CONSTRUCTION OF A SMOOTH MAPPING WITH PRESCRIBED JACOBIAN. I

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1. FORMULATION OF THE PROBLEM AND STATEMENT OF RESULTS

1.1. Introduction

1.1.1. Let M and N be smooth n-dimensional manifolds, the second being provided with a nondegenerate (i.e., nonvanishing) n-form ω. We ask when a given (possibly degenerate) n-form can be induced from ω onto M by a smooth mapping $M \to N$.

1.1.2. The appearance of the present paper is motivated by a question raised by V. I. Arnold' (see [1]): Can an exact 2-form be induced onto a two-dimensional sphere by a mapping onto a plane? Later we learned from V. I. Arnold' that A. B. Katok has an affirmative answer to this problem and then D. V. Anosov informed us that A. B. Krygin has carried Katok's results over to other orientable surfaces.

1.1.3. In the first part of our paper, published here, it is shown that every exact n-form can be induced onto a stably parallelizable manifold M by a mapping $M \to R^n$.

This proposition is generalized and refined by the formulations in Sec. 1.2, which are proved in Sections 2.3 and 2.4. One necessary condition for inducibility is formulated in Sec. 1.3 and is demonstrated in Sec. 2.5.

In the second part of the paper inducibility conditions will be analyzed in detail, the case where M and N are of different dimensionality will be considered, and applications to other geometrical problems will be given.

1.2. Affirmative Results

1.2.1. In this paper the term "smooth" always refers to smoothness of the class C^∞.

In the case of a smooth mapping f of the manifold M into the manifold N with the form ω the form induced on M is denoted by $f^*(\omega)$. For fixed n-forms σ on M and ω on N the mapping f: $M \to N$ is called exact if the difference $\sigma - f^*(\omega)$ is exact. If the manifolds are closed, then exactness is equivalent to the equality $\int\limits_M \sigma = \deg f \int\limits_N \omega$, where deg is the degree of the mapping. If N is open exactness is equivalent to exactness of the form σ and if M is open every mapping $M \to N$ is exact.

A mapping f: $M \to N$ is called stably parallelizable if the fibration induced by it out of T(N) is stably equivalent to the tangential fibration T(M).

We call a form σ on an orientable manifold a form of variable sign if it takes both positive and negative values. An equivalent condition is the fulfillment of the strict inequality $\int\limits_M |\sigma| > \left|\int\limits_M \sigma\right|$.

1.2.2. Let M and N be connected n-dimensional ($n > 1$) manifolds with smooth n-forms σ and ω, the second being nondegenerate, and let f_0: $M \to N$ be an exact, stably parallelizable mapping. Then in the following three cases there exists a smooth mapping f: $M \to N$ with $f^*(\omega) = \sigma$ which is homotopic to the mapping f_0:

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1) the form σ is of variable sign;

2) f_0 is a smooth imbedding;

3) M is an open manifold.

Remark 1. Case 3) follows both from 1) and from 2). The reduction $1 \Rightarrow 3)$ is obvious since a form with an open manifold can be extended with the appearance of variable sign onto a larger connected manifold. The reduction $2 \Rightarrow 3)$ is based on Hirsch's theorem (see [6]), according to which f_0 can be deformed into an imbedding in our arrangement.

Remark 2. As will become apparent from the construction of the mapping f (see Sections 2.2, 2.3), it has a rank not less than $n-1$ at every point $m \in M$.

Corollaries

1.2.3. Let σ and ω be smooth n-form on an n-dimensional $(n > 1)$ manifold M, the form ω being nondegenerate and the difference $\sigma - \omega$ being exact. Then there exists a smooth mapping f: $M \to M$ with $f^*(\omega) = \sigma$, which is homotopically identical and has a rank not less than $n-1$ at every point.

This follows from case 2) and Remark 2 of Section 1.2.2.

