Annals of Mathematics, **136** (1992), 123–135

Embeddings of Stein manifolds of dimension n into the affine space of dimension $3n/2 + 1$

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Introduction

We prove in this paper the following theorem:

EMBEDDING THEOREM. *There exist proper holomorphic embeddings of Stein manifolds of dimension n into* $\mathbb{C}^q$ *for the minimal integer* $q > (3n+1)/2$.

This was announced in our paper [GE1] twenty years ago, and we apologize for the delay (which is due, in part, to nonmathematical circumstances).

It is known since the works of R. Remmert [Re], R. Narasimhan [Na1], and E. Bishop [B] that an n-dimensional Stein manifold X can be properly embedded into C^q for $q = 2n+1$. In 1970, O. Forster [F] proved embeddability into C^q for $q = [5n/3] + 2$. This dimension was improved in 1984 by U. Schaft (see [S]); he got $q = 2n - [3n/7 - 14]$ for sufficiently large n.

We began thinking about the problem back in 1970 before Forster's paper appeared. Using the method of removal of singularities (see [GE1-3]), we obtained $q = [(1 + \ln 2)n] + 2$. Note that $1 + \ln 2 = 1.69\ldots > 5/3 = 1,66\ldots$. We had not yet finished our paper when we saw Forster's article and learned about Narasimhan's theorem [N2] (the "Lefschetz theorem" for singular Stein spaces). When we confronted the removal of singularities with Narasimhan's result, we obtained the Embedding Theorem into $\mathbb{C}^q$ for $q = [3n/2] + 2$, as is sketched in [GE1]. Our present estimate is the same in the case where n is odd and it is better by 1 for n even. Note that simple examples show (see [F]) that q cannot be made less than $[3n/2] + 1$. Therefore our q is the best possible for even n, but one can, probably, improve q by 1 for n odd (see the discussion in subsection 2.D below).

On the positive side, the twenty years that have passed since our previous paper [GE1] brought along a drastic simplification of the proof (as well as an improvement of the result by 1/2). This is due, mainly, to a generalization of

*The first author was partially supported by NSF Grant DMS-9006179.

H. Grauert's "Oka principle" (see [Gr1]) for holomorphic sections of submersions, which was proven in [G1]. In the next section we discuss a partial case of this theorem, which we use as a basic ingredient in our new simpler proof.

1. Oka's principle for submersions

Consider a holomorphic vector bundle $p : V \to Y$, where Y is Stein, and let $\Sigma \subset V$ be a (properly embedded, complex analytic) subvariety such that the projection $p(\Sigma) \subset Y$ equals the difference of two subvarieties in Y, say, $p(\Sigma) = Y_0 \setminus Y_1$ for $Y_1 \subset Y_0 \subset Y$. Let us restrict our bundle V to $Y_0^* = Y_0 \setminus Y_1 = p(\Sigma)$ and then look at the complement $Z_0^* = p^{-1}(Y_0^*) \setminus \Sigma$ over Y_0^*. We make the following assumptions concerning the map $p : Z_0^* \to Y_0^*$:

(1) The map p is a locally trivial holomorphic fibration. (Notice that p is a *submersion* to start with.)

(2) All fibers $Z_y^* \subset Z_0^*$, $y \in Y_0^* = Y_0 \setminus Y_1$ satisfy either the property

(2 hom) Z_y^* admits the structure of a complex homogeneous space;

or the property

(2 cod) The complement of Z_y^* in V_y, that is $\Sigma_y = \Sigma \cap p^{-1}(y)$, is a complex *algebraic* subvariety in $V_y = \mathbb{C}^k$ (for $k = \operatorname{rank} V$) of codimension ≥ 2.

Now one has (see Theorem 4.5 in [G1]) the following *h-principle* (or Oka's principle) for holomorphic sections $Y \to V$ missing $\Sigma \subset V$:

EXISTENCE LEMMA 1.A. *If there exists a continuous section $\alpha : Y \to V$ missing Σ (i.e., $\alpha(Y) \cap \Sigma = \emptyset$), then there exists a holomorphic section with this property. Moreover every continuous section $\alpha : Y \to V$ missing Σ can be deformed to a holomorphic section such that the whole homotopy misses Σ.*

Remark 1.B. (a) The second part of the statement concerning the homotopy is not needed for our construction of the holomorphic embeddings. Yet the presence of such a homotopy clarifies the relation between holomorphic and continuous maps, which we call the *h-principle* (h for homotopy) for holomorphic sections missing Σ.

