Soft differential equations

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ABSTRACT: There are systems of partial differential equations which have amazingly little visible effect on the functions solving the equations. To put it another way, the solutions of such a system seem nearly unrestricted by the equations they obey. The first instance of this paradoxal softness was discovered by John Nash (1954) for isometric deformations of submanifolds in $\mathbb{R}^n$. Years later it was realized that the softness is quite common among non-linear P.D.E. For example, generic underdetermined systems turned out soft.

1. INTRODUCTION

Solutions of most P.D.E. systems appearing in mathematical physics display definite global features which are specific for a given system. For example, the solutions of the Laplace equation $\Delta f = 0$ on $\mathbb{R}^m$ satisfy the maximum principle. Thus one may rule out a function like

$$f = \sum_{i=1}^{m} a_i u_i^2 \text{ , where } 0 < a_i \leq \text{const},$$

without ever bothering to compute the derivatives of f.

As another example consider self-mappings of the plane, $f: \mathbb{R}^2 \to \mathbb{R}^2$, with unit Jacobian. That is
$(\partial_1 f_1)(\partial_2 f_2) - (\partial_1 f_2)(\partial_2 f_1) = 1$, where $f = (f_1, f_2)$ and $\partial_i = \frac{\partial}{\partial u_i}$ for $i = 1,2$. Here we have a single non-linear equation imposed on two unknown functions. The space of solutions is quite large and the solutions are by far more flexible than those of $\Delta f = 0$. Yet every solution (or non-solution) is immediately recognizable by the way it transforms the Lebesgue measure in $\mathbb{R}^2$. For example, no map close to $(u_1, u_2) \mapsto (2u_1, 2u_2)$ can have a Jacobian one.

Finally, consider <u>isometric immersions</u> f of $\mathbb{R}^2$ into $\mathbb{R}^3$. These are solutions of the following system of three equations $\langle \partial_i f, \partial_j f \rangle = \delta_{ij}$ for $1 \leq i, j \leq 2$, where $\langle\ ,\ \rangle$ denotes the scalar product in $\mathbb{R}^3$ and $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. If such an f is C^2-smooth then it can be as earlier characterized by certain global properties. Namely every local isometric C^2-immersion is represented (apart from a few irrelevant degenerate cases) by a developable surface that is a union of straight segments tangent to a space curve. Furthermore, every global isometric C^2-immersion $f: \mathbb{R}^2 \to \mathbb{R}^3$ is cylindrical, that is f sends the lines parallel to some direction in $\mathbb{R}^2$ to a family of parallel lines in $\mathbb{R}^3$.

2. SOFTNESS OF ISOMETRIC C^1-IMMERSIONS AND GENERALIZATIONS

The equations $\Delta f = 0$ and Jacobian = 1 can be solved by a kind of integration transforming the infinitesimal information incoded in these equations to global properties of the solutions. In particular, an apriori smoothness condition on the solutions plays no essential role. The following striking result shows that the situation is quite different for isometric immersions where the descent from C^2 to C^1 brings forth a new smoothness phenomenon which undermines the classical P.D.E. phylosophy.

THOEREM (Nash 1954, Kuiper 1955):

Every distance strictly decreasing map $f_o: \mathbb{R}^2 \to \mathbb{R}^3$ can be uniformly approximated by isometric C^1-immersions $f: \mathbb{R}^2 \to \mathbb{R}^3$. The same remains true for isometric immersions of $\mathbb{R}^m$ with an arbitrary Riemannian metric to a given Riemannian manifold of dimension $n > m$.

A generalization of this theorem is proven in the author's book (1986) for sufficiently non-linear systems having m independent characteristics. Note that the non-linearity condition is satisfied by generic systems while the characteristics requirement is more stringent. Yet it is generically satisfied by underdetermined systems. It is also satisfied by the isometric immersion system which is not underdetermined for $n < \frac{m(m+1)}{2}$. (See my book for the precise statements and proofs).

3. SOFT LINEAR SYSTEMS AND DELINEARIZATION

There is another method of establishing softness of P.D.E. which starts with the analysis of the linearized system. If the linear system can be solved purely algebraically, (which is possible for generic underdetermined systems) then sufficiently smooth (e.g. C^∞ or real analytic) solutions of the original system are locally soft by the Nash implicit function theorem. In many interesting cases (it is unknown if these are generic) one can assemble local solutions by topological techniques and thus obtain the global softness of the P.D.E. in question. Here is an application motivated by isometric immersions.

THEOREM (Gromov 1986) :

Let G denote the linear space of quadratic differential C^∞-smooth forms on a compact connected manifold V and let $G' \subset G$ be a convex cone invariant under the natural action of $\mathrm{Diff}^{\infty}(V)$ on G. If G' contains a non-zero positive semidefinite form, then G' contains the whole cone G^{+} of the positive definite C^{∞}-forms, that are the Riemannian C^{∞}-metrics on V (We do not apriori assume that G' is open or closed).

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