1.2.4. Let M be a connected, orientable manifold and $A \subseteq M$ be a closed set with nonempty complement. Then there exists a smooth mapping f: $M \to M$, which has the rank $n-1$ in points of A and the rank n on the complement $M \setminus A$. Moreover there exists a smooth mapping g: $M \times [0, 1] \to M$ such that the contraction $g|_{M \times 0}$ is the identical mapping, and all $g|_{M \times 1} = f$, $g|_{M \times t}$ with $t < 1$ are diffeomorphisms.

If g is ignored, 1.2.4 follows easily from 1.2.3, and the existence of g will become evident in Sections 2.2 and 2.3.

1.2.5. Proposition 1.2.4 is also valid in the non-orientable case, a fact that will become evident from the proofs. The same can also be said of Propositions 1.2.2 and 1.2.3, but in addition to the proofs of these, it is necessary to refine the formulation.

Mappings with Folds

1.2.6. Suppose that the set of zeros of the form σ is a smooth manifold $Z \subset M$ of codimensionality 1. This rather typical situation provides an example of a form whose density (with respect to any nondegenerate form) is a function without a zero critical value. For the induction of a similar form it would be ideal to achieve a mapping $M \to N$ having no other singularity than a fold on Z.

We shall describe the situation more precisely. Let M and N be connected, orientable, n-dimensional $(n > 1)$ manifolds having no boundaries, and suppose that M is divided by a non-empty manifold Z into two (necessarily connected) manifolds M_+ and M_- with the common boundary Z. We call a smooth form on admissible if it is positive within M_- and negative with M_-. We assume that the smooth n-form ω on N is positive. The smooth mapping f: $M \to N$ is admissible if the induced form $f^*(\omega)$ is admissible and the contraction $f|_Z : Z \to M$ is a smooth imbedding.

1.2.7. For any admissible form σ and any exact admissible mapping $M \to N$ there exists an admissible mapping f: $M \to N$ with $f^*(\omega) = \sigma$ that is homotopic to it.

1.2.8. The necessary and sufficient condition for the existence of an admissible mapping (i.e., a mapping with a prescribed fold) is to be found in [4]. Referring to [4] for the general case, we state only a corollary stemming from 1.2.7 and case a) of Theorem 3.10 in [4].

1.2.9. If $n = 3.7$, then for any admissible form σ and any exact, stably parallelizable mapping $M \to N$ there is an admissible mapping f: $M \to N$ with $f^*(\omega) = \sigma$ that is homotopic to it.

1.3. Negative Results

1.3.1. There is an essential obstacle to the induction of a form on M. For example, the induction of a form σ by the mapping $M \to R^n$ is equivalent to its representation by the external derivative of n exact 1-forms and the necessary condition for this is that the form σ can be factored into the product of n linear (through not necessarily exact) forms.

If the form σ is not degenerate and the manifold M is open, then this condition is also sufficient (which follows from case 3) of Proposition 1.2.2). On the contrary, in the degenerate case this condition is far from sufficient, as follows from Proposition 1.3.3.

1.3.2. We agree to call the dimensionality of the set $Z \subset M$ the smallest number k for which there is a smooth k-dimensional manifold A and a smooth mapping $A \to M$ such that its image contains Z.

1.3.3. Let σ and ω be smooth n-forms on n-dimensional manifolds M and N, the form ω being nondegenerate, and the form σ having a compact set of zeros which does not intersect the boundary of the manifold M, this set having a dimensionality less than n/2. If $n \geq 6$, then any smooth mapping f: $M \to N$ with $f^*(\omega) = \sigma$ can be C^0-approximated by continuous, locally homeomorphic mappings $M \to N$.

Remark 1. In most cases Proposition 1.3.3 does not apply to the manifold M itself, but applies only to those of its open parts where the zeros of the form σ have the appropriate dimensionality.

Remark 2. The dimensionality n/2 cannot be replaced by a larger one; on every closed k-connection ($k < n/2$) it is easy to find a form with an $(n - k - 1)$dimensional set of zeros which can be induced by a mapping in the sphere S^n with a nondegenerate form.

Remark 3. If Poincaré's hypothesis is taken to be valid in a dimensionality of four, then the condition $n \geq 6$ can be weakened to $n \geq 4$.

