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Synthetic Geometry in Riemannian Manifolds

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1. We measure *deviation* of a map $f: X\to X'$ from isometry by

$$\sup_{x,y\in X}\left|\log\left(\operatorname{dist}(f(x), f(y))/\operatorname{dist}(x, y)\right)\right|.$$

For Riemannian manifolds V, V' we define $\operatorname{dev}(V, V')$ to be the "inf" of the deviations of all *diffeomorphisms* $V\to V'$. We treat "dev" as a metric in the set of isometry classes of Riemannian manifolds, though "dev" takes infinite values (say, when V and V' are not diffeomorphic).

The sectional curvature $K=K(V)$ is solely responsible for the local deviation of V from $\boldsymbol{R}^n$: when $|K|\leqslant\varkappa$ each point $v\in V$ has an arbitrary small ε-neighborhood U_ε such that its deviation from the ε-ball in $\boldsymbol{R}^n$ ($n=\dim V$) does not exceed $\varkappa\varepsilon^2$. (The converse is true up to a constant.)

A priori localization. Start with choosing a very small but fixed number ε. Neighborhoods U_ε can look very different from usual balls, no matter how small the curvature is.

Split tori. Take the product of n circles of lengths $l_1\geqslant l_2\geqslant\ldots\geqslant l_n>0$. This is a flat manifold (i.e. $K\equiv 0$). Look at the ε-neighborhood U_ε of a point (ε-neighborhoods of different points are, obviously, isometric). Suppose that the ratio l_k/l_{k+1} is very large (about n^n) and ε is just in the middle between l_k and l_{k+1}. Such a U_ε looks approximately as the product of an $(n-k)$-dimensional torus (product of the "short" circles of lengths $l_{k+1}, \ldots, l_n$) by the k-dimensional ε-ball.

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When $l_1 \leq \varepsilon/n$ then U_ε coincides with the whole torus; thus tori (and flat manifolds in general) must be viewed as *local* geometric objects.

Nontrivial local geometry is always accompanied ($|K|$ is kept small) by nontrivial local topology: one defines *the injectivity radius* $\mathrm{rad}_v\, V$ as the maximal number r such that the ε-neighborhoods U_ε of $v \in V$ with $\varepsilon < r$ have smooth boundaries. These U_ε are automatically smooth topological balls and their deviation from Euclidean balls depends only on ε and $|K|$.

When $|K|$ and $(\mathrm{rad})^{-1}$ are kept bounded the sheer size of V (say volume or diameter) determines the overall geometric and topological complexity of V as follows.

Strong compactness. The set of all closed n-dimensional Riemannian manifolds satisfying (a) $|K| \leq \varkappa$, (b) $\mathrm{rad} \geq \varrho > 0$, (c) $\mathrm{Volume} \leq C$, ($\varkappa, \varrho, C$ are arbitrary numbers) is compact with respect to metric "dev". (A short proof can be found in [4].)

This fact generalizes the Mahler compactness theorem for flat tori (see [2]) and sharpens Cheeger's theorem (see [3]) on the finiteness of the number of topological types under conditions (a), (b), and (c).

2. *Flat manifolds* are the simplest nontrivial local objects. Our understanding of their structure is based on the following classical theorems of Bieberbach and Hermite (see [8]).

(a) there are finitely many topologically distinct flat manifolds of given dimension;

(b) every compact flat manifold can be covered by a torus;

(c) every flat torus T stays close to a split torus, i.e. there is a split torus T' such that $\mathrm{dev}(T', T) \leq \mathrm{const}$ ($\leq n^n$, $n = \dim T$).

Further examples of manifolds V_ε with $\mathrm{rad} \leq \varepsilon$ and $|K| \leq \mathrm{const}$ can be obtained by multiplying a fixed V_0 by a flat manifold with diameter $\leq \varepsilon$, say, by the circle of length ε. This phenomenon can also be observed (Berger, see [1]) on general circle bundles: realize V_0 as a totally geodesic manifold of codimension 2 with prescribed normal bundle in W and take for V_ε the boundary of the ε-neighborhood of V.

Iterating this construction we arrive at an inductive definition of *nilmanifolds* of dimension n as circle bundles over $(n-1)$-dimensional nilmanifolds. Each nilmanifold carries a family of Riemannian structures with $|K| \leq \mathrm{const}$, $\mathrm{Diam} \to 0$.

Nilmanifolds are characterized homotopically as manifolds with nilpotent fundamental groups and contractible universal coverings.

