issuing "transcendental" constraints on G that are not expressible (at least not directly) in terms of subgroups of finite index in G and/or finite dimensional representation of G.

1.2. The basic examples of Kähler groups are surface groups that are the fundamental groups of Riemann surfaces S, i.e. complex algebraic curves, and Cartesian products of surface groups, e.g. free Abelian groups of even rank. Less obvious examples are provided by discrete co-compact groups acting on Hermitian symmetric spaces, such as the unit ball in $\mathbb{C}^n$ with the Bergman metric, for instance.
Let $V \subset \mathbb{P}^N$ be a projective algebraic manifold of dimension n. Due to the Lefschetz theorem, if $V' = V \cap P$ is the intersection of V by a transverse projective subspace of dimension $N - n + 2$, then the inclusion $V' \subset V$ induces an isomorphism on the fundamental groups, $\pi_1(V') \simeq \pi_1(V)$. However, such a surface -called Lefschetz surface- usually is more complicated than the original V; to see this try to visualize such an (hyper)surface in the Cartesian product of three Riemann surfaces of positive genera, $V' \subset V = S_1 \times S_2 \times S_3$.

1.3. The above central problem is accompanied by its relative version: determine characteristic features of homomorphisms between Kähler groups induced by holomorphic maps f.
Observe in this regard, that an arbitrary proper holomorphic map $f : V \to W$ factors trough a surjective holomorphic map $f' : V \to W'$ with non empty connected fibers $f^{-1}(v), w \in W$, followed by a finite-to-one map $W' \to W$. The connectedness of the fibers makes f' surjective on the fundamental groups ; furthermore, if W is non-singular and f is onto, then the map $W' \to W$, being a ramified covering, sends the fundamental group of W' (and, hence of V) onto a subgroup of finite index in $\pi_1(W)$ (here W is assumed compact). In particular, if V fibers over a non-singular curve (Riemann surface) of genus g, i.e. admits a surjective holomorphic map to such curve S, then the fundamental group of V surjects onto a surface group $\pi_1(S')$ of genus $g' \geq g$.
A particular seemingly innocuous instance of the relative problem concerns subgroups in the products of surface groups, $G \subset \pi_1(W = S_1 \times S_2, \ldots, \times S_N)$. When is such G Kähler ?When does there exist, for some choice of conformal structures in S_i, an algebraic sub-variety $V \subset W$ (singular or non-singular) of a given dimension n such that the image of the fundamental group of V in $\pi_1(W)$ equals G? (Notice that for a general non-Lefschetz V the inclusion homomorphism is not injective on the fundamental groups.) This question seems non-trivial already for $n = 1$ where it has a purely topological counterpart: find a real surface V in W such that the projections of V to all S_i's are ramified coverings and such that the image of $\pi_1(V)$ in $\pi_1(W)$ equals G.
We shall see later on that some algebraic condition (existence of cuts) on an general Kähler group makes it a subgroup in a product of surface groups.

2. The Kähler nature of X becomes metrically discernible when X is harmonically mapped into a (globally) H-non positive space. Here are the necessary definitions.

2.1. The energy density E of a smooth map f between Euclidean balls at a point x is defined as one half of the squared norm of its differential D at x,

$$E(x) = \frac{1}{2} trace D^* D(x).$$

This generalizes to maps where the differential exists on a dense set of points x', e.g. to Lipschitz maps f, as $\limsup_{x' \to x} E(x')$.

2.2. For a Lipschitz map between arbitrary metric spaces, $f : X \to Y$, the (Euclidean-like) energy density is defined as the infimum of those e such that, for every two Euclidean balls B_1 and B_2 and arbitrary 1-Lipschitz maps $B_1 \to X$ and $Y \to B_2$, where the center 0 of B_1 goes to x, the composed map $B_1 \to B_2$ has $E(0) \leq e$.

2.3. If the space X is endowed with a measure then the energy of an f, denoted $E(f)$ is defined as the integral of the energy density with this measure, where for Riemannian, e.g. Kähler manifolds one uses the ordinary Riemannian measure for this purpose (see [Gr]$_{FPP}$).

2.4. A map f is called harmonic if it is locally energy minimizing, i.e minimizing under variations of f which are non trivial on small balls in X.

2.5. If X is a Riemann surface, i.e. a 1-dimensional complex manifold, then the energy of an arbitrary map as well as the harmonicity obviously are conformal invariants, i.e. are independent of the Kähler metric compatible with the complex structure. This allows one to define *pluriharmonic* maps from an arbitrary complex space to a metric space as those f whose restrictions to all holomorphic curves in X are harmonic.
One knows (this is easy, at least for Riemannian targets) that every pluriharmonic map of a Kähler manifold is harmonic but for $dim X = n > 1$ most harmonic maps are not pluriharmonic. For example a real valued function f on a Kähler manifold is pluriharmonic if and only if its gradient (vector field) is Hamiltonian, i.e. preserving the symplectic part ω of the Kähler metric while harmonicity amounts to preservation of the corresponding volume form ω^n under the gradient flow of f.

2.6. If a space X is properly (e.g. discretely) acted upon by a group G and the energy density $E(x)$ of some map f is G-invariant, then the G-energy of f is defined as the integral of E descended to the quotient space X/G. This applies, in particular, to G-equivariant (harmonic and non-harmonic) maps between G-spaces.

2.7. A metric G-space Y, i.e. a space Y isometrically acted upon by G, is called (globally) H-non-positive if every G-equivariant harmonic map f of finite G-energy from an arbitrary Kähler G-manifold X to Y is pluriharmonic.
This definition (albeit provisional) is justified by the following fundamental (and amazing)

2.8. Hodge Lemma (see [ABCKT]). *Flat Hilbertian manifolds are H-non-positive.*

Recall that "flat Hilbertian" signifies that Y is locally isometric to a finite or infinite dimensional Hilbert space.

2.9. Basic corollary. *Let X be a complete Kähler G-manifold and suppose the group G is represented by isometries of a Hilbert space Y. Then every harmonic G-equivariant map $f : X \to Y$ of finite G-energy is pluriharmonic.*

Remarks.

2.10. The relevant actions on the Hilbert space Y above are *affine* rather than linear. For example, they may be free and discrete (Haagerup property, see [CCJV]).

2.11. If the actions of G on X and Y are discrete, then the G-energy equals the ordinary energy of the corresponding map between the quotient spaces. In particular, this energy is necessarily finite if the action of G on X is co-compact.

2.12. In some cases (e.g. if X is simply connected and Y is a Hilbert space) every pluriharmonic map from X to Y analytically extends to a holomorphic map from X to a suitable complexification of Y.

2.13. The existence of a G-equivariant harmonic map often (but not always) comes cheap: for example, it follows in many cases from the existence of a continuous G-equivariant map with finite G-energy. But the issuing pluriharmonic (and even more so holomorphic) map carries a much higher price tag and the presence of such a map imposes strong geometric restrictions on the manifolds and groups in question.
The idea of Hermitian sectional curvature, as well as the following non-linear Hodge Lemma has been discovered by Y.T. Siu ([Siu]), and further developed in [He],[C-T],[Sam] and [Gr-Sc] ; see [ABCKT] for additional informations and references.

2.14. Non-linear Hodge lemma (see [ABCKT]). *The H-non-positivity property, remains valid for the following non-flat target spaces Y.*

(A) *Metric graphs, e.g. trees (including $\mathbb{R}$-trees with no local finiteness condition).*
(B) *Euclidean buildings (these generalize trees).*
(C) *Riemannian and Hilbertian symmetric spaces of non-positive sectional curvature.*
(D) *Riemannian and Hilbertian manifolds with point-wise $1/4$-pinched negative sectional curvature, e.g. Riemann surfaces with negative sectional curvature.*
(E) *Riemannian and Hilbertian manifolds with non-positive Hermitian sectional curvature. (Riemann surfaces with non-positive sectional curvature, symmetric spaces and $1/4$-pinched manifolds fall into this category.)*
(F) *Metric spaces locally isometric to finite and infinite Cartesian products of the above* (A)–(E).

The maximal class of known H-non positive spaces Y admits a local characterization saying in effect that these are locally $CAT(0)$, their non-singular loci have non-positive Hermitian sectional curvature and the singularities of Y are quasi-regular in the sense of [Gr-Sc], i.e. they have no more negativity (of singular sectional) curvature than Euclidean buildings do. Observe that this class is closed under scaling and Cartesian products but it is unclear how stable this class (and H-negativity in general) is under conventional limits of metric spaces.

2.15. Examples. Let V be a compact connected Kähler manifold and Y a compact Riemannian manifold of non-positive sectional curvature. Then every continuous map $f_0 : V \to Y$ is homotopic to a (essentially unique) harmonic map f and this f is pluriharmonic if Y is H-non-positive.

(a) For instance, if Y is a flat torus (where the harmonicity and pluriharmonicity of f does not depend on a choice of a flat metric compatible with the affine structure) then one obtains a harmonic, hence pluriharmonic, map f homotopic to a given f_0, where this f is unique up to a toral translation. This applies, in particular, to the the Jacobian (torus) of V, that is $J(V) = H_1(V;\mathbb{R})/H_1(V;\mathbb{Z})$, where one concludes to the existence of, a unique up to a translation, pluriharmonic Abel-Jacobi map $f_0 : V \to J(V)$ that induces the identity isomorphism on the 1-dimensional real homology (for the canonical identification of the 1-dimensional homology $H_1(V;\mathbb{R})$ with the homology of the Jacobian $J(V)$). Furthermore, according to the Albanese-Abel-Jacobi theorem, there exists a unique invariant complex structure on $J(V)$ for which f_0, called Albanese map, is holomorphic and such that every holomorphic map from V to a flat Kähler manifold A with Abelian fundamental group (compact complex torus) factors via the Albanese map followed by an affine holomorphic map $J(V) \to A$. The existence of the complex structure on the Jacobian makes the first Betti number of V even : this is the first basic constraint on the fundamental group of V.

