NONSINGULAR MAPPINGS OF STEIN MANIFOLDS

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1. Let M be a Stein manifold, and $\mathscr{O}$ a bundle of germs of holomorphic functions on this manifold. For the vector fibering $X \to M$ we shall denote by $\mathscr{A}(X)$ a bundle of germs of its holomorphic cuts, by $X^s \to M$ a fibering of s-jets of cuts belonging to $\mathscr{A}(X)$, and by $J^s: \mathscr{A}(X) \to \mathscr{A}(X^s)$ the operator of taking an s-jet. For fiberings X, $Z \to M$ the differential operator $D: \mathscr{A}(X) \to \mathscr{A}(Z)$ and the $\mathscr{O}$-homomorphism $\Delta: \mathscr{A}(X^s) \to \mathscr{A}(Z)$ are said to be associated if $D = \Delta \circ J^s$.

For an array $\{\psi_i\}_{1 \leq i \leq q}$ of (not necessarily holomorphic) cuts $M \to Z$ we shall denote by $\Sigma_k\{\psi_i\} \subset M$ a set of points $m \in M$ such that these cuts generate in fibers $Z_m \subset Z$ over these points a space of dimension smaller than k. An array $\{\psi_i\}$ is said to be Σ_k-nonsingular if $\Sigma_k\{\psi_j\} = \varnothing$ $k = 1, 2, \ldots, \min(q, \dim Z_m)$.

2. Let $\Delta: \mathscr{A}(X^s) \to \mathscr{A}(Z)$ and $D: \mathscr{A}(X) \to \mathscr{A}(Z)$ be a $\mathscr{O}$-homomorphism and an operator that are associated. By $I_k(p;\ q-p)$ we shall denote the space of arrays $\{\gamma_i, \psi_j\}$ of holomorphic cuts γ_i: $M \to X$, ψ_j: $M \to X^s$, $1 \leq i \leq p$, $p+1 \leq j \leq q$, such that the array $\{D(\gamma_i);\ \Delta(\psi_j)\}$ is Σ_k-nonsingular.

2A. If $k < q$, the mappings $I_k(p+1;\ q-p-1) \to I_k(p;\ q-p)$, consisting of a replacement of the cut γ_{p+1} by a jet $\varphi_{p+1} = J^s(\gamma_{p+1})$, will be weak homotopic equivalences.

2B. COROLLARY. If Δ is an epimorphism, $q > k$, and there exists a Σ_k-nonsingular array of q holomorphic cuts $M \to Z$, the space $I_k(q;\ 0)$ will be nonempty.

It is convenient to compare 2B with two corollaries of the Ika-Grauert principle.

2C. In three cases, a) $k = 1$, b) $k = q$, and c) $k = \dim Z_m$ ($m \in M$), the presence of a Σ_k-nonsingular array of q continuous cuts $M \to Z$ implies the existence of the same type of holomorphic array.

2D. Let us write $n = \dim M$ and $r = \dim Z_m$. In the following five cases there exists a Σ_k-nonsingular array of q holomorphic cuts $M \to Z$: a) $k = 1$, $2rq > n$; b) $k = q$, $2(r-q+1) > n$; c) $k = r$, $2(q-r+1) > n$; d) $r \geq q$, $n < (r-k+1)(q-k+2)$; e) $q-r-2 > \sum_{i=0}^{v} 2^{-(i+1)} a_i^{-1}(n - a_i^2)$, где $a_i = r - k + 2i + 1$, $v = \left[\frac{1}{2}(\sqrt{n} + k - r - 1)\right]$.

3. Outline of Proof of Proposition 2A. For $\Pi \subset M$ we shall denote by $\mathrm{Id}\,\Pi \subset \mathscr{O}$ an associated bundle of ideals. A differential operator D_0: $\mathscr{A}(X) \to \mathscr{O}$ is said to semitransverse to an analytic set $\Sigma \subset M$ if for any natural t, any analytic $\Pi \subset \Sigma$ with $D_0((\mathrm{Id}\,\Pi)^t \mathscr{A}(X)) \subset \mathrm{Id}\,\Pi$ and any germs $f \in \mathscr{O}$, $\gamma \in (\mathrm{Id}\,\Pi)^{t-1}\mathscr{A}(X)$ we have $D_0(f_\gamma) - f D_0(\gamma) \in \mathrm{ID}\,\Pi$.