(b) The Existence Lemma, as stated, is a rather special case of a general h-principle for submersions in [G1]. In the (2 hom)-case this is very close to the classical *Oka principle of Grauert* (see [G1], [C], [Ra] and 1.1.2 in [G2]), but there is no obvious (at least to the authors) reduction of the Existence Lemma to Grauert's theorem. However the basic ideas and constructions in *Cartan's proof* of the Grauert theorem (see [C]) do apply to the (2 hom)-case (under certain restrictions), although the details still appear quite messy (as they did back in 1970 when we needed the proof for the Embedding Theorem). The proof of the general h-principle in [G1] uses Cartan's ideas along with later improvements due to G.M. Henkin and J. Leiterer [HL]). In fact, one could use in the proof of the Embedding Theorem only a special case of the Existence Lemma, which follows by the somewhat more elementary techniques of Section 2 in [G1].

The Existence Lemma applies to embeddings via the following corollary:

EXISTENCE COROLLARY 1.C. *If* $\dim Y_0 < 2 \operatorname{codim} \Sigma_y$ *for* $\Sigma_y = \Sigma \cap V_y \subset V_y$, $y \in Y_0 \setminus Y_1$, *then there exists a holomorphic section* $Y \to V$ *missing* Σ.

Proof. Since Y_0 and Y_1 are Stein, they are, by the Lefschetz theorem (see [AF]), homotopy equivalent to polyhedra of *real* dimensions $\dim_{\mathbb{C}} Y_0$ and $\dim_{\mathbb{C}} Y_1$, respectively, where we may assume that $\dim Y_1 < \dim Y_0$. On the other hand, the fiber $Z_y^* = V_y \setminus \Sigma_y$ is k-connected for $k = 2 \operatorname{codim} \Sigma_y - 2$. This ensures the existence of a continuous section missing Σ, which can then be made holomorphic according to the Existence Lemma. □

1.D. *Extension of holomorphic sections.* We shall formulate here relative versions of Lemma 1.A and Corollary 1.C. They are not formally needed to prove the Embedding Theorem, but the extension picture better reflects the spirit of the situation. The following Lemma $1.\mathrm{D}_1$ can be deduced from the results of Section 4 in [G1].

EXTENSION LEMMA $1.\mathrm{D}_1$. *Let* $\alpha_1 : Y_1 \to V$ *be a holomorphic section. Then* α_1 *extends to a holomorphic section* $\alpha : Y \to V$ *missing* Σ *(i.e.,* $\alpha(Y) \cap \Sigma = \emptyset$*) if and only if there is a continuous extension with this property. Moreover every continuous section* $\varphi_0 : Y \to V$ *extending* α_1 *and missing* Σ *can be deformed to a holomorphic section such that the whole homotopy is fixed over* Y_1 *and misses* Σ.

EXTENSION COROLLARY $1.\mathrm{D}_2$. *If* $\dim Y_0 < 2 \operatorname{codim} \Sigma_y$ *for* $\Sigma_y = \Sigma \cap V_y \subset V_y$, $y \in Y_0 \setminus Y_1$, *then every holomorphic section* $Y_1 \to V$ *holomorphically extends to a section* $Y \to V$ *missing* Σ.

2. Relative embeddings

Suppose that we are given a Stein manifold X *over* Y; that is, X comes along with the holomorphic map $b : X \to Y$ (where the complex manifold Y does not have to be Stein at this stage). Then a map $f_0 : X \to \mathbb{C}^{q_0}$ is called an *embedding over* Y if the Cartesian sum of f_0 with the *background* map b is a (proper holomorphic) embedding of X into $Y \times \mathbb{C}^{q_0}$, denoted $f = b \oplus f_0 : X \to Y \times \mathbb{C}^{q_0}$.

An important (for the embedding problem) invariant of b is the *set of double points* $\Delta_b^2 \subset X \times X$, which consists of those pairs of mutually distinct points x_1 and $x_2 \neq x_1$ for which $b(x_1) = b(x_2)$. We shall also need the *singularities* $\Sigma_b^i \subset X, i = 1, \ldots n$, where $x \in \Sigma_b^i$ if and only if the differential D_x of b at x has

$$\dim \ker D_x b = i.$$

The following theorem is the main technical result of our paper:

RELATIVE EMBEDDING THEOREM 2.A. *Let the background map $b : X \to Y$ be proper and suppose that*

(a) $\dim \Delta_b^2 < 2q_0$,

(b) $\dim \Sigma_b^i < 2(q_0 - i + 1)$ *for* $i = 1, 2, \ldots, n$. *Then X admits an embedding $f_0 : X \to \mathbb{C}^{q_0}$ over Y, provided that $q_0 \geq 2$. The same is true for $q_0 = 1$ if the set of triple points of b (i.e., of those $y \in Y$, where $\#b^{-1}(y) \geq 3$) is 0-dimensional.*

We construct f_0 in Section 3 with an appropriate stratification of Y by successively extending the required f_0 from stratum to stratum, where each extension step is based on the Existence Corollary.