(b) If Y has constant negative curvature then harmonic maps $f : V \to Y$, besides being pluriharmonic, necessarily have rank (of their differentials at all point in V) at most two ([Sam]). Moreover, every harmonic map $V \to Y$of rank 2 factors via a holomorphic map of V to a hyperbolic (i.e. of genus¿1) Riemann surface, $V \to S \to Y$, by a theorem of Sampson ([Sam]) see also [ABCKT],[C-T] for generalizations.

(c) If Y is itself Kähler and moreover, has constant Hermitian curvature (i.e. covered by the unit ball in $\mathbb{C}^n$ with the Bergman metric) and if a harmonic map f has $rank > 2$ at some point in V then, by Siu's theorem, f is either holomorphic or anti-holomorphic. (A map f is anti-holomorphic if it maps each holomorphic curve $C \subset V$ to a holomorphic curve $C' \subset Y$ and the maps $C \to C'$ are conformal and orientation reversing for the canonical orientation on holomorphic curves).

2.16. The above (a), (b), (c) remain valid for infinite dimensional (Hilbertian) manifolds Y (with accordingly constant curvatures), where one may needs a certain stability conditions (depending on Y, V and the homotopy class) that ensure the existence of a harmonic map in a given homotopy class of maps $f : V \to Y$.
Let x_0 a base-point in V, and let $G = \pi_1(X, x_0)$ acts on the universal cover $\tilde{Y}$ by $f_* : G \to \pi_1(Y, f(x_0))$. Let $g_1, \dots g_r$ be a generating system of G. One says ([Gr]$_{RWRG}$ 3.7.A′) that the action is *stable* if for every $K > 0$, any sequence

$y_n \in \tilde{Y}$ s.t for all $1 \leqslant i \leqslant r, d(g_i y_n, y_n) < K$ admits a convergent subsequence. This property only depends on the action of G on $\tilde{Y}$ induced by f_*.

We shall return on this stability condition latter on, but even in the absence of stability one still can obtain harmonic maps starting from maps of finite energy and by applying an energy minimizing process. Such a process, which may diverge in ordinary sense, often (essentially always) converges in a generalized sense where the target space Y need to be eventually replaced by a suitable limit of pointed spaces (Y, y_i) ([Mo],[Ko-Sc], $[\mathrm{Gr}]_{RWRG}$). In particular, N. Mok [Mo] proved that, if the fundamental group of a Kähler manifold X does not satisfy Kazhdan T property, then the universal cover of Xcarries a non constant holomorphic function to some Hilbert space, equivariant for some affine isometric representation of the fundamental group ; consequently the universal cover of X support a non constant holomorphic function with bounded differential (and thus, of at most linear growth).

Basic examples of infinite dimensional symmetric spaces have been introduced by P. de la Harpe [Ha].

2.17. Strict H-negativity. This means, by definition, that Y satisfies the conclusion of the above (b) : every harmonic map $V \to Y$ (or, in general, every harmonic H-equivariant with finite H-energy map $X \to Y$) of rank $\geqslant 2$ is of rank 2 and factors via a holomorphic map to a hyperbolic Riemann surface S followed by a harmonic map $S \to Y$.

The basic examples of such Y's are Riemannian manifold with strictly negative Hermitian curvature. These include strictly $\frac{1}{4}$-pinched manifold by a Siu-Sampson-Hernandez theorem (see [ABCKT], Chap.6 and references therein).

Furthermore, the piecewise Riemannian spaces built of simplexes of negative Hermitian curvature with geodesic faces and with the links of all faces of diameter $> \pi$ (compare the regularity assumption in [Gr-Sc]) are H-negative. Moreover, the strictness of negativity is needed not everywhere but only on a "sufficiently large" part of Y. For example, every 2-dimensional $CAT(0)$-polyhedron with the above assumption on the links is H-strictly negative, provided its fundamental group is hyperbolic.

It follows, for instance, that if Y is obtained by ramified covering of a Euclidean 2-dimensional building Y_0 (e.g. the product of two graphs), where the ramification locus lies away from the 1-skeleton of Y_0, and meets all 2-simplexes in Y_0, then every harmonic map of a Kähler manifold to Y factors via a holomorphic map to a Riemann surface. Notice that the singularities of Y at the ramification points have links of diameters$> \pi$ (in fact $\geq \pi$ but these can be smoothed and therefore, the H-negativity does not suffer. Similarly, the ramified covers of manifolds of H-non-positive (H-negative) curvature along totally geodesic submanifolds of codimension two are H-non-positive (H-negative). In fact, the presence of ramification enhances H-negativity. For example, if a complex surface Y_0 of constant Hermitian curvature< 0 is ramified over a totally real geodesic surface, then every harmonic map of X to the resulting $Y \to Y_0$ that transversally meets the ramification locus factors via a holomorphic map to some $S \to Y$. In particular, the fundamental group of Y itself is non-Kähler. Similarly one sees that the majority of ramified coverings of Abelian varieties over unions of (mutually intersecting) flat real codimension two sub-tori have non-Kähler fundamental groups.

3. Let V be a compact Riemannian manifold with fundamental group G and X be a covering of V with the fundamental group $H \subseteq G$. The existence/non-existence of a non-constant harmonic function f on X with finite energy, i.e. with a square integrable differential, depends only on G and $H \subseteq G$ but not on X per se.

Examples.

3.1. If $H = \{id\}$, i.e. X equals the universal covering of V, then the existence of such f is equivalent by De Rham-Hodge theory to non-vanishing of the reduced 1-dimensional cohomology of V and/or of G with coefficients in the regular representation, $H^1(V; l_2(G)) = H^1(G; l_2(G)) \neq 0$.

3.2. On reduced cohomology. When defining cohomology with infinite dimensional coefficients one may factorize the kernel of d by the closure of the image of d, where the resulting cohomology is referred to as reduced. For example, a square integrable closed 1-form a on X represents a non-zero reduced cohomology class (in $H^1(V; l_2(G))$ if and only if there exists a square integrable 1-cycle (closed 1-current) b on X such that $a(b) \neq 0$. In what follows the cohomology is understood as reduced unless otherwise stated.
One can express this property in terms of the *stability* of a certain unitary action.

Stability and reduced cohomology. *Let $\mathcal{H}$ be a Hilbert space. Let $\rho : G \to Isom(\mathcal{H})$ be some affine isometric action of G, and let $\pi : G \to U(\mathcal{H})$ its unitary part. Then the following properties are equivalent :*

- -i *The affine action ρ is stable in the sense of* 2.16.
- -ii *The unitary representation π has no almost fixed vector, in other words there is no sequence ξ_n of unit vectors such that $\|\rho(g_i)\xi_n - \xi_n\| \to 0$for all $1 \leqslant i \leqslant r$*
- -iii *The unitary representation π has no fixed vectors, and $H^1(G, \pi) = H^1(G, \pi)$.*

Proof. Suppose ρ is not stable. Then there exists a constant K and a sequence y_n s.t for all$1 \leqslant i \leqslant r$, $\|\rho(g_i)y_n - y_n\| \leqslant K$, but y_n has no convergent subsequence. Therefore $\|\pi(g_i)y_n - y_n\| \leqslant K'$ for some constant K'. As the sequence y_n has no convergent subsequence for the weak topology, it is unbounded. Thus the sequence $\xi_n = \frac{y_n}{\|y_n\|}$ is an almost fixed vector. Conversely, if ξ_n is an almost fixed vector, and π has no fixed vector$\max_i \|\rho(g_i)\xi_n - \xi_n\| = \varepsilon_i \to 0$. Let $y_n = \varepsilon_n^{-1}\xi_n$, then $\|\rho(g_i)y_n - y_n\| \leqslant 1$, but y_n is not bounded. This proves i⇔ii.
Let us prove i⇒ iii : as G is finitely generated $Z^1(G, \pi)$ has a structure of Hilbert space. If π has no fixed vector, the boundary map $\beta : \mathcal{H} \to Z^1(G, \pi)$ is injective ; if $Im\beta$ is not closed, there exists a sequence y_n such that $\rho(g_i)y_n - y_n \to b(g_i)$ where b is some 1-boundary not homologous to zero. Thus the sequence y_n has no convergent subsequence and ρ is not stable. The implication iii⇒ii is due to

Guichardet (Lemma page 48 in [HV]) : if $H^1(G,\pi) = H^1(G,\pi)$, $B^1(G,\pi)$ is closed in $Z^1(G,\pi)$ and is a Hilbert space. The map β is thus an isomorphism of Hilbert spaces, and if $\|\beta(\xi_n)\| \to 0$ then $\|\xi_n\| \to 0$. □

3.3. Stability at infinity. Let X be a complete connected Riemannian manifold. An *end* of X is a non compact connected component of the complementary of some non empty relatively compact open set B with smooth boundary (for instance a ball). If E is an end, one defines (see [Gri]) its capacity : $cap(E) = \inf_{\varphi\in\Phi} \int_M |\nabla\varphi|^2$, where Φ is the set of smooth maps such that $0 \leqslant \varphi \leqslant 1$, $\varphi|_{E^c} = 0$, and $\varphi = 1$ outside a compact subset of E. If $x \in M$ let $c(x,R)$ be the capacity of the complementary of $B(x,R)$.