Remark 4. From Proposition 1.3.3 it follows that a mapping f induces corresponding classes in M from Shtiefel classes and rational Pontryagin classes of the manifold N without the need of making the as sumption $n \geq 6$, since in these cases Lemma 2.5.2 can be used in place of Proposition 1.3.3.

§ 2. PROOFS

2.1. Nondegenerate Forms

2.1.1. With reference to a set $A \subset M$ the term "near A" means "in some neighborhood of the set A."

2.1.2. Let M be a connected, compact, n-dimensional manifold with the boundary B, and let ω_0 and ω_1 be nondegenerate n-forms on M whose values coincide near B. If $\int_M \omega_1 = \int_M \omega_0$, then there exists a diffeomorphism f: $M \to M$, joined with the identity diffeotopy and fixed near B, such that $f^*(\omega_0) = \omega_1$.

For the proof see [7] and [2].

2.1.3. Let M be a connected n-dimensional manifold that factors into the direct product $M = M_0 \times [0, 1]$, and let ω_0 and ω_1 be nondegenerate n-forms on M whose values coincide near $M_0 \times 0$. If $n > 1$, then there exists an imbedding f: $M \to M$ with $f^*(\omega_0) = \omega_1$ that is joined with the identity by a regular homotopy fixed near $M_0 \times 0$.

This obviously follows from 2.1.2.

2.2. Short Fibrations

2.2.1. A one-dimensional fibration L on a compact manifold M with boundary B is called short if every fiber l is a segment having its ends on B. Tangency of l and B in internal points of the segment l is not excluded. The set of points of tangency between fibers of the fibration and B is denote by $\Sigma = \Sigma(L) \subset B$.

The one-dimensional subfibration of a tangential fibration T(M) consisting of vectors that are tangent to the fiber of the fibration L is denoted by T(L). The factor fibration T(M)/T(L) is denoted by S(L).

If L is a short fibration fiber of the fibration S(L) lying on one fiber l of the fibration L can be identified by means of a holonomy along l. The result of the identification is an $(n - 1)$-dimensional ($n = \dim M$) vector space, which we denote by S_l. By R_l we denote the (one-dimensional) vector space of the $(n - 1)$-vectors of the space S_l, i.e., R_l is the external degree $\Lambda^{n-1}S_l$.

2.2.2. For an n-form σ on a manifold M with a short fibration L, having a fiber l and an $(n - 1)$-vector $\lambda \in R_l$, we denote by $\lambda\sigma$ the linear form on l defined (properly, as can be seen) by the formula $\lambda\sigma(x) = \sigma(\lambda \wedge x)$, where x is a vector tangent to l.

For an orientable segment $l_0 \subset l$ we denote by $\int_{l_0} \sigma$ an $(n-1)$-form in S_l whose value on $\lambda \in R_l$ is equal to the integral $\int_{l_0} \lambda\sigma$.

2.2.3. A manifold M with boundary B is conveniently represented as part of a larger manifold $\tilde{M}$ with boundary $\tilde{B}$, which is obtained by affixing to B the cylinder B × [0, 1]. A short fibration L on M can be extended to some short fibration $\tilde{L}$ on $\tilde{M}$.

2.2.4. Consider smooth n-forms σ on M and ω on $\tilde{M}$, the second being nondegenerate.

a) We call σ and ω congruent along L if for any segment l_0, lying entirely in one layer and having its ends on B, the identity $\int_{l_0} \sigma = \int_{l_0} \omega$ is fulfilled.

b) We say that the form σ is dominated along L by the form ω if, for any fiber $\tilde{l}$ of the fibration $\tilde{L}$, the intersection $l = \tilde{l} \cap M$, and any segment $\tilde{l}_0 \subset \tilde{l}$ having one end on B and the other on $\tilde{B}$, the inequality $\int_{l} |\sigma| < \int_{\tilde{l}_0} |\omega|$ is satisfied.

2.2.5. A mapping of part of a manifold having a fibration is called laminar in the entire manifold if every fiber goes over into itself.