The above V_ε do not "dev"-converge to V_0 but there is a coarser metric which provides such convergence. This is the Hausdorff metric defined in the set of isometry classes of all metric spaces as follows: $H(X, X')$ is the lower bound of the numbers δ satisfying the following property: there are isometrical imbeddings $X, X' \to Y$ into a metric space Y such that X is contained in the δ-neighborhood of X' and X' is contained in the δ-neighborhood of X.

Convergence $V_\varepsilon \to V_0$ in our examples is not surprising in view of the following fact.

Weak compactness. Let be given a sequence of n-dimensional closed Riemannian manifolds satisfying (a) $|K| \leqslant \varkappa$, (b) Diameter $\leqslant C$ ($\varkappa$, C are arbitrary). Then there is a subsequence which H-converges (i.e. relative to the H-metric) to a metric space X_0 (which is in general not a manifold).

Condition (a) can be relaxed to $K \geqslant -\varkappa$, $\varkappa \geqslant 0$, and in this more general form weak compactness follows directly from the Toponogov comparison theorem (see [1], [3]).

The following theorem discloses the geometry of the convergence $V_\varepsilon \to X_0$ in the simplest case when X_0 is the single point.

3. *Near flat manifolds.* A closed Riemannian manifold is called ε-near flat if its sectional curvature K and diameter satisfy $|K|\,(\mathrm{Diam})^2 \leqslant \varepsilon$.

If V is ε-near flat with $\varepsilon \leqslant \varepsilon_n$ ($\approx n^{-n^n}$), $n = \dim V$ then there exists a k-sheeted covering $\tilde{V} \to V$ with $k \leqslant n^{n^n}$ such that V is diffeomorphic to a nilmanifold and the induced metric in $\tilde{V}$ is "dev"-close to a locally homogeneous metric, i.e., there is a locally homogeneous V' with $\mathrm{dev}(V', \tilde{V}) \to 0$ as $\varepsilon \to 0$. (See [4].)

Probably V itself is diffeomorphic (and "dev"-close) to a locally homogeneous manifold. When the fundamental group $\pi_1(\tilde{V})$ is Abelian V is known to be diffeomorphic to a flat manifold.

Observe that the fundamental group of a near flat manifold contains a nilpotent subgroup of finite index. This property is probably shared by all *near elliptic manifolds,* i.e., when $K(\mathrm{Diam})^2 \geqslant -\varepsilon$, $0 \leqslant \varepsilon \leqslant \varepsilon_n$. It is known that $\mathrm{rank}\, H_1(V, R) \leqslant n = \dim V$, when V is near elliptic.

EXAMPLES. Products of near flat and elliptic (i.e. with nonnegative curvature) manifolds are near elliptic; circle bundles over elliptic manifolds are near elliptic.

4. *Micromanifolds.* Consider the set $\mathcal{M}_\varkappa$ of the isometry classes of the n-dimensional Riemannian manifolds with curvature bounded by $\varkappa$ i.e. with $|K| \leqslant \varkappa$. Manifolds from $\mathcal{M}_\varkappa$ display their most interesting features when rad$\to 0$. We are tempted to introduce new objects—*manifolds M with infinitely small injectivity radius.* We view every such M as an element from an ideal boundary $\partial\mathcal{M}_\varkappa$. Each M is represented by a sequence of $V_\varepsilon \in \mathcal{M}_\varkappa$, $\varepsilon \to 0$, converging relative to the H-metric to a metric space X_0. Our M is "fibered" over X_0; the fibers look like "infinitely small" near flat manifolds, but geometry and topology of the "fibers" can, in general, jump when $x \in X_0$ varies. In physical terms, M carries not only the macroscopic structure of X_0 but also additional microscopic information hidden in the "fibers".

When this description is made precise it yields the following "macroscopic" theorems I, II and III.

THEOREM I. ESTIMATES FOR BETTI NUMBERS. *Suppose that the sectional curvature of a closed n-dimensional Riemannian manifold V satisfies $|K| \leqslant 1$. Denote by $\sum_0^n b_i$ the sum of the Betti numbers of V (with any coefficients).*

(a) $\sum_0^n b_i \leqslant C_m$, $m=2+n+\text{Diam } V$, $n=\dim V$, $C_m \approx m^{m^m}$.

(b) *If* V *is homeomorphic to a connected sum of manifolds of constant negative curvature then* $\sum_0^n b_i \leqslant C_n \cdot \text{Volume } V$, $(C_n \approx n^{n^n})$. (See [5].) Probably, the word "constant" can be omitted.