Definition. The manifold X is stable at infinity if, as $R \to \infty$, $c(x,R) \to \infty$ uniformly in x.

3.4. Example (See [Gri] Thm. 8.1). Suppose that X satisfies an isoperimetric inequality : for every compact domain $A \subset X$, one has $(vol_{n-1}(\partial A)) > f(vol_n A)$, and suppose that the integral $\int^{+\infty} \frac{dt}{f^2(t)}$ is convergent. Then for every ball $c(x,R) > (\int_{vol(B(x,R))}^{+\infty} \frac{dt}{f^2(t)})^{-1}$, and therefore X is stable at infinity if $volB(x,R)) \to \infty$ uniformly in X.
Recall also the fundamental result of Eells Sampson, also valid for harmonic maps with values in trees ([Gr-Sc] 2.4).

3.5. *Suppose M has bounded geometry and a lower bound ρ on the injectivity radius, then, there exists a constant c s.t. if u is an harmonic map :*
$\sup_{x\in B(x,\rho/2)} |\nabla u| < c(\int_{B(x,\rho)} |\nabla u|^2 dx)^{1/2}$

The stability condition insures the existence of *proper* harmonic maps.

3.6. *Let E be some end. If $cap(E) > 0$, there exists a non constant harmonic map $u : E \to [0,1[$. If furthermore M has bounded geometry, a lower bound ρon the injectivity radius, and is stable at infinity, there exists a proper harmonic map $u : E \to [0,1[$.*

Proof. Suppose $cap(E) > 0$, and let $F : E \to [0,+\infty[$ be a proper C^∞ map with $F^{-1}(0) = \partial E$, and such that $F^{-1}\{n\}$is smooth for all n. For all n let u_n be the solution of the Dirichlet problem $u|_{F^{-1}(0)} = 0$, $u|_{F^{-1}(n)} = 1$, with minimal energy $e(u) = \int_E |\nabla u_n|^2$. The sequence of harmonic maps u_n is uniformly Lipschitz (3.5), therefore converges to some harmonic function $u : X \to [0,1]$, this convergence is uniform on each compact subset, and the sequence $|\nabla u_n|$ also converges uniformly to $|\nabla u|$ on each compact set.
Let D be the Dirichlet space of function $f : E \to \mathbb{R}$ with $f|_{\partial E} = 0$ and $\|f\| = \int_E |\nabla f|^2 < +\infty$. If $m \geqslant n$, u_m is the projection of u_n to the affine subspace $f|_{F(x)\geqslant m} = 1$, and u_m is also the projection of 0 on this space, thus $\|u_n - u_m\|^2 = 2\|u_n\|^2 + 2\|u_m\|^2 - 4\|\frac{u_n+u_m}{2}\|^2 \leqslant 2\|u_n\|^2 - 2\|u_m\|^2$.

The sequence $\|u_n\|^2$ is decreasing, hence converges to some e. Thus $\|u_n - u_m\|^2 \to 0$ as $n \to \infty$, $n \leqslant m$. Choose n so that $\|u_n\|^2 - c \leqslant \varepsilon$. Then if $m \geqslant n$, $\int_{n \leqslant F(x)} |\nabla u_m|^2 \leqslant \varepsilon$.
By definition if $e(u_n) \to 0$, the capacity of E is zero. Therefore if $c(E) \neq 0$, $e(f_n) \to e \neq 0$, and $e(u) \geqslant \int_{n \geqslant F(x)} |\nabla u|^2 \geqslant e - \varepsilon$, thus $e(u) = e$ and u is a non constant harmonic map.
Let us suppose now that M has bounded geometry, a lower bound on the injectivity radius and is stable at infinity, and let us prove that in fact u is a *proper* map $E \to [0,1[$. If not there exists a sequence $x_k \to \infty$, and $u(x_k) \to \alpha < 1$. Let $\beta = \frac{1+\alpha}{2}$. Choose R such that for all x, $cap(B(x,R) > (\frac{1}{1 - \frac{\alpha+\beta}{2}})^2 e(u)$. Let $\varepsilon = \frac{\alpha-\beta}{1000Rc}$, and choose a compact set K so that for all n,$\int_{K^c} |\nabla u_n|^2 \leqslant \varepsilon$.
Choose $k_1 \geqslant k_0$ large enough so that $B(x_{k_1}, R) \subset K^c$. Choose n_1 large enough so that all the function u_n are harmonic on $B(x_{k_1}, R)$ for $n \geqslant n_1$, and such that $u_n(x_{k_1}) \leqslant \frac{3\alpha+1}{4}$ for all $n \geqslant n_0$.
On the set $\Omega = \{x/d(x,K) \geqslant \rho\}$,the function u_n are $c.\varepsilon = \frac{\alpha-\beta}{1000R}$ Lipschitz by 3.5. Thus, for all $n \geqslant n_0$, one has $|x - x_k| \leqslant R \Rightarrow |u_n(x) - u_n(x_{k_1})| \leqslant \frac{\alpha-\beta}{1000}$, and $u_n(x) \leqslant \frac{\alpha+\beta}{2}$.
On the ball $B(x_{k_1}, R)$ the function$v_n = \frac{u_n - \frac{\alpha+\beta}{2}}{1 - \frac{\alpha+\beta}{2}}$ is negative, and this function is 1 outside a compact set of M. Therefore using $\max(v_n, 0)$ to evaluate the capacity, one gets $cap(B(x_n, R)) \leqslant (\frac{1}{1 - \frac{\alpha+\beta}{2}})^2 e(u_n)$, contradiction. □

3.7. Recall that X has more than one end if the complementary of some open relatively compact set B has several connected non compact component. This is equivalent to the existence of a proper surjective C^∞ function f_0 from X onto the open interval $]-1,1[$. Denote by $E[f_0]$ the energy of the proper homotopy class $[f_0]$ of f_0 that is the infimum of the energies of the maps properly homotopic to f_0. The following version of the Dirichlet Riemann Kelvin theorem will be useful.

DRK Theorem. *If X has bounded geometry, a lower bound on its injectivity radius and is stable at infinity, there exists a unique proper harmonic map $f : X \to]-1,1[$ in the same proper homotopy class than f_0.*

The existence follows from the argument of 3.6 ; the uniqueness follows from the convexity of the function $t \to e(tf_0 + (1-t)f_1)$ (which is also valid for maps f_i with values in trees, [Gr-Sc], prop.4.1).

3.8. Let $G = \pi_1(M)$ be the fundamental group of a compact manifold M, and H a subgroup such $X = \tilde{M}/H$ has several ends. The stability of the representation $l^2(G/H)$is equivalent to the non amenability of the Shreier graph $Cay(G)/H$, where $Cay(G)$ is the Cayley graph of G relative to a fixed system of generators Σ. This is equivalent also to the fact that X (or $Cay(G)/H$) satisfies a strong isoperimetric inequality $(vol_{n-1}(\partial A)) > k(vol_n A)$. This linear isoperimetric inequality implies that X is stable at infinity. In this case we say that (G/H) is *stable*.

More concretely, if $R_n \to \infty$, and $B(x_n, R_n)$ is a family of balls in this graph with bounded capacity, there exists a family of functions f_n such that $f_n|_{B(x_n,R_n)} = 1$, f_n=0 outside a compact set and $e(f_n)$ is bounded by some constant C. Then $u_n = \frac{f_n}{\|f_n\|} \in l^2(G/H)$ is an almost invariant vector, as $e(u_n) = \frac{1}{\|f_n\|^2}\Sigma_{g\in\Sigma}\|f_n - f_n \circ g\|^2 \leqslant \frac{C}{vol(B(x_n,R_n))} \to 0$.
Therefore the stability of the representation $l^2(G/H)$ is *stronger* that the stability at infinity of X/H. For example, if $G = \mathbb{Z}^n$, $H = \{e\}, l^2(G/H)$ is not stable but, if $n \geqslant 3$, $\mathbb{R}^n$ is stable at infinity as it satisfies the isoperimetric inequality $vol_{n-1}(\partial A) > k(vol_n A)^{\frac{n-1}{n}}$, and $2\frac{n-1}{n} > 1$.

3.9. Let X be a Riemannian manifold with bounded geometry discretely acted upon by a group G with $H^1(G; l_2(G)) \neq 0$. If X/G has finite volume, then X supports a non-zero exact square integrable harmonic 1-form, i.e. the differential of a non-constant harmonic function f with finite energy (see [Ch-Gr1]).
The above concept of capacity defined via the L_2-norm of the gradient (quadratic energy) extends to all L_p, where the most studied case after $p = 2$ is that of the conformally invariant energy for $p = dim_{\mathbb{R}}(V)$. (This conformal p equals 2 for Riemann surfaces.) What is badly missing for $p \neq 2$ is a Hodge lemma.