Remark 2.A$_1$. (a) By using the Extension Lemma rather than the Existence Corollary, one can obtain the *h-principle* for relative embeddings, where one only assumes that b is proper (and there is no dimension condition). This h-principle claims (very) roughly speaking, that every topological embedding $X \to \mathbb{C}^{q_0}$ can be deformed to a holomorphic one. (See §5 in [G1] for a more precise formulation.)

(b) Special cases of the Relative Embedding Theorem appeared in [F], [GE1] and [S]. The proof we give in this paper is much more straightforward than the more elaborate procedure of *removal of singularities* indicated in our old paper [GE1] and elaborated (for somewhat different ends) in [GE2], [GE3], [Sz] and [G2].

2.B. *Construction of embeddings $X \to \mathbb{C}^q$ for* $(3n+1)/2$. A classical theorem of Whitney–Thom says that a *generic map $b : X \to \mathbb{C}^m$* has

(a) $\dim \Delta_b^2 \leq 2n - m$,

(b) $\operatorname{codim} \Sigma_b^i \geq i(m - n + i)$ for $m \geq n$

and $i = 1, 2, \ldots, n$. We need the following "proper version" of this result:

GENERIC LEMMA 2.B$_1$. *Every Stein manifold admits a proper holomorphic map $b : X \to \mathbb{C}^m$ for a given $m > n$, which satisfies the above inequalities* (a) *and* (b).

Proof. We start with the following lemma, which is a slight modification of Bishop's theorem (see [B]) about proper maps $X \to \mathbb{C}^{n+1}$. In the proof we follow Bishop's scheme (compare [GuR]).

LEMMA 2.B$_2$. *Let $a : X \to \mathbb{C}^N$ be a proper map of a Stein manifold X. Then for $m > n$ there exists a proper map $b_0 : X \to \mathbb{C}^m$, which grows faster than a; i.e., $\|b_0(x)\|/\|a(x)\| \to \infty$ for $x \to \infty$.*

Proof. It is sufficient, of course, to take $m = n+1$. Let $X = \bigcup_{k=0}^{\infty} P_k$, $\bar{P}_k \subset P_{k+1}$, for $k = 0, 1, \dots$, be an exhaustion of X by special analytic polyhedra; i.e., $P_k \subset X$, $k = 0, 1, \dots$, are relatively compact sets defined by inequalities $|f_i^k| < 1, i = 1, \dots, n$, for some analytic functions $f_i^k : X \to \mathbb{C}$. Let $\beta_k = \max_{P_k} \|a(x)\|, k = 0, 1 \dots$. Choosing, by induction, real numbers α_k and integers $t_k, k = 0, 1 \dots$, we can construct functions $F_j = \sum_{k=0}^{\infty} \alpha_k (f_j^k)^{t_k}$, $j = 1, \dots, n$, such that the map $F = (F_1, \dots, F_n) : X \to \mathbb{C}^n$ satisfies inequalities $\|F(x)\| > k\beta_k$ for $x \in \partial P_k$, $k = 0, 1, \dots$. Let $B_R \subset \mathbb{C}^n$ be the ball of radius R centered at the origin. Note that $F^{-1}(B_{k\beta_k}) \cap \partial P_n = 0$ for $n \geq k$. Let H_k be the union of those components of $F^{-1}(B_{k\beta_k})$ that are contained in $P_{k+1} \setminus P_k$. Then H_k and P_k are disjoint, holomorphically convex domains. So we can apply Oka's approximation theorem and construct holomorphic functions $\varphi_k : X \to \mathbb{C}, k = 0, 1, \dots$, such that $|\varphi_k(x)| \leq 2^{-k}$ for $x \in P_k$ and $|\varphi_k(x)| > (k+1)\beta_{k+1}$ for $x \in H_k$. Let $\varphi = \sum_{k=0}^{\infty} \varphi_k$. Then the map $b_0 = F \oplus \varphi : X \to \mathbb{C}^{n+1}$ is proper and, moreover, satisfies the inequality $\|b_0(x)\|/\|a(x)\| > k$ for $x \in X \setminus P_k$, $k = 0, 1, \dots$. □