2.2.6. Let the forms σ and ω coincide near $\Sigma = \Sigma(L)$ and suppose that conditions a) and b) of 2.2.4 are fulfilled. Then there exists a unique, smooth, laminar mapping f: $M \to \tilde{M}$ with $f^*(\omega) = \sigma$, which is fixed on B and joined with the identity homotopy fixed on B.

Proof. Take any fiber l of the fibration L, its extension to the fiber $\tilde{l}$ of the fibration $\tilde{L}$, and a vector $\lambda \in R_l$. Construct the mapping $f_l : l \to \tilde{l}$ with $f_l^* (\lambda\omega) = \lambda\sigma$, fixed in points of $\tilde{l} \cap B$. Evidently such a mapping exists, is unique, and among the aggregate of mappings f_l, defines the required f.

2.2.7. We call a form σ on an orientable manifold M with a short fibration L laminarly positive if for any segment l_0, lying in one fiber and having its ends on B, the inequality $\int_{l_0} \sigma > 0$ is fulfilled.

2.2.8. Let M be a connected, compact, orientable manifold with the short fibration L and the form σ, which is positive near $\Sigma(L)$. If $\int_M \sigma > 0$, there exists a diffeomorphism f: $M \to M$, associated with the identity diffeotopy, fixed near the boundary B, such that the form $f^*(\sigma)$ is laminarly positive.

Proof. Let $U_1, \ldots, U_k$ be connected components of a region where the form σ has positive values such that their union U contains $\Sigma(L)$, and $\int_U \sigma > \int_{M \setminus U} |\sigma|$. As a distance from the boundary B select a tube $T \subset M$, i.e., a set smoothly developed as the direct product of a sphere D^{n-1} and the segment [0, 1], so that every segment $d \times [0, 1] \subset T \subset M$ $(d \in D^{n-1})$ lies entirely in one fiber of the fibration L. We assume that each segment of the tube intersects each of the sets U_i and the inequality $\int_{M \setminus T} |\sigma| < \int_{U_i} \sigma$ $(i = 1, \ldots, k)$ is fulfilled. This is easily verified if σ is subjected to the effect of an appropriate diffeomorphism. Under the assumptions made, it is possible to construct a laminarly positive form σ' which is positive in U, coincides with σ near $B \cup (M \setminus U)$, and is of such a character that the equality $\int_{U_i} \sigma' - \sigma = 0$ $(i = 1, \ldots, k)$ is satisfied. From 2.1.2 it follows that the required diffeomorphism converting σ' to σ (in addition to being fixed near $M \setminus U$) exists.

2.2.9. Let M and L be the same as in 2.2.8, let σ be a laminarly positive form, and let ω be a positive form on M that coincides with σ near $\Sigma(L)$. Then there exists a positive form ω_0 on M which is congruent to σ (see a) of 2.2.4) along L and which coincides with ω near B.

For the proof one must cover the complement up to a small neighborhood of the set $\Sigma(L)$ in an appropriate way with a finite number of tubes, then achieve the required positive character of the form by changing it sequentially in each of the tubes.

2.3. Induction of Forms

2.3.1. Consider the M and $\tilde{M}$ of paragraph 2.2.3 and provide $\tilde{M}$ with a smooth Riemann metric. We call the mapping f: $M \to \tilde{M}$ normal if the following two conditions are fulfilled:

1) the mapping f is fixed on the boundary B and is joined with the identity homotopy that is fixed on B;

2) for every point $x \in M$ lying near B the image $f(x) \in \tilde{M}$ is located on the geodesic that passes through x and is normal to B.

2.3.2. Fundamental Lemma. Let M be a compact, connected, n-dimensional manifold with $n > 1$ and let σ be a smooth n-form on M. Moreover, let the manifold $\tilde{M}$ be provided with a Riemann metric and a nondegenerate n-form ω such that $\int_M \omega = \int_M \sigma$. Then there exists a normal mapping f: $M \to \tilde{M}$ with $f^*(\omega) = \sigma$ that is uniquely determined near B.