3.10. Questions. Let X be an infinite (not necessarily Galois) covering of a compact manifold V with the fundamental group $H \subset \pi_1(V)$. When does a homotopy class of maps $X \to Y$ have a representative f_0 of finite L_p-energy for a given $p \in [1, \infty]$? Here, Y may be some standard metric space, e.g. a flat torus; another possibility is where $Y = X$ and f_0 is homotopic to identity. Among examples one singles out aspherical spaces V, e.g. those with non-positive curvature, and finitely generated groups H such as $H = \mathbb{Z}^n$, for instance.
A closely related question concerns the structure of the L_p-subspaces in the cohomology of X, denoted $L_pH^*(X) \subset H^*(X; \mathbb{R})$, of the cohomology classes of X realizable by closed L_p-forms on X. For example, can such subspace be irrational in the case where $H^*(Y; \mathbb{R})$ comes with a natural $\mathbb{Q}$-structure, e.g. for aspherical X with finitely generated Abelian fundamental group H? (This subspace in the aspherical case is determined solely by $G = \pi_1(V)$, the subgroup $H \subset G$ and the number p).

3.11. If the group G "branches", i.e. has at least three (and hence, infinitely many) ends, then its Cayley graph is stable (at infinity). This can be seen, for example, by exhibiting square integrable 1-cycles b that flow (as currents) from a given non-empty open subset of ends to another, say from ∂_- to ∂_+, where the underlying X is some manifold or polyhedron with a co-compact discrete action of G such as the Cayley graph of G, see for instance [ABCKT] page 50. This stability implies the existence of a harmonic function on X of finite energy separating ∂_- from ∂_+ and thus, the non-vanishing of $H^1(G; l_2(G))$. Notice, one does not use here the Stallings theorem on groups with an infinity of ends. In fact Stallings theorem follows from the existence of a harmonic function separating two complementary open ends, see p.228 in $[\mathrm{Gr}]_{HG}$.

3.12. The non-vanishing of the first l_2-cohomology group $H^1(G; l_2(G))$ in other known examples is established with a use of Atiyah-Euler-Poincaré formula applied to the first two l^2-Betti numbers of infinite Galois G-coverings X' of connected 2-polyhedra V',

$$l_2b^2(V') - l_2b^1(G) = \chi(V')$$

where $\chi(V')$ is the ordinary Euler characteristics,

$$\chi(V') = b^2(V') - b^1(V') + 1.$$

For instance, the inequality $\chi(V') < 0$ for some V' with the fundamental group G yields non-vanishing of $l_2b^1(G)$ and hence, of $H^1(G; l_2(G))$, which amounts to the existence of a non-zero harmonic square integrable 1-cocycle in X'.
A more convincing example is given by G obtained by adding l relations $R_i$to the free product of k infinite groups G_p, such that the natural homomorphisms $G_p \to G$ are injective. Then, by applying the l^2-Mayer-Vietoris sequence, like in [Ch-Gr 2], one checks that the first l_2-Betti number satisfies $l_2b^1(G) \geq k - l - 1$. If the maps $G_p \to G$ have *infinite* image H_p, the same result applies : indeed, G is the quotient of the free products of H_p by the images S_i of the relations R_i in this free product. This infinite image condition can be, probably, removed with a use of Romanovskii Freiheitssatz ; on the other hand, the injectivity holds for generic relations added to free products, by small cancellation theory over free products [L-S].

4. Clean Functions and Maps

4.1. Let f be a pluri-harmonic *function* on a complex manifold X or, more generally, a pluri-harmonic map of X to a *metric graph* Y e.g. to a tree. Then there exists a unique holomorphic 1-codimensional foliation $\mathcal{F}$on X such that f is constant on the leaves of $\mathcal{F}$. We call f clean if the leaves L of $\mathcal{F}$ are closed and say that f is properly clean if the leaves are compact. Notice that "clean" $\Longrightarrow$ "properly clean" if f is proper and thus has compact level sets.
Let use note that, if *some* leaf of $\mathcal{F}$ is compact, and if X is *Kähler*, complete and with bounded geometry, then f is properly clean. Furthermore, the leaves have uniformly bounded volume and diameter.

Proof. (compare [Gr-Sc] p.240). Let us check that the set $Y$of points s.t the leaf through this point is compact is an open set. If L is a compact leaf, as the restriction of $idf$to L is 0, L has a neighborhood where $idf = dg$ is exact. In this neighborhood the foliation $\mathcal{F}$is defined by an holomorphic function F : thus Y is open. Let $Y' \subset Y$ be the set of compact non-singular leaves. On each component of Y' the homology class $[L_x]$ of the leaf through x is constant. If $x \in Y \backslash Y'$, L_x is a singular fiber, and its homology class is $\frac{1}{m}[L_y]$, where L_y is a non singular leaf close to x and $m \in \mathbb{N}$ is the multiplicity. Let Y_1 be a connected component of the open set Y. Let us check that Y_1 is also closed. As Y_1 is connected, and as Y' has codimension 2 in Y, $[L_x] = \frac{1}{m_x}[L]$ for L a generic leaf in Y_1. In particular the volumes of the leaves in Y_1 are uniformly bounded. Let $x_n \to x^*$ be a converging sequence of points of Y_1.

As X is Kähler with bounded geometry, the leaves L_{x_n} have uniformly bounded diameter R. Thus, for n large enough L_{x_n} is in the compact set $K = B(x^*, R+1)$. Let $(B(z_i, \eta))_{1\leqslant i\leqslant k}$ be a finite cover of this compact set by closed balls of radius η, such that on each $B(z_i, \eta)$ the foliation is defined by a function F_i, and such that the balls $B(z_i, \eta/100)$ cover K. If L is a leaf of our foliation, and L pass through some $y \in B(z_i, \eta/100)$, the volume of the connected component of $L \cap B(z_i, \eta/2)$ through y is$\geqslant \alpha$ for some universal constant α . From the bound on the volume of $[L_{x_n}]$, we deduce that there exists a uniform bound on the number of connected component of L_{x_n}through$B(z_i, \eta/100)$. Thus, one can extract a subsequence such that all of these components converge to the leaf through x which is therefore compact □

In the special case where $Y = \mathbb{R}$, the function f locally serves as the real part of a holomorphic function on X whose level sets define the above foliation. Globally, one has a holomorphic 1-form, say a on X with the real part df. This form becomes exact on some Abelian covering $\tilde{X}$ of X; therefore, the lift of f to$\widetilde{X}$ becomes clean. As for the function f itself, it can be clean without a being necessarily exact; in fact, f is clean iff a represents a multiple of a rational (possibly non-zero) cohomology class on X.

4.2. If a harmonic function $f : X \to \mathbb{R}$ is clean then (by an easy argument) there exists, a Riemann surface S, a surjective holomorphic map with connected fibers $h : X \to S$ and a harmonic function $S \to \mathbb{R}$ such that f equals the composed map $X \to S \to \mathbb{R}$. Moreover, this remains true with an arbitrary one-dimensional target space Y in place of R.
If f is properly clean then, clearly, the factorization map $h : X \to S$ is proper. Furthermore, the group H of holomorphic automorphisms of X sends (compact!) leaves to leaves. Indeed, if $g \in H$ and L is a compact leaf, then $h|_{g.L}$ is holomorphic from a compact manifold to a Stein manifold hence it is constant, and $g.L$is another leaf. Thus the group H acts on S and the map h is equivariant. For example, if X serves as a Galois covering of a compact manifold V, then the Galois group G acts on S and V fibers (i.e. admits a surjective holomorphic map with connected fibers) over a Riemann surface. (See [ABCKT] for details and references).

4.3. If $rank H^1(X;\mathbb{R}) < 2$, e.g. if X is simply connected, then, by the above, every pluriharmonic function f on X is clean. Furthermore, if X is Kähler and f has finite energy then some (generic) leaf L_0 has finite volume (due to the co-area formula, see $[\mathrm{Gr}]_{GFK}$). If X is complete and has bounded geometry then "finite volume" $\Longrightarrow$ "compact" for complete holomorphic submanifolds in X. It follows that L_0 is compact; consequently all leaves L are compact (4.1). (All one needs here of the bounded geometry is a slow decay of the convexity radius of X).

4.4. Corollary *Let a countable group G discretely act on a Kähler manifold X such that the quotient X/G has finite volume. If $H^1(G, l_2(G)) \neq 0$ while $H^1(X, \mathbb{R}) = 0$ (e.g. X is simply connected) and and if X has bounded geometry (e.g. the action is co-compact) then X/G fibers over a Riemann surface: there is a holomorphic* *action of G on a Riemann surface S and an equivariant holomorphic map h from X onto S with compact connected fibers.*

This corollary applies, for instance, to non-singular projective varieties V and yields the following (see $[\mathrm{Gr}]_{GFK}$):

4.5. Theorem. *If the fundamental group G of V has $l_2 b^1 \neq 0$ or, equivalently, $H^1(G; l_2(G)) \neq 0$, then G is commensurable to a surface group.*

4.6. If V is singular then the above considerations can be applied to a nonsingular G-equivariant resolution of the universal covering X of V (induced from a resolution of V), say to X, where the corresponding map $X \to S$ necessarily factors via a holomorphic map $X \to S$.

4.7. The simplest way to handle a smooth quasi-projective variety $V_0 = V \setminus W$ would be by constructing a complete Kähler metric on V_0 such that the induced metric on the universal covering X_0 of V_0 had bounded geometry. By the Hironaka theorem, one may assume that V_0 is smooth and W is a divisor with normal crossings where handy candidates for the desired metric come readily; yet, one has to check that the curvatures of such metrics are bounded.