End of the proof of Lemma 2.B$_1$. Let $b_0 : X \to \mathbb{C}^m$ be the map constructed in Lemma 2.B$_2$. Then for every affine map $l : \mathbb{C}^n \to \mathbb{C}^m$, the perturbed map $b_l = b_0 + l \circ a$ is proper. Now the Thom transversality theorem shows that b_l satisfies (a) and (b) for generic l, and the corresponding b_l is our b. □

2.C. *Proof of the Embedding Theorem.* This is now obvious. Start with a proper map $b : X \to \mathbb{C}^{n+1}$ with

$$\dim \Delta_b^2 \leq n-1, \text{ and}$$
$$\dim \Sigma_b^i \leq n - i(i+1).$$

Then for $q > (3n+1)/2$ we have $q_0 = q - n - 1 > (n-1)/2$, which implies inequalities (a) and (b) in Theorem 2.A and, hence, ensures the existence of an embedding $f = b \oplus f_0 : X \to \mathbb{C}^q = \mathbb{C}^{n+1} \times \mathbb{C}^{q_0}$ for $q_0 \geq 2$. Notice that the condition $q_0 \geq 2$ rules out the cases $\dim X = 1$ and $\dim X = 2$, where one can apply the Relative Embedding Theorem only if the triple points set of b is 0-dimensional. But a simple refinement of the Generic Lemma provides a map b satisfying, besides (a) and (b), the relation

$$\dim \Delta_b^3 \leq 3n - 2m$$

for the set Δ_b^2 of the triple points, which amounts to the required condition

$$\dim \Delta_b^3 \leq 0$$

for $n = 1$ and 2.

2.D. *Embedding to* $\mathbb{C}^{(3n+1)/2}$. It seems likely that every X^n properly embeds into $\mathbb{C}^q$ for the first $q > 3n/2$. This would improve our result for odd n. For example, every open Riemann surface should embed into $\mathbb{C}^2$. The Relative Embedding Theorem provides such embeddings whenever X admits a proper map $b : X \to \mathbb{C}^n$ such that

$$\dim \Sigma_b^i < n - 2i + 2, \ i = 2, 3, \ldots, n .$$

(The inequalities for Δ_b^2 and Σ_b^1 are automatic here.) For example, if X is obtained by a finite ramified covering of $\mathbb{C}^n$ with nonsingular branching locus, then X properly embeds into $\mathbb{C}^q$ whenever $q > 3n/2$.

In the general case, one may start with a *quasiproper* map $b : X \to \mathbb{C}^n$ ("quasiproper" meaning that the pullback of every compact subset in $\mathbb{C}^n$ is a *disjoint* union of compact subsets in X) and then try to prove the Relative Embedding Theorem for such b. For this, one will probably need a generalization of the h-principle in [G1] to submersions with infinite-dimensional fibers.

2.E. *Immersions to* $\mathbb{C}^{3n/2}$. Recall that a holomorphic map $f : X \to W$ is called an *immersion* if the differential Df is injective on each tangent space $T_x(X)$, $x \in X$, of X. Notice that, by definition, every embedding $X \to W$ satisfies this condition.

Now let $b : X \to Y$ be a holomorphic (possibly nonproper) map and define

$$\dim_Y X = \sup_{y \in Y} \dim b^{-1}(y).$$

2.E$_1$. Relative h-Principle for Immersions Theorem. *If* $\dim_Y X = 0$, *then immersions* $f_0 : X \to \mathbb{C}^{q_0}$ *over* Y (*i.e.*, $f = b \oplus f_0 : X \to Y \times \mathbb{C}^{q_0}$ are immersions) *satisfy the h-principle in the sense of* [G2]. *In particular, such an immersion exists if and only if there exists a continuous homomorphism* $\varphi_0 : T(X) \to T(\mathbb{C}^{q_0})$ *such that* $D(b) \oplus \varphi_0 : T(X) \to T(Y \times \mathbb{C}^{q_0})$ *is injective on every tangent space* $T_x(X), x \in X$.

The proof of this is similar to (in fact, significantly simpler than) our proof of the Relative Embedding Theorem in Section 3 and, as in the embedding case, we have the following theorem:

Relative Immersion Theorem 2.E$_2$. *If* $\dim \Sigma_b^i < 2(q_0 - i + 1)$, $i = 1, 2, \ldots n$, *then* X *admits an immersion into* $\mathbb{C}^{q_0}$ *over* Y.