Proof. We fix an orientation in M relative to which the form ω is positive. Without loss of generality we assume that near some smooth, closed sphere $D^n \subset \operatorname{Int} M$ the forms ω and σ coincide, and we denote by M_0 the manifold $M \setminus \operatorname{Int} D^n$ with boundary $B \cup \partial D^n$. We construct a short fibration L_0 on M_0 with extension $\tilde{L}_0$, which, near B, consists of geodesics normal to B. By virtue of 2.2.8 the form σ can be considered laminarly positive, and, according to 2.2.9, there exists a positive form ω_0 on $\tilde{M}_0$ that is congruent to σ along L_0. Augmenting ω_0 outside of M_0, one can obtain a dominance (see 2.2.4), and so, applying 2.2.6, construct f_0: $M_0 \to \tilde{M}_0$ with $f_0(\omega_0) = \sigma$. According to 2.1.2 and 2.1.3 there exists a mapping f_1: $\tilde{M}_0 \to \tilde{M}$ with $f_1^*(\omega) = \omega_0$, so that one can take the composition $f_0 \circ f_1$ as the desired f on M_0 and extend it onto D^n by the identity transformation, this being possible since f_0 and f_1 were constructed so as to be fixed near ∂D^n.

2.3.3. Proof of Theorem 1.2.2. Without loss of generality we assume that M has no boundary. Using [8] or [4], we deform f_0 into a continuous mapping f^0: $M \to N$, for which there exist n-dimensional, connected, compact manifolds $M_1, M_2, \ldots, M_i, \ldots$ (whose number is finite or denumerable, depending on whether M is compact or not) that cover M, no pair having interesecting interiors, such that the contractions $f^0|_{M_i}$ are smooth imbeddings. Simple additional considerations enable one to assume that the equality $\int_{M_i} \sigma = \int_{M_0} (f^0)^*(\omega)$ holds. We enlarge each M_i to $\tilde{M}_i$ and extend the imbedding $f^0|_{M_i}$ to the imbedding f^i: $M_i \to N$. We fix a smooth Riemann metric in N and assume smoothness of the normal (with respect to the metric generated in M upon affixing the manifolds M_i with Riemann metrics induced by the mapping f^0) decomposition of the tubular neighborhood of the manifold $\bigcup_i \partial M_i \subset M$. This can be done with a slight modification of f^0.

We provide each $\tilde{M}_i$ with an induced metric and an induced form $\omega_i = (f^i)^*(\omega)$. Applying Lemma 2.3.2, we construct normal mappings f_i: $M_i \to \tilde{M}_i$ with $f_i^*(\omega_i) = \sigma$. The mappings $f^i \circ f_i$: $M_i \to N$ determine, in the aggregate, the required f.

2.4. Folds

2.4.1. In addition to the hypotheses in 1.2.6 it will be supposed that the manifold M is closed, and we denote by $M_1, \ldots, M_s \subset M_+$; $M_{s+1}, \ldots, M_t \subset M_-$ connected components of the complement M/Z. Moreover, we fix an integer d and limit the term admissible to admissible (in the sense of 1.2.6) mappings of degree d.

2.4.2. By $H \subset R^t$ we shall denote the hyperplane given by the equation $\sum_{i=1}^{s} x_i - \sum_{i=s+1}^{t} x_i = d$, by $H^+ \subset H$ the set of vectors with nonnegative components, and by $ZH^+ \subset H^+$ the set of vectors with integer components.

The following fact is obvious.

2.4.3. The convex hull of the set ZH^+ is H^+.

2.4.4. For an admissible mapping g: $M \to N$ we denote by $N_\mu = N_\mu(g) \subset N$, where $1 \leq \mu < \nu = \nu(g) \leq \infty$, the connected components of the complement $N \setminus g(Z)$, and by $h_\mu(g) \in ZH^+$ the vector whose i-th ($i = 1, \ldots, t$) component is equal to the number of points in the intersection $g^{-1}(a) \cap M_i$, where a is some point of N_μ. By $H(g) \subset ZH^+$ we denote the union $\bigcup_{1 \leq \mu < \nu} h_\mu(g)$.

2.4.5. For any finite set $A \subset ZH^+$ and any admissible mapping $M \to N$ there exists an admissible mapping g with $H(g) \supset A$ that is homotopic to it.