5. Cleanness and branching

Let f be a proper harmonic function $X \to]-1, +1[$ with finite energy separating two open ends on a Kähler manifold X as in 3.4. Cleanness and proper cleanness, as was mentioned earlier, are equivalent for such f; more significantly one has the following :

5.1. *First cleanness criterion.* If X has at least three ends then f is properly clean. This follows from L^2-version of the Castelnuovo de Franchis theorem discussed in [ABCKT] pp. 60-62.

5.2. There is a more general geometric version of this result. Let T be the tree obtained by joining several (finite or infinite number of) copies of the segment $[0, 1[$ at 0 and let a group H isometrically act on T, while fixing the origin 0, so that T/H is a finite tree, perhaps reduced to $[0, 1[$. (Such an action amounts to permuting the copies of $[0, 1[$ with finitely many orbits. This action is not necessarily discrete, not even proper for infinite H since the action fixes 0).

Second cleanness criterion. Let H discretely and isometrically act on a Kähler manifold X and $f : X \to T$ be a surjective H-equivariant pluriharmonic map of H-finite energy. If the tree T has at least three branches (i.e. there are at least three copies of $[0, 1[$ in the above construction) and f is H-proper, i.e. the corresponding map $X/H \to T/H$ is proper, then f is clean; moreover, f is H-properly clean, i.e. the holomorphic leaves L from X go to compact (complex analytic) subsets in X/H under the quotient map $X \to X/H$.
This follows from the argument on pp 239-40 in [Gr-Sch], (see also [Si]) : note that as the number of branches of the tree is at least three, the foliation defined by f

must have a singular leaf (i.e a leaf whose image is a branch point of the tree). This leaf is compact modulo H and therefore (4.1)f is properly clean.

5.3. In order to apply the second criterion one needs a useful version of the DRK-theorem. As X/H is not a manifold but an orbifold, we must modify our assumption on the geometry of X/H. We suppose that X as bounded geometry, a lower bound on the injectivity radius ρ. We also assume that there exist a $r < \rho$ such that for every x in X the ball $B(x,r)$ is the quotient of $B(x,r)$ by a finite group of bounded cardinality. This r plays the role of the injectivity radius in the orbifold case. The definition of capacity and uniform stability at infinity of 3.3 remains valid in this case, as well as the Ells-Sampson Theorem.

Theorem. *If X/H is uniformly stable at infinity, then there exists a proper H-equivariant harmonic map $u : X \to T$ with finite non-zero H-energy.*

Proof. We explain the modifications needed in the proof of 3.6. Let $T - \{0\} =]0,1[\times\Sigma$ where Σ is some set, and let T the metric completion of T obtained by adjoining a point 1 to each interval $[0,1[$. Let $F : T/H \to [0,+1[$ be the folding map, so that $u : F \circ f : X/H \to [0,1[$ is proper. For every n,let $U_n : X \to T$ be the unique solution of the Dirichlet problem $U_n|_{f^{-1}([1/n,1[\times\sigma} = 1 \times \sigma$, U_n is H invariant, and of minimal energy. By the argument of 3.6, U_n converges to some harmonic map $U : X \to T$. It remains to check that $U(X) \subset T$ and U is proper. If $U(x) = 1 \times \sigma$ for some point x, by the maximum principle, U is constant of zero energy, but the G-energy of f is the energy of U, hence the capacity of X/H is zero and this manifold would not be stable at infinity.
Hence the harmonic map U sends X in T. In order to prove that U is proper, let us check that if $x_n \in X$ is such that $f(x_n) \to 1 \times \sigma$, then $U(x_n) \to 1 \times \sigma$. If not, there exists an α such that $U(x_n)$ remains on the complement of $H \times [\alpha, 1[\times\sigma$. Let $e(U)$ be the energy of this harmonic map U, and let $\beta = \frac{1+\alpha}{2}$.
Let $u_n(resp.u) : X/H \to [0,1[$ be the map induced from $F \circ U_n(resp.F \circ U)$.Choose R such that for all $x \in X/H$, $cap(B(x,R)) \geqslant (\frac{1}{1-\frac{\alpha+\beta}{2}})^2 e(u)$. Let $\varepsilon = \frac{\alpha-\beta}{1000Rc}$, and choose a set $K \subset X/H$ so that for all n,$\int_{K^c} |\nabla u_n|^2 \leqslant \varepsilon$. As the image of K by u is compact, one can choose $k_1 \geqslant k_0$ large enough so that $B(x_{k_1}, R) \subset K^c$. Choose n_1 large enough so that all the functions U_n are harmonic on $B(x_{k_1}, R)$ for $n \geqslant n_1$, and such that $u_n(x_{k_1}) \notin [\frac{3\alpha+1}{4}, 1[\times\Sigma$ for all $n \geqslant n_0$. On the set $\Omega = \{x/d(x,K) \geqslant R\}$,the function U_n are $c.\varepsilon = \frac{\alpha-\beta}{1000R}$ Lipschitz. Thus, for all $n \geqslant n_0$, one has $|x - x_k| \leqslant R \Rightarrow |U_n(x) - U_n(x_{k_1})| \leqslant \frac{\alpha-\beta}{1000}$, and $U_n(x) \notin [\frac{\alpha+\beta}{2}, 1[\times\Sigma \leqslant \frac{\alpha+\beta}{2}$. Let $v_n = \frac{u_n - \frac{\alpha+\beta}{2}}{1-\frac{\alpha+\beta}{2}}$ if $u_n(x) \in [\frac{\alpha+\beta}{2}, 1] \times \sigma$, and 0 otherwise : this function is 1 outside a compact set of X/H, and 0 on $B(x_n, R)$. Therefore using v_n to evaluate the capacity, one gets $cap(B(x_n, R)) \leqslant (\frac{1}{1-\frac{\alpha+\beta}{2}})^2 e(u_n)$, contradiction. □

5.4. Hyperbolic example. Let H be a quasi-convex subgroup in a hyperbolic group G such that ∂G is connected and the limit set $\partial(H) \subset \partial(G)$ divides the ideal boundary $\partial(G)$ into at least three components. Then every Riemannian manifold

X with a discrete isometric co-compact action of G admits the above H-equivariant harmonic map f to Y. We postpone the proof to 6.3 and 6.6. Furthermore, if X is Kähler, 5.2 proves that f is H-properly clean.

5.5. The desired branching, i.e. the strict inequality $card(\pi_0(\partial(G)\backslash\partial(H))) > 2$, can be always achieved in the hyperbolic case by enlarging the subgroup H according to the following simple version of the "ping-pong" lemma.

Lemma ([Gr]$_{HG}$5.3.C_1, see also [Ar].) *Let H be a quasi-convex subgroup in a non-elementary hyperbolic group G where $card(\pi_0(\partial(G) \setminus \partial(H))) = 2$. Let f be an hyperbolic element such that no power of f is in H. Then for some power f^k of f the subgroup H' of G generated by f^k and H is quasi-convex; if F is the finite subgroup of H of elements commuting with f^k, H' is the free product $H' = H *_F < f^k \times F >$ amalgamated along F. Furthermore, $card(\pi_0(\partial(G)\backslash\partial(H')) = \infty$.*

Note that the group H' contains H as well as all its conjugates $f^{nk}Hf^{-nk}$. In the boundary of Gall the boundaries of these groups are disjoint (see 6.2 below) and each of them cuts ∂G in (at least) two components, therefore $\partial G\backslash\partial H'$ has infinity many of connected components.
The above lemma can be generalized to many other (e.g. "nearly hyperbolic") groups where H is contained in a larger subgroup H' that usually cuts G into more than three pieces ; yet, the overall picture remains unclear.

5.6. The above cleanness criteria deliver holomorphic fibrations of open Kähler manifolds over Riemann surfaces with compact connected fibers, denoted $f : X \to S$. In the cases of interest such an X comes as a (possibly non-Galois) covering of a compact (e.g. projective algebraic) manifold V, say $p : X \to V$, where one can induce the fibration f from a surface fibration of a finite covering of V according to the following simple lemma. (For a similar statement, see [Ca] 1.2.3 p. 490, or [Ko] Prop. 1.2.11).

Lemma. *Given the above $f : X \to S$ and $p : X \to V$. Then, the normalizer G' of the image of the fundamental group of a generic fiber $f^{-1}(x_0)$is of finite indexin $\pi_1(V, p(x_0)) = G$. The finite covering $V' \to V$ of group G' fibers over a compact Riemann surface S' where the images of the fibers under the covering map $V' \to V$ equal the p-images $p(L) \subset V$ of the f-fibers $L \subset X$.*

Proof. Let us identify X with the quotient $f : X = \tilde{V}/H \to S$, for $H \subset G = \pi_1(V)$. Let y_0 be some pre-image of x_0 and N be the image of $\pi_1(f^{-1}(x_0), y_0)$in $H \subset G$. In order to prove that the normalizer G_1 of N is of finite index in G, it is enough to show that N has only a finite number of conjugate in G. Indeed, the conjugate gNg^{-1} is represented by the image in X of the fiber of F through some point in $p^{-1}(p(y_0)) = G/H$. But all the fibers of f are analytic submanifolds with the *same volume*, in a Kähler manifold of bounded geometry. So their fundamental groups (at the point $g.x_0$)are generated by loops of uniformly bounded length (independent of g). Therefore the images of these groups in $\pi_1(V, p(x_0))$ is generated by elements of bounded length, and can only take a finite number of values. Thus $\tilde{V}/N$ is a Galois cover of the finite cover V' of V of group G', and 4.2 applies. □

6. Cutting Groups by Subgroups

Given a subspace in a proper geodesic metric space, $X_0 \subset X$ let $X_{-r} \subset X$ be the set of points $x \in X$ with distance $\geq r$ from X_0. We take the projective limit for $r \to \infty$ of the projective system of the sets of connected components of X_{-r} and call this limit the space of (relative) ends of $X|X_0$, denoted $Ends(X|X_0)$. Observe that if X is proper (bounded sets in X are relatively compact) and X_0 is bounded then the space $Ends(X|X_0)$ equals the ordinary space of ends $Ends(X)$.
If X is acted upon by a group H we take an orbit X_0 of H and set $Ends(X|H) = Ends(X|X_0)$. If H serves as a subgroup of a finitely generated group G we apply the above to $X = Cayl(G)$ that is the Cayley graph of G and abbreviate by putting $Ends(G|H) = Ends(Cayl(G)|H)$, and $Ends(G/H) = Ends(Cayl(G)/H)$.