This theorem applies to *generic* background maps $b : X \to \mathbb{C}^n$ and to *proper generic* maps $b : X \to \mathbb{C}^{n+1}$, thus yielding the following result:

IMMERSION THEOREM 2.E$_3$. *If $q > (3n-1)/2$, then X^n admits an immersion into $\mathbb{C}^q$; if $q > 3n/2$, then there exists a proper immersion $X \to \mathbb{C}^q$.*

Remark 2.E$_4$. (a) The original proof of Theorem 2.E$_3$ (indicated in [GE1] and presented in detail in [G2]) relies on the removal of singularities. In fact, the removal of singularities also applies to the case $\dim_Y X > 0$, provided that $q_0 > \dim_Y X$. For example, one gets by the removal of singularities *an immersion of every topologically contractible manifold X^n into $\mathbb{C}^{n+1}$*. No such result is known for embeddings, or for proper immersions, where the removal does not (seem to) work.

(b) The h-principles for embeddings and immersions of X over Y with $\dim_Y X = 0$ remain valid in the real C^∞-category. In fact, the proof turns out to be rather trivial in the C^∞-case, as Oka's principle becomes a tautology. (Compare the 0-dimensional h-principle in 1.4.3 of [G2].) On the other hand, if $\dim_Y X > 0$, then constructing C^∞-immersion and embeddings requires an indirect construction (e.g., the removal of singularities; see [GE3], [Sz] and [G2]). Notice that one can understand pretty well here the case of immersions, but the embeddings $X \to \mathbb{R}^q$, $q \leq 3n/2$, are not amenable to the h-principle as seen today.

3. The proof of the Relative Embedding Theorem

We are given a proper holomorphic map $b : X \to Y$ and we look for a map $f_0 : X \to \mathbb{C}^{q_0}$ such that $b \oplus f_0$ is an embedding of X to $Y \times \mathbb{C}^{q_0}$. Let us denote by $\mathcal{F}_0$ the sheaf of (germs of) holomorphic maps $X \to \mathbb{C}^{q_0}$ and take the direct image $\mathcal{F}_*$ of $\mathcal{F}_0$. This $\mathcal{F}_*$ is the sheaf over Y whose sections over every open $U \subset Y$ are, by definition, the same as the sections of $\mathcal{F}_0$ over $b^{-1}(U)$. That is, $\mathcal{F}_*(U) = \mathcal{F}_0(b^{-1}(U)) =$ (the space of holomorphic maps $b^{-1}(U) \to \mathbb{C}^{q_0}$).

Denote by $\mathcal{E}_* \subset \mathcal{F}_*$ the subsheaf whose sections correspond to relative embeddings of X to $\mathbb{C}^{q_0}$. That is, $\mathcal{E}_*(U) =$ (the space of embeddings $b^{-1}(U) \to \mathbb{C}^{q_0}$ over U). Notice that embeddings $X \to \mathbb{C}^{q_0}$ do not form a sheaf over X (only a presheaf), but they do form a sheaf over Y. (Immersions, unlike embeddings, do form a subsheaf in $\mathcal{F}_0$, which makes their treatment significantly easier.)

By a famous theorem of Grauert (see [Gr2]) the sheaf $\mathcal{F}_*$ is *coherent*, since b is proper. To clarify the idea we assume for the moment that $\mathcal{F}_*$ is (globally) generated by finitely many sections $\psi_1 \dots \psi_k$ over Y. (In fact, this is the case for generic proper maps $b : X \to \mathbb{C}^{n+1}$, but this will not be used in the sequel.) This means that every section φ of $\mathcal{F}_*$ is of the form $\Sigma_{i=1}^k \alpha_i \psi_i$ for some holomorphic functions α_i on Y. Now the condition that φ belong to $\mathcal{E}_* \subset \mathcal{F}_*$ can be expressed in terms of certain relations between the functions α_i. In fact, there are relations of two different types:

I. *Difference relations.* These are due to the presence of double points of b. Namely, whenever $b(x_1) = b(x_2)$ for $x_1 \neq x_2$, the map $f_0 : X \to \mathbb{C}^q$ corresponding to a section φ must have $f_0(x_1) \neq f_0(x_2)$ in order to be an embedding. This condition can be expressed in terms of a trivial fibration $Z = Y \times \mathbb{C}^k \to Y$ (whose sections correspond to the strings of functions $\alpha_1, \dots, \alpha_k$) with an appropriate subvariety $\Sigma \subset Z$ such that $f_0(x_1) \neq f_0(x_2)$ whenever the corresponding section $Y \to Z$ misses Σ.