This follows from Theorem 2.2 of [4].

2.4.6. For an n-form ω on N and an admissible g we denote by $h(\omega, g) \in R^t$ the vector whose i-th component is equal to the absolute magnitude of the integral $\int_{M_i} g^*(\omega)$. We denote by Ω the set of smooth nondegenerate forms ω for which $\int_N \omega = 1$, and by $\Omega g \subset H$ the union $\bigcup_{\omega \in \Omega} h(\omega, g)$.

The following fact is obvious.

2.4.7. The set Ω^g coincides with the interior of the convex hull of the set $H(g) \subset H$.

2.4.8. Let $\omega \in \Omega$ and let σ be an admissible (see 1.2.6) form on M. Then for any exact, admissible mapping $M \to N$ there is an admissible mapping g that is homotopic to it, such that $\int_{M_i} g^*(\omega) = \int_{M_i} \sigma$ for $i = 1, \ldots, t$.

Proof. We denote by $y \in H^+$ the vector with components $y_i = \left| \int_{M_i} \sigma \right|$. Using Propositions 2.4.3 and 2.4.5, we first construct a g_0: $M \to N$ for which the convex hull of the set $H(g_0)$ contains the vector y within itself. By virtue of 2.4.7 there exists a form f: $N \to N$ with $f^*(\omega) = \omega$ such that $f \circ g_0$ is the required mapping.

2.4.9. Proof of Theorem 1.2.7. If M is closed, Theorem 1.2.7 follows immediately from 2.4.8 and 2.1.2. If M is open, one requires an analog of Lemma 2.4.8, which is easily derived from the Smale-Hirsch immersion theory (see [5, 6]).

2.5. Mappings with Small Singularities

2.5.1. For a smooth mapping f: $M \to N$ we denote by $Z(f) \subset M$ the set of zeros of its jacobian and by $BZ(f)$ the intersection of the boundary of the manifold M with $Z(f)$.

2.5.2. Let M and N_0 be compact, connected, n-dimensional manifolds and let f_0: $M \to N_0$ be a smooth mapping, converting the boundary of the manifold M into the boundary of the manifold N_0, such that the inverse images $f_0^{-1}(x)$ with $x \in N_0$ are connected and $\dim Z(f_0) < n/2$, $\dim BZ(f_0) < (n/2) - 1$ (dimensionality is understood to be in conformity with 1.3.2). Then f_0 is a homotopic equivalence.

Proof. The contraction of the mapping f_0 on $M \setminus Z(f)$ maps the set $M \setminus Z(f)$ onto its image diffeomorphically and induces an isomorphism in all homotopies and homologs. Consequently, the same is true of the f_0 itself and homologs with dimensions up to half that possessed by M and for all required homologs, in virtue of the Poincaré-Lefshetz duality.

2.5.3. If the inequality $n \geq 6$ is satisfied under the conditions of Proposition 2.5.2, then the mapping f_0 can be approximated by the homeomorphism $M \to N_0$.

Proof. We choose a sufficiently fine smooth triangulation on N_0, this being in a common position with respect to f_0. It suffices to show that the inverse image of every k-dimensional simplex of this triangulation is homeomorphic to a k-dimensional simplex. For $k \leq (n/2) + 1$ the homeomorphism is realized by f_0 itself, since the mapping f_0 on k-dimensional simplices has, in view of the common position, no more than zero-dimensional singularities. On the remaining simplices the mapping f_0 is a homotopic equivalence in virtue of 2.5.2, but for $n \geq 6$ the inequality $(n/2) + 2 \geq 5$, holds, and this guarantees the applicability of Smale's theorem (see [3]), according to which, for $k \geq 5$, a homotopic k-dimensional sphere is a topological sphere.

2.5.4. Proof of Theorem 1.3.3. We subdivide M into connected components of the inverse images $f^{-1}(x)$ $(x \in N)$. The corresponding factor space N_0 is evidently an n-dimensional manifold, which is of course imbedded in N, and Lemma 2.5.3 applies to the factor mapping f_0: $M \to N_0$.

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