6.1. Definitions. Say that $X_0 \subset X$ cuts X (at infinity) if $X|X_0$ has at least two ends, i.e. $card(Ends(X|X_0)) > 1$. In future, the noun "cut" may refers to the fact that X is being cut by X_0 or to an actual division of $Ends(X|X_0)$ into two (or more) non-empty open subsets.
The disjoint union $X \cup Ends(X|X_0)$ carries a natural topology. Thus for every subset $X' \subset X$ one can take its closure in $X \cup Ends(X|X_0)$ and then intersect this closure with $Ends(X|X_0)$. We denote the resulting subset by $\partial_{end}(X') \subset Ends(X|X_0)$ and say that X_0 cuts X' in X if $card \partial_{end}(X') > 1$.
We say that a subgroup $H \subset G$ cuts a group G if $card(Ends(G|H)) > 1$ and an H-cut is called branched if $card(Ends(G|H)) > 2$. A cut of H by G is called stable if the Schreier graph $(Cayl(G)/H)$ is uniformly stable (at infinity) in the sense of 3.3. For this it is enough that $l^2(G/H)$ is stable by 3.8.
If H cuts G, H acts on the set of relative ends $Ends(G|H)$. One can distinguish Schreier cuts where the disconnectedness persists under the action of H, i.e. where the action of H on $Ends(G|H)$ is not transitive. Note that if H cuts G, the cut is a Shreier cut if and only if the Schreier graph $(Cayl(G)/H)$ is disconnected at infinity (see also [CCJV] where such cuts are called "walls"). In particular if $Ends(G|H)$ is finite there exists a subgroup H' of finite index in H such that $Ends(G/H') = Ends(G|H) = Ends(G|H')$, and H' is a Schreier cut of G.

Example. Let S be a compact Riemann surface of genus $\geqslant 2$, C be a simple closed curve separating S in two connected components $S^{\pm}$. If $G = \pi_1(S, x_0)$ operates on the hyperbolic plane D, $H = \pi_1(S^+, x_0)$, then D/H is connected at infinity, whereas $Ends(D|Hx_{_0})$ is infinite, thus H cuts G at infinity, but in fact is transitive on the set of relative ends (see also 6.3).

Historical remarks. The set of ends of a topological space has been introduced by Freudenthal ; for homogeneous space of Lie groups, it has been firstly studied by Borel to prove that there are no action of a Lie group on a simply connected manifold which is 4-transitive [Bo]. After Stalling's famous paper on the structure of groups with an infinity of ends, C. Houghton [Ho] and P. Scott [Sc] began to study ends of pairs of groups.

6.2. Induced cuts. Cuts and their properties (obviously) lift under *surjective* homomorphisms $G \to G$: if H cuts G the so does the pullback $H \subset G$ of H to G; furthermore, the invariance (by the action of H) and stability pass from H-cuts to H-cuts, in other words if H is a Shreier cut of G then H is a Schreier cut of G. Cuts also pass to subgroups $G' \subset G$. In fact, if a finitely generated $G' \subset G$ is cut in G by a subgroup $H \subset G$, then G' is also cut by the intersection (subgroup) $H \cap G' \subset G'$ as follows from the following simple :

Lemma. *Given subgroups G' and H in a finitely generated group with G endowed with the word metric, there exists a function $\epsilon(\delta)$ with $\epsilon(\delta) \to_{\delta \to +\infty} +\infty$, such that the intersection of the δ-neighborhoods of G' and H in the Cayley graph of G is contained in the ϵ-neighborhood of the intersection $G' \cap H$.*

Proof. As the ball in G of radius 2δ is finite, there exists an $r < \infty$ (depending on G' and H) such that if some $g \in G'$ and $h \in H$ are s.t. $|g - h| < 2\delta$,then there exists a pair $(g_0, h_0) in G' \times H$ in the ball of radius r of G s.t. $g^{-1}h = g_0^{-1}h_0$. Then $k = gg_0^{-1} = hh_0^{-1} \in G' \cap H$. This h is at distance $< r$ of g, hence the result with $\epsilon = r + \delta$. □

Thus arbitrary (non necessary surjective) homomorphisms $G_1 \to G$ induce cuts in G_1 from those in G.

6.3. Convex hyperbolic cuts. Let X be a proper geodesic δ-hyperbolic space. Recall (see for instance [CDP]) that subset $Y \subset X$ is *quasi-convex* if there exists a constant A s.t. for every pair $y, y' \in Y$ and any point z in a geodesic segment $[y, y']$, the distance of z to Y is $\leqslant A$. It is known that if Y is A-quasi convex, and $B > A + 100\delta$ the set $Y^{+B} = \{x/d(x, Y) \leqslant B\}$ is 100δ-quasi-convex.
Let H be a group of isometry acting on X. Recall that H is *quasi-convex co-compact* if there exists a geodesic subspace $Y \subset X$ which is quasi-convex and such that the action of H on Y is discrete co-compact. This is equivalent to the fact that the orbit $H.x_0$ of any point is quasi-convex. In the case where X is the hyperbolic space of constant curvature, a *quasi-convex co-compact* group is a geometrically finite group without parabolics.
If H is quasi convex co-compact, it is an hyperbolic group, and its boundary ∂H embeds as a closed subset in ∂X ; it is also the limit set of the action of H on X. It is known (see [Coo]) that the action of H on $\partial X / \partial Y$ is discrete co-compact.
One says that a X is *thin* if there exists a constant B s.t. every point in X is at a distance $\leqslant B$ of a bi-infinite geodesic.

Lemma. *Suppose that H is quasi-convex co-compact in some thin proper geodesic hyperbolic space X. Let $Y = H.x_0$, so that Y is a quasi-convex subset of X. The set of relative ends $Ends(X|Y)$ is the set of "connected components" of $\partial X \setminus \partial Y$.*

Remark. It is possible that $\partial X \setminus \partial Y$is not locally connected. Thus, the expression "connected component" needs an explanation. If $\mathcal{O}$ is an open cover of $\partial X \setminus \partial Y$ by open subsets, let $|\mathcal{O}|^0$ be set of connected components of the nerve of this cover. If $\mathcal{O}'$ is finer than $\mathcal{O}$, $|\mathcal{O}'|^0$ projects onto $|\mathcal{O}|^0$. A limit point of this projective system is a *connected component*. If $\mathcal{O}$ is such a cover, if one replace each $O \in \mathcal{O}$ by $O = \cup_{O' \in n(0)} O'$, where $n(O)$ is the set of O' such that O' belongs to the same connected component of $|\mathcal{O}|$, we get a new cover by *disjoints* open sets having the same set of connected components. So we can restrict our attention to covers by *disjoint* open subsets.