II. *Differential relations.* These appear because the differential Db may have a nontrivial kernel (i.e., $\Sigma^1_b \neq \emptyset$); and so the differential of f_0 must be injective on this kernel (as we want $b \oplus f_0$ to be an immersion). We can express such relations with certain subvarieties Σ' in the jet space of sections of Z by insisting that the jets of pertinent sections $Y \to Z$ miss Σ'. In general, such relations do not abide by any h-principles. However it is all right in our present case, as these relations are 0-*dimensional* in the sense of 1.4.3 in [G2]. Namely, one can stratify Y such that over each stratum $Y_i \subset Y$ the relation is expressed with certain vector fields $\partial_1, \dots, \partial_l$ *transversal* to Y_i by asking that some combinations of the derivatives $\partial_j \alpha$ of our sections $\alpha : Y \to Z$ do not vanish on Y_i. Then the construction of α satisfying such a condition is usually quite easy (at least it is infinitely easier than in the case where the fields are tangent to Y_i; see subsection 3.C below).

3.A. *Equisingular stratification.* We start the actual proof of the Relative Embedding Theorem by stratifying Y_0 such that the basic topological characteristics of the map b (and, hence, of the sheaf $\mathcal{F}_*$) are constant along each (locally closed) stratum in the sense of the following definition:

Definition 3.A$_1$. A locally closed, nonsingular subvariety $S \subset Y$ is called *equisingular* (for b) if the following three conditions are satisfied:

A. The map b has constant geometric multiplicity over S. That is, the number $\#(b^{-1}(y))$ is constant for $y \in S$. (Notice that this number is finite for b proper and X Stein.)

B. The rank of b is locally constant over S. That is, the rank of Db is constant on each connected component of $\tilde{S} = b^{-1}(S) \subset X$.

C. The subvariety $\tilde{S}$ is regular (i.e., smooth) and the kernel of Db is everywhere transversal to $\tilde{S}$. Equivalently, the map b is an immersion of $\tilde{S}$ onto S. In fact, this map (being proper) is a finite nonramified covering $\tilde{S} \to S$.

PROPOSITION 3.A$_2$. *Every subvariety $Y_0 \subset Y$ admits an equisingular stratification, that is, a descending sequence of (closed) subvarieties in Y, say, $Y_0 \supset Y_1 \supset \dots \supset Y_k$, where the strata (i.e., the complements $S_i = Y_i \setminus Y_{i+1}$) are smooth, equisingular (locally closed) subvarieties in Y.*

Proof. First we stratify Y_0 to provide property A and then substratify it consequently to arrange B and C. To insure the convergence at the last step one needs to use Sard's lemma (which is trivial in the analytic situation). $\square$

3.B. *Separation of points over S.* Let Y_0 and $Y_1 \subset Y_0$ be subvarieties in Y, where $S = Y_1 \setminus Y_0$ is smooth and equisingular, as earlier. Let $\varphi \in \mathcal{F}_*$ be a section for which the corresponding map $f : X \to \mathbb{C}^{q_0}$ is an *embedding* over Y_1. This means that $f(x_1) \neq f(x_2)$ for the distinct points x_1 and x_2 with $b(x_1) = b(x_2) \in Y_1$, and that the differential of f does not vanish on $(\ker Db)|X_1$ for $X_1 = b^{-1}(Y_1)$.

SEPARATION LEMMA 3.B$_1$. *If Y is Stein and if $2q_0 > \dim Y_0$, then for $q_0 \geq 2$ there exists a section $\varphi' \in \mathcal{F}_*$, which agrees with φ on Y_1 with second order (as explained below) and for which the corresponding map $f' : X \to \mathbb{C}^{q_0}$ separates the points over Y_0; i.e., $f'(x_1) \neq f'(x_2)$ for the distinct points x_1 and x_2 with $b(x_1) = b(x_2) \in Y_0$. Furthermore, if $q_0 = 1$, then such a φ' exists provided that the map b has no triple points over $S = Y_0 \setminus Y_1$; (i.e., $\#b^{-1}(s) \leq 2$* for $s \in S$).