3A. If D_0 is semitransverse to Σ and the homomorphism over Σ associated with D_0 is epimorphic, there exists a holomorphic cut γ: $M \to X$, such that the function $D_0(\gamma)$ does not vanish on Σ.

Proof. By successive continuation to the appropriate layers of the set Σ we construct γ and a function ψ on Σ with $\exp \psi = D_0(\gamma) | \Sigma$.

We show how to construct on the basis of an array belonging to $I_k(p;\ q-p)$ an array belonging to $I_k(p+1;\ q-p-1)$ (thus we prove 2B, but not 2A). We utilize the fact that for a typical array $\alpha = \{\gamma_i;\ \varphi_j\} \in I_k(p;\ q-p)$ there exists a $\mathscr{O}$-homomorphism $\delta: \mathscr{A}(Z) \to \mathscr{O}$, for which the operator D_0: $\mathscr{A}(X) \to \mathscr{O}$ associated with $\Delta_0 = \delta \circ \Delta$ is semitransverse to $\Sigma = \Sigma_k\{D(\gamma_i);\ \Delta(\varphi_j)\}$, $1 \leq i \leq p$, $p+2 \leq j \leq q$, all the $D_0(\gamma_i)$, $\Delta_0(\varphi_j)$, $j \geq p+2$, vanish on Σ, whereas $\Delta_0(\varphi_{p+1})$ does not vanish on Σ. By replacing in the array α the cut φ_{p+1} by the same γ as in 3A, we obtain the sought-for array.

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4. Mappings $M \to C^q$. For a holomorphic mapping f: $M \to C^q$ with coordinates $f_i : M \to C^1$, $1 \leq i \leq q$, we shall write $\Sigma_k(f) = \Sigma_k \{df_i\}$ [the differentials df_i are cuts $M \to T^*(M)$] and call it a Σ_k-nonsingular mapping if $\Sigma_k(f) = \varnothing$. Mappings which are Σ_k-nonsingular with $k = \dim M$ are immersions.

4A. An n-dimensional Stein manifold can be immersed in C^q with $q = [3n/2]$.

Proof. For the case $n = 1$, see [2]; the case $n > 1$ reduces to 2B and 2D.

A more thorough analysis (see 4B) leads to a proper immersion in C^q with $q = [3n/2] + 1$ and to a proper Σ_{n-1}-nonsingular mapping in C^q with $q = 1 + n + [n \ln \sqrt{2}]$.

4B. An n-dimensional Stein manifold can be properly embedded in C^q with $q = [3n/2] + 2$. (For $q = [5n/3] + 2$, see [1]).

Outline of Proof. Let H^q be a set of holomorphic mappings $h : M \times M \to C^q$ with $h(x, y) = -h(y, x)$, $(x, y) \in M \times M$. For f: $M \to C^q$ we shall define $Df \in H^q$ by the formula $Df(x, y) = f(x) - f(y)$. Let $IH^q \subset H^q$ be a set of $h \in H^q$ for which the pre-image $h^{-1}(0)$ coincides with the diagonal in $M \times M$, and the restriction of the differential dh to the diagonal has rank n. We shall begin with a proper mapping f: $M \to C^{n+1}$, for which codim $\Sigma_i(f) \geq (n-i+1)(n-i+2)$, $1 \leq i \leq n$, whereas the dimensions of double and triple self-intersections are not larger than $n-1$ and $n-2$. Then we construct an $h \in H^p$, $p = [n/2] + 1$, such that the pair $(Df, h) \in H^{n+1} \times H^p = H^q$ belongs to IH^q. Finally, by successively replacing (see Section 3) the coordinates $h_i \in H^1$, $1 \leq i \leq p$, we transform h into a g: $M \to C^p$ such that the pair $(Df, Dg) \in H^q = H^{n+1} \times H^p$ will also belong to IH^q. The holomorphic mapping $f \times g : M \to C^{n+1} \times C^p$ is the desired one.

5. The above results remain valid for coherent bundles over Stein spaces. Moreover, Proposition 2B can be partially extended to affine manifolds over an algebraically closed field K. For example, for linearly independent vector fields $X_1, \ldots, X_n$ over such a manifold M there exists a regular mapping $f : M \to K^{n+1}$, for which the vectors $(X_i f)(m) \in K^{n+1}$, $m \in M$, are linearly independent. In particular, an n-dimensional linear algebraic group can be immersed in K^{n+1}.

LITERATURE CITED

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2. R. C. Gunning and R. Narasimhan, Math. Ann., 174, No. 2, 103-108 (1967).