Proof of the lemma. Changing the value of the hyperbolicity constant δ, one may suppose that X is δ-hyperbolic, δ-thin and that $Y$is δ-quasi-convex.
One chooses a H-equivariant projection of $p : X \to Y$ such that $d(x, p(x)) = \min_{y \in Y} d(x, y)$. A ray $[x, y]$ is called a vertical ray if y is a projection of x on Y. If $w \in \partial X/\partial Y$, a ray $\rho = [y, w[$ is called vertical if for every $x \in \rho$,the point y is a projection of x on Y. By properness and δ hyperbolicity every $w \in \partial X/\partial Y$is the end of some ray, and two such rays are 10δ close one to each other. One can extend p to $\partial X/\partial Y$by choosinf once for all and for every ω a vertical ray $\rho_\omega = [y, \omega[$ which ends at ω, and setting $p(w) = y$. This choice can be make equivariant. If $w, w' \in \partial X/\partial Y$, let $\rho_w(t), \rho'_w(t) : [0, \infty[\to X$ be two geodesic parameterizations of the rays $[p(\omega), \omega[, [p(w'), w'[$. One sets $< w, w' >_Y = \max\{t / d(\rho(t), \rho'(t)) \leqslant 10\delta\}$.
Let us first prove :
(1) as X is thin every point x s.t. $d(x, Y) > 200\delta$ is at the distance $\leqslant 100\delta$ to some vertical ray $[y, w[$ with $y \in Y$.
Choose some bi-infinite geodesic $]w, w'[$ s.t. $d(x,]w, w'[) \leqslant \delta$. By δ-quasi convexity, w or w' do not belong to ∂Y. If $w' \in \partial Y$, and p is a projection of w on Y, so that $]p, w[$is a vertical ray, the hyperbolicity proves that $]w, w'[$ is 10δ close to $]w, p[\cup]p, w'[$ therefore x is 11δ close to the vertical ray $]w, p[$. If neither w nor w' are in ∂Y, let p and p' be projections of w, w' on Y, so that $]p, w[$and $]p', w'[$ are vertical rays. The hyperbolicity proves that $]w, w'[$ is 10δ close to $]w, p[\cup]p, p'[\cup]p', w'[$. By quasi-convexity, x cannot be close to $]p, p'[$, and is therefore close to one of the two vertical rays $]w, p[,]p', w'[$.
Le C be a component of Y_{-r} s.t. there exists a point in C with $d(x_0, C) > r + 10000\delta)$. Let $O(C) = \{w$: $\rho_w \cap C$ is not compact$\}$. Let us check that $O(C) \neq \emptyset$. By (1) x is a the distance $\leqslant 100\delta$ of some geodesic ray $[y, w[$ with $y = p(w) \in Y$. Let $x' \in [y, w[$ with $d(x, x') \leqslant 100\delta$. Then $d(x', y) \geqslant r$ and therefore $[x', w[\subset C$. It is easy to see that $O(C)$ is an open set, and that the collection $\mathcal{O}_r$ of all these sets is a cover of $\partial X \backslash \partial Y$. These sets are *disjoint* and $\mathcal{O}_r$ is an open cover of $\partial X \backslash \partial Y$ by disjoint sets. Thus $|O_r|$ is the set of connected components of Y_{-r} which are the image of a connected component of $Y_{-r'}$, $r' > r + 100\delta$ under the natural projection $Y^0_{-r'} \to Y^0_{-r}$.
Let $\mathcal{O} = (O_i)_{i \in I}$ be a H-invariant, H-finite cover of $\partial X/\partial Y$ by non empty *disjoint* open subsets. In order to conclude, its is enough to prove the following : there exists an r s.t. $\mathcal{O}_r$ is finer than $\mathcal{O}$.
As H is *co-compact* in $\partial X \backslash \partial Y$, and as our cover H-invariant, there exists an $s > 0$ s.t. for every w there exist an i s.t if $< w, w' >_Y > s \Rightarrow w, w' \in O_i$.

Let us choose such an s. Let $r = s + 1000\delta$. Let C be some component of $\mathcal{O}_r$. For each x in C, let $O(x)$ be the (non-empty, due to (1)) set of endpoints of vertical rays $\rho = [y, \omega[$s.t. $d(x, \rho) \leqslant 100\delta$. If $d(x, x') \leqslant \delta$, then the product $< \omega_x, \omega_{x'} >$is bigger than s. By connexity there exist an i s.t. for all $x \in C$, $w_x \in O_i$. If $w \in O(C)$ there exist a ray $[y, \omega[$ which contains a point x of C. Then $w \in O_i$, and $O(C) \subset O_i$. □

A particular instance of this is the Cayley graph X of a non-elementary word hyperbolic group $G \supset H$.

6.4. Full systems of convex cuts. Say that a (usually infinite but with finitely many mutually non-conjugate members) collection of convex subgroups H_i in a word hyperbolic group G fully cuts G if for every pair of distinct points in the ideal boundary $\partial(G)$ there is some H_{i_0} among H_i whose limit set $\partial(H_{i_0}) \subset \partial(G)$ separates these points, i.e. they lie in different connected components of the complement $\partial(G) \setminus \partial(H_{i_0})$. (This definition can be extended to general groups and spaces but we are mostly concerned with convex cuts in hyperbolic groups and in $CAT(0)$-spaces).

Examples. (a) If some immersed compact totally geodesic hypersurfaces W_j cut a compact manifold V of negative curvature into simply connected pieces then the conjugates of the fundamental (sub)groups of W_j's fully and convexly cuts $\pi_1(V)$. (This, with an appropriate definition, remains valid for arbitrary $CAT(0)$-spaces).

(b) If G is a reflection group then the isotropy subgroups of the walls provide a full system of convex Schreier (as in 6.1) cuts of G (where G does not even has to be hyperbolic for this matter, see [BJS]).

(c) The above generalizes to cubical $CAT(0)$-polyhedra and their isometry groups (see [Sa] [CCJJV]). In particular hyperbolic groups co-compactly acting on such polyhedra admit full systems of convex Shreier cuts.

(d) There are compact (arithmetic) n-manifolds for all $n > 1$ of constant negative curvature with a full system of Shreier hyperplane cuts of its universal covering and hence, of their fundamental groups. (It is unlikely that the fundamental group of each n-manifold of constant negative curvature admits a convex cut for $n > 2$ but no counter example seems to be known even for $n = 3$.)

(e) Dani Wise (see [Wi]) has shown that many small cancellation groups G, including geometric $C'(1/6)$-groups, admit full systems of convex Schreier cuts H_i with at most finitely many mutually non-conjugate among them. In conjunction with Sageev's theorem his result provides, for all such G, a cubical $CAT(0)$-polyhedra with fundamental groups G assuming G has no torsion.

6.5. Stability of hyperbolically induced cuts. If a f.g subgroup $A \subset G$ in a word hyperbolic group G is cut by a quasi-convex subgroup $H \subset G$ then the induced cut of A by $H \cap A$ is stable unless A is virtually cyclic. It follows that the cut of an arbitrary finitely generated group G_1 induced from a convex cut of a hyperbolic group by a homomorphisms $G_1 \to G$ is *stable* (hence G/H is uniformly stable at infinity) except for the virtually cyclic image case.

Proof. (Compare [K] for a discussion of the case where A is quasi-convex.) In order to prove this proposition it is enough to show:

Proposition. *Let A be a subgroup of a hyperbolic group G, and H be a quasi-convex subgroup in G. If $A/A \cap H$ and $A \cap H$ are infinite, or if A is non elementary and $A \cap H$ is finite, then$A$ contains a free group F such that F meets no A-conjugate of $A \cap H$, i.e. F freely operates in $A/A \cap H$.*

If A is non elementary and $A \cap H$ is finite, as$A$ contains free subgroups, the result is obvious. So we may assume that that $A/A \cap H$ and $A \cap H$ are infinite.
In order to prove this proposition, we think of G as a uniform convergence group on its boundary ∂G. Our proof is therefore also valid if G is a *geometrically finite* convergence group on a compact set M provided that H is *fully quasi-convex* in the sense of Dahmani [Da], i.e. H is quasi-convex and meets each parabolic subgroup of G either in a finite group of a subgroup of finite index.
Recall that $\partial^2(G)$ denotes the set of *distinct* pairs of elements in ∂G. As H is quasi-convex it is hyperbolic, and its limit set is equivariantly homeomorphic to ∂H.Let $\Lambda^2(A)$ be the closure in $\partial^2(G)$of the set of pairs (a^+, a^-) of fixed points of hyperbolic elements in A. If A is quasi-convex, $\Lambda^2(A) = \partial^2(A)$.

Lemma. *$\Lambda^2(A) \cap \partial^2 H$ is of empty interior in $\Lambda^2(A)$.*

Before proving this lemma let us recall basic facts about quasi-convex subgroups (for a proof also valid in the case of geometrically finite convergence groups and fully quasi-convex subgroups, see [Da]).

Proposition. *Let H be a quasi-convex subgroup of G, and let $g_n \in G/H$ be an infinite sequence of distinct elements.*

i) *The intersection $\cap g_n \partial H$ is empty.*
ii) *Furthermore, if g_n is a representative of g_n of minimal length mod H, i.e. $d(g_n, H) = d(g_n, e)$, then $d(g_n, e) \to \infty$. Suppose that $g_n \to \alpha \in \partial G$. Then $g_n \partial H \to \alpha$ as well. In particular the set $\mathcal{L} = \cup_n g_n \partial H^2$is closed in $\partial^2 G$.*
iii) *Let H_1, H_2 be two quasi-convex subgroups of G, then $H_1 \cap H_2$ is quasi-convex and $\partial H_1 \cap \partial H_2 = \partial(H_1 \cap H_2)$. Furthermore $\partial H_1 \cap \partial H_2$ is of empty interior in ∂H_1 unless $H_1 \cap H_2$ is of finite index in H_1.* □

Proof of the lemma. Suppose first that A is quasi-convex . Assume that the lemma is false ; as A is quasi-convex, the set of pairs $(a^+, a^-), a \in A$, a hyperbolic is dense in $\partial^2 A, ([Gr]_{HG} 8.2.G)$. Thus, in this case, $\Lambda^2(A) = \partial^2(A)$. But $\partial A \cap \partial H = \partial(A \cap H)$, and this set is of empty interior in ∂A, i.e nowhere dense, in ∂A, unless $A/A \cap H$ is finite.
Let A be not necessary quasi-convex and assume again the lemma is false. Then there exists an hyperbolic element u in A s.t (u^+, u^-)belongs to the interior of $\Lambda^2(A) \cap \partial^2 H$. Let $u = u_1, u_2, \ldots\ldots. u_n \ldots.$be the list of hyperbolic elements of A, and let n be a fixed integer. For n_i large enough, $(u_i^{n_i})_{1 \leqslant i \leqslant n}$ generate a free q.c group A_n : if the lemma is false, $\Lambda^2(A_n) \cap \partial^2 H$ is not of empty interior in $\Lambda^2(A_n)$. Thus $A_n \cap H$ is of finite index in A_n, and every element of A has a power in H.