Problem. What happens to (a) and (b) when condition $|K| \leqslant 1$ is replaced by $K \geqslant -1$?

THEOREM II. HYPERBOLIC MANIFOLDS *(Sectional curvature nonpositive). There are only finitely many topologically different manifolds satisfying:* (a) $0 \geqslant K \geqslant -\varkappa$, (b) $\text{Diam} \leqslant C$ *($\varkappa$, C are arbitrary). When $\varkappa=0$ this is the Bieberbach finiteness theorem,* §2.

THEOREM III. NEAR HYPERBOLIC MANIFOLDS. *When* $\varepsilon \geqslant K \geqslant -1$, $\varepsilon \geqslant 0$, *and* ε *is small compared to diameter (say* $\varepsilon \leqslant m^{-m^m}$, $m=n+\text{Diam}$) *then the fundamental group* $\pi_1(V)$ *is infinite. (I am certain that* V *is covered by* $\mathbf{R}^n$ *but the proof is not completed yet.)*

When $n \geqslant 3$ the restriction $K \geqslant -1$ can not be omitted (see [4]).

Locally homogeneous manifolds constitute a very rare set in $\mathcal{M}$ but the amount of the related mathematics is enormous (Lie groups etc.). The study of manifolds that are locally near homogeneous is conducted in the disguise of the Pinching Problem. In the heart of the problem we find again "rad"$\to 0$. (See [1,] [7] for further information.)

5. *Noncompact manifolds and their ends.* Let V be a complete noncompact connected manifold with bounded curvature, i.e. $|K| < \infty$. When $\text{rad}_v \to 0$ as $v \to \infty$, for example, when the total volume is finite, V carries at infinity nontrivial "microstructure", but only in very few cases is this structure completely understood.

Pinched negative curvature. Let $-p\varkappa \leqslant K \leqslant -\varkappa$, and $p, \varkappa \geqslant 0$. If volume of V is finite then V can be exhausted by compact manifolds V_i such that each inclusion $V_i \to V$ is a homotopy equivalence and each component of the boundary ∂V_i (with the induced metric) is ε-near flat with $\varepsilon \to 0$ as $i \to \infty$, and its degree of nilpotency (i.e. the nilpotency degree of the fundamental group of the associated nilmanifold) does not exceed $\sqrt{p}$. In particular, when $p<4$ each component is diffeomorphic to a flat manifold.

The complex hyperbolic space forms provide examples with $p=4$ and with non-Abelian nilpotent ends.

Incompressible ends. The next theorem provides us with many examples of noncompact manifolds supporting no complete metric with bounded curvature and finite volume.

THEOREM IV. *If* $|K| < \infty$, *Volume* $< \infty$ *and* V *is diffeomorphic to the interior of a compact manifold with boundary* B *then* B *has no metric of negative curvature. In* *particular when $n=3$ and B is a closed surface the Euler characteristic of B must be nonnegative.*

It is unclear whether $\boldsymbol{R}^{2n+1}$, $n>0$, supports complete metrics with bounded curvature and finite volume.

6. *Manifolds with boundary.* We must take into account the norm of the second quadratic form K^{∂} of the boundary (Say, for the Euclidean ε-ball $\|K^{\partial}\|=\varepsilon^{-2}$.) We measure the interior size of V by $\mathrm{Int}=\sup_{v\in V}\operatorname{dist}(v,\partial V)$. Lower estimates for "Int" by $\|K^{\partial}\|$ were established in [6] for domains in $\boldsymbol{R}^n$. In general we have:

THEOREM V. *If* $\mathrm{Int}^2(|K|+\|K^{\partial}\|)\leq\varepsilon_n$ $(\leq n^{-n^n})$ *then V is diffeomorphic (and dev-close) to the product of a manifold V' without boundary and an interval* $[0,\delta]$, *or V can be doubly covered by such product.*

There are further relations between topology and the interior size of V. The following simple example points in the right direction:

THEOREM VI. *Let $V\subset\boldsymbol{R}^n$ be a compact domain with $\|K^{\partial}\|\leq 1$. If n is even then* $|\chi(V)|\leq C_n\operatorname{Vol}(V)$ $(C_n\approx n^n$, *χ is the Euler characteristic, "Vol" means volume*).

When V is the complement of the union of distinct unit balls, C_n is equal to the packing constant (see [9]).

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