Explanation 3.B$_2$. The second order agreement means that

$$\varphi - \varphi' \in (\mathcal{I}Y_1)^2 \mathcal{F}_*,$$

where $\mathcal{I}Y_1$ is the ideal sheaf defining Y_1. In fact, for our purpose we could use the weaker relation

$$f - f' \in (\mathcal{I}X_1)^2 \mathcal{F}_0,$$

which says, in effect, that f and f' agree on X_1 with their first derivatives. In our case, where f is an embedding over Y_1, the agreement condition makes f' an embedding over Y_1 as well.

Proof of Lemma 3.B$_1$. Call a system of sections $\psi_1, \ldots, \psi_N \in \mathcal{F}_*$ *generating at $y \in Y$* if for every map $f_0 : X \to \mathbb{C}^q$ there exists a linear combination $\psi = \Sigma_{i=1}^N c_i \psi_i$ such that the corresponding section $f : X \to \mathbb{C}^q$ equals f_0 on the (finite) set $b^{-1}(y) \subset X$. Then the coherence of $\mathcal{F}_*$ and the standard application of Cartan's Theorems A and B give the following result:

SUBLEMMA 3.B$_3$. *There exist finitely many sections $\psi_1, \ldots, \psi_N$ of $\mathcal{F}_*$, which vanish on Y_1 with second order and which are generating at all points of $S = Y_0 \setminus Y_1$.*

Now we are looking for the required φ' in the form $\varphi' = \varphi + \Sigma_{i=1}^N \alpha_i \psi_i$, where the string of functions α_i is thought of as the section α of the trivial fibration $p : V = Y \times \mathbb{C}^N \to Y$. The separation-of-points condition is automatically satisfied over Y_1, while over $S = Y_0 \setminus Y_1$ it is obviously expressed with a subvariety $\Sigma \subset V$ defined by the condition

$$\alpha(s) \in \Sigma \Leftrightarrow \text{ there exist } x_1 \text{ and } x_2 \neq x_1$$

in $b^{-1}(s)$, $s \in S$, such that $f'(x_1) = f'(x_2)$ for the map f' corresponding to φ'. The generating property ensures the equality

$$\operatorname{codim} \Sigma_s = q_0$$

for $\Sigma_s = p^{-1}(s) \cap \Sigma \subset V_s$, $s \in S$. In fact, Σ_s is a finite union of the affine subspaces of codimension q_0, which are obtained as the pullbacks of the diagonals in

$$\mathbb{C}^{dq_0} = \underbrace{\mathbb{C}^{q_0} \times \mathbb{C}^{q_0} \times \ldots \times \mathbb{C}^{q_0}}_{d},$$

for $d = \# b^{-1}(s)$ by a *surjective* (because the ψ_i are generating at s) affine map $\mathbb{C}^N = V_s \to \mathbb{C}^{dq_0}$. Notice that $V_s \setminus \Sigma_s$ has an obvious homogeneous structure for $d = 2$ (and there is no such structure for $d \geq 3$). Now Existence Corollary 1.C provides a section $\alpha : Y \to V$ missing Σ, which gives us the required $\varphi' = \varphi + \Sigma \alpha_i \psi_i$. □

3.C. *Making f' an immersion over S.* The map f' delivered by Lemma 3.B$_1$ is *not* an embedding over Y_0, as the differential Df' may have a nontrivial kernel over $\tilde{S} = b^{-1}(S)$. The next lemma allows us to eliminate this kernel without changing $f'|X_0$ for $X_0 = b^{-1}(Y_0)$ and thus retaining the separation property of f'. To set the stage we consider analytic subsets X_0 and $X_1 \subset X_0$ in a Stein manifold X and consider a homomorphism Δ of $T(X)$ to some holomorphic bundle T_0 over X (this is $\Delta = Db : T(X) \to b^*T(Y)$ in our concrete situation). We assume that the rank of Δ is locally constant on $\tilde{S} = X_0 \backslash X_1$. Moreover we assume that $\tilde{S}$ is nonsingular and that the subbundle $K = \ker \Delta \subset T(X)|X_0$ is transversal to $\tilde{S}$, which means that $K \cap T(\tilde{S}) = 0$. Finally we consider a holomorphic map $f' : X \to \mathbb{C}^{q_0}$ for which the differential Df' is injective on $K|X_1$.