Therefore $\Lambda^2(A) \subset \partial^2(H)$. Suppose $A \cap H$ is not of finite index in A, and let a_n be an infinite sequence in $A/A \cap H$. We may assume that $a_n\partial(H)$ converges to some point $\alpha \in \partial G$, but this is impossible as $a_n(\partial(H)^2) \supset a_n\Lambda^2(A) = \Lambda^2(A)$. □

Proof of the proposition. Let $\mathcal{L} = \cup_{a\in A} a\partial H^2 = \cup_{a\in A}(\partial aHa^{-1})^2$. This set is closed in $\partial^2(G)$,therefore its intersection with $\Lambda^2(A)$ is closed, and applying the lemma to the family $(aHa^{-1})_{a\in A}$, nowhere dense in $\Lambda^2(A)$. Therefore, we can choose an hyperbolic element $a \in A$ such that $(a^+, a^-) \in \mathcal{L}^c$. Choose $m \in A$ be some element s.t. $ma^+ \neq a^\pm$ (for instance any element of infinite order in the infinite group $A \cap H$). For N large enough, the group $< a^N, ma^N m^{-1} >$ has its limit set in the neighbourhood of the four points a^+, a^-, ma^+, ma^-, and no element of this group is conjugate in H. □

6.6. Implementation of cuts by maps into trees. A Schreier cut, i.e. a partition of the space of ends of X/H for a Riemannian H-manifold X (or a general geodesic space for this matter, e.g., the Cayley graph of a group G) into two open subsets, say ∂_- and ∂_+, can be implemented by a proper function $f_0 : X/H \to]0,1[$ with finite energy (compare 3.4) which lifts to an H-invariant function on X with finite H-energy.
The latter function can be defined for general non-Schreier cuts with $]0,1[$ replaced by a tree Y as in 5.2. Namely, for each $r > 0$ we denote by $Comp_r$ the set of connected components of the subset $X_{-r} \subset X$ of points within distance$> r$ from an H-orbit $X_0 \subset X$ of a base point x_0 in X (compare 6.1) and denote by $c : X_{-r} \to Comp_r$ the tautological map. We assign a copy of $[0,1[$ to each point c of $Comp_r$, denoted $[0,1[_c$, and choose a proper monotone function $d : [0,\infty[\to [0,1[$ that vanishes on $[0.r]$. Then we construct the map f_r from X to the tree Y obtained by identifying the copies $[0,1[_c, c \in Comp_r$ of $[0,1[$ at 0 as the composition of the maps c, $dist(.,X_0)$ and d, that is each $x \in X$ goes to $d(dist(x,X_0)) \in [0,1[_c \subset Y$ for $c = c(x)$.
This map f_r is H-proper as well as H-invariant and it can be easily adjusted to have finite H-energy. Among the branches $[0,1[_c$ of Y not all are essential, i.e. totally covered by the image of f_r. The non-essential branches can be removed by retracting them to the root 0 of Y; as $r \to \infty$ the number of essential branches converges to $cardEnds(X|H)$.

7. Cuts in Kähler groups

7.1. A Riemann surface S and its fundamental group can be cut in many ways and these cuts pass to complex manifolds fibered over S. Conversely, by combining the above and 5.6 (compare [De-Gr]), one conclude to the following:

7.2. Cut Kähler Theorem. *Let the fundamental group G of a Kähler manifold V be cut by a subgroup $H \subset G$, where this cut, call it C, satisfies the following two conditions.*

(1) *The cut C is stable.*
(2) *The cut C is branched, i.e. $card(Ends(G|H)) > 2$.*

Then C is virtually induced from a Riemann surface: a finite cover V of V admits a surjective holomorphic map to a Riemann surface with connected fibers, $V \to S$, such that the pullback to $\pi_1(V) \subset \pi_1(V)$ of some subgroup in $\pi_1(S)$ equals $H \cap \pi_1(V)$. In particular, the kernel of the induced homomorphism $\pi_1(V) \to \pi_1(S)$ is contained in $H \cap \pi_1(V)$.

Remarks. (a) The stability condition is violated, for instance, for cuts of Abelian groups G; yet, the conclusion of the theorem, when properly (and obviously) modified, holds in this case. But, it remains unclear how the general non-stable picture looks like.

(b) It seems that the desired cleanness does not truly need the branching condition (introduced solely for cleanness sake) but it is unclear how to remove or significantly relax it in the general case. However, branchings come cheap in the hyperbolic case (see 5.5) that brings along the following corollary where there is no explicit reference to any branching.

(c) In the case of a Shreier cut, this result has been proved by Napier and Ramachandran [N-R]

Corollary. *If a Kähler group is hyperbolic and admits a convex cut then it is commensurable to a surface group. Moreover, let $h : G \to G_0$ be a homomorphism where G is a Kähler (not necessarily hyperbolic) group, G_0 is a hyperbolic one admitting a full system of convex cuts, e.g. a Wise small cancellation group and $h(G)$is not virtually cyclic. Then the restriction of h to a subgroup $G' \subset G$ of finite index factors through an epimorphism $G' \to \pi_1(S')$ induced by a holomorphic map $V' \to S'$ followed by a homomorphism $\pi_1(S) \to G_0$, where V is the finite covering of V with the fundamental group G' and where S' is a Riemann surface.*

Proof. From a convex cut of $h(G)$ by a subgroup H, we construct a convex branch cut of $h(G)$(5.5) by a group H', s.t. $H' \supset H$. Thus $h^{-1}(H')$ is a branched stable (6.5) cut of G, and we get a proper holomorphic map with connected fibers $f : X = \tilde{V}/h^{-1}(H') \to S$ for some non compact Riemann surface S. By 5.6 one gets a finite cover V' of V and an holomorphic map to a compact Riemann surface $f' : V' \to S'$, s.t the fibers of f' are the images of the fibers of f. But the image of the fundamental group of the generic fiber of f' is normal in $G' = \pi_1(G)$. Its image is a normal subgroup of $h(G)$ contained in the convex subgroup H'. But no convex group in a hyperbolic group contains an infinite normal subgroup of infinite index, therefore $h(S')$is of finite index in $h(G)$. □

7.3. Remarks. (a) Probably, most hyperbolic groups, including the majority of small cancellation ones admitting convex cuts, have $l_2 b^1 = 0$ and thus, at this point conjecturally, the above applies to a much wider class of groups than the $l_2 b^1$-theorem (see 4.5).

(b) There are Kähler hyperbolic groups that admit non-convex cuts but no convex ones. In fact, by a construction of D. Kazhdan, there are compact Kähler manifold of constant Hermitian curvature of any dimension n with infinite 1-dimensional homology groups and hence, with cuts induced from Z. These have no convex cuts for $n > 1$ by the above Corollary (or by a direct application of Grauert's solution to the Levi problem).

7.4. sbc-Groups. Consider all stable branched cuts of a group G and denote by $K = K_{sbc} \subset G$ the intersection of the subgroups $H \subset G$ implementing these cuts. We call this K the sbc-kernel of G and say that it has finite type if there finitely many subgroups among H's such the intersection of all conjugates of these equals K. We say that G is of sbc-type if K equals the identity element id in G, where "finite sbc-type" means the finiteness of the type of $K = id$ of stability, branching and finiteness conditions).

Example. A finitely generated subgroup G in the product of surface groups is of finite sbc-type, unless it admits a splitting $G = G_0 \times \mathbb{Z}$.

The above Theorem yields that the following converse to this example.

7.5. sbc-Theorem. *If the sbc-kernel K of a Kähler group G has finite type, then a finite covering $V \to V$ admits a holomorphic map to a finite product of Riemann surfaces $f : V \to W = (S_1 \times S_2, \dots, \times S_N)$ where the kernel of the induced homomorphism of the fundamental groups equals $K \cap \pi_1(V) \subset \pi_1(V)$. Moreover, one can choose a Galois covering $V \to V$ and a G-equivariant map f for some holomorphic action of the Galois group G of $V \to V$ on W.*

sbc-Corollary. *Let G be a torsion free Kähler group with no non-trivial Abelian normal subgroups. Then G admits a subgroup G of finite index isomorphic to a subgroup in the product of N surface groups if and only if G is of finite sbc-type.*

Remark. The minimal N in this Corollary (obviously) equals the maximum of the ranks of the free Abelian subgroups in G. There are only finitely many S_i for a given V but the relations between these S_i for different finite coverings V seems rather obscure.

7.6. Cut-Kähler conjecture. Probably, the stability, branching and finiteness conditions are not truly needed and the above theorem could be generalized as follows. Let K_c denote the intersection of all cutting subgroups in G. Then a finite covering of V admits a holomorphic map $f : V \to W$, where W is a flat Kähler torus bundle over the product of a several Riemann surfaces S_i and where the the kernel of the induced homomorphism of the fundamental groups equal $K_c \cap \pi_1(V)$. (If V is algebraic then W is a product of S_i's with an Abelian variety). We shall return to this problem in [De-Gr].

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Thomas Delzant
Institut de Recherche Mathématique Avancée, Université Louis Pasteur, 7 rue René Descartes, 67084 Strasbourg, France
e-mail: `delzant@math.u-strasbg.fr`

Misha Gromov
Institut des Hautes Études Scientifiques, 35 route de Chartres, 91140 Bures-sur-Yvette, France
e-mail: `gromov@ihes.fr`