Ker-ELIMINATION LEMMA 3.C$_1$. *If $\dim \tilde{S} < 2(q_0 - \dim K + 1)$, then there exists a map $f'' : X \to \mathbb{C}^{q_0}$, which equals f' on X_0 and which, moreover, agrees with f' on X_1 with second order and such that Df'' is injective on $K|X_0$.*

Proof. One easily sees, as in Sublemma 3.B$_3$, that there exist maps $f_1, \ldots, f_N : X \to \mathbb{C}^{q_0}$ such that

(a) the f_i vanish on X_0 for $i = 1, \ldots, N$;

(b) the f_i vanish on X_1 with second order, which means that their coordinate function belongs to the ideal $(\mathcal{I}X_1)^2$;

(c) the homomorphisms $Df_i|K$ span the bundle $\mathrm{Hom}(K, T_0)$ over $\tilde{S}$, where T_0 is the trivial bundle of dimension q_0 over X identified with $f_i^*(T(\mathbb{C}^{q_0}))$.

In fact, by Thom's transversality theorem, property (c) is satisfied by *generic* f_i for

$$N > \dim \tilde{S} + \dim \mathrm{Hom}(K, T^*)$$

even if the f_i are restricted by properties (a) and (b). Then we look for our f'' in the form

$$f'' = f' + \Sigma_{i=1}^N \alpha_i f_i$$

and observe that derivations of f'' along K are expressed by algebraic (non-differential!) conditions on α_i. In fact,

$$\partial f''(s) = \partial f'(s) + \Sigma_{i=1}^N \alpha_i \partial f_i(s),$$

as $f_i(s) = 0$ for $s \in S$. It follows that the injectivity of Df'' on K can be expressed with some $\Sigma \subset V = X \times \mathbb{C}^N \to X$. Namely, let Σ_0, $\Sigma_0 \subset \mathrm{Hom}(K, T_0)$, consist, by definition, of the *noninjective* homomorphisms $K \to T_0$. Then Σ is the inverse image of Σ_0 by the affine morphism $\delta : V|S \to \mathrm{Hom}(K, T^*)$ defined by the differential of f'',

$$\delta(\alpha) = Df''|K\ .$$

Now we clearly see that Df'' is injective on $K|S$ if and only if the section $\alpha : X \to V$ defined by $\alpha_1, \dots, \alpha_N$ misses Σ. It is equally clear that $V \setminus \Sigma$ is a locally trivial fibration with homogeneous fibers and that

$$\operatorname{codim} \Sigma_s = q_0 - \dim K.$$

This allows us to apply Existence Corollary 1.C and to "eliminate" K. □

3.D. *Conclusion of the proof of the Relative Embedding Theorem.* We start with an arbitrary holomorphic section φ of the bundle $\mathcal{F}_*$ over Y (see the beginning of Section 3), which we then move to the subsheaf $\mathcal{E}_* \subset \mathcal{F}_*$ (whose sections correspond to relative embeddings $X \to \mathbb{C}^{q_0}$). This moving is done in $2(k+1)$ steps for $k = \dim Y$ with some equisingular stratification of Y provided by Proposition 3.A$_2$.

At the first stage we do what we need over the 0-dimensional stratum $Y_k \subset Y$; that is, we modify the original section of $\mathcal{F}_*$ so that the corresponding map $X \to \mathbb{C}^q$ becomes an embedding over Y_k. (This is, obviously, possible in our case, as $\dim Y_k = 0$.) Then we improve our section in succession by making it first the embedding over $Y_{k-1} \supset Y_k$, then over $Y_{k-2} \supset Y_{k-1}$, and so on until we come to $Y = Y_0$. Each passage from Y_{i-1} to Y_i is advised in two steps: First we modify φ to make it separate points over Y_{i-1} (see Lemma 3.B$_1$) and, secondly, we make it an immersion (and, hence, an embedding) over Y_{i-1} with Lemma 3.C$_1$.

Notice that in Lemmas 3.B_1 and 3.C_1 we assumed Y to be Stein. But this is not a serious restriction for the embedding problem, where the action takes place on the image $b(X) \subset Y$, which is Stein. Unfortunately $b(X)$ is a singular space, while the Existence Lemma (and its corollaries) was stated only for smooth manifolds Y. Here either one can notice that this lemma remains valid in the singular case, or one can take some Stein neighborhood of $b(X)$ in Y.

Finally we observe that all this discussion becomes irrelevant for the embeddings $X \to \mathbb{C}^q$, as our $Y = \mathbb{C}^{n+1}$ is Stein in this case. Thus the proof of the Relative Embedding Theorem is concluded, which gives us the proof of the Embedding Theorem $X \to \mathbb{C}^q$ for $q > 3n/2 + 1/2$. □

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(Received October 5, 1990)