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WIDTH AND RELATED INVARIANTS OF RIEMANNIAN MANIFOLDS.

by

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INTRODUCTION. There are many (geo)metric invariants characterizing the size and shape of a subset X in $\mathbb{R}^n$. For example, solids in $\mathbb{R}^3$ have three measurements : length, Width and hight. Various characteristics of convex subsets $X \subset \mathbb{R}^n$ are obtained by looking at linear projections and sections of X of dimension $k < n$.

In Riemannian geometry one is usually concerned only with two measurements of a manifold X. These are the total volume of X and the diameter of X. One may think of $\mathrm{Vol}\, X$ as a measure of "the n-spread" of X for $n = \dim X$, while $\mathrm{Diam}\, X$ measures "the 1-spread".

We discuss in these lectures intermediate diameters $\mathrm{Diam}_k X$ for all $k = 0,1,\dots,n-1$ introduced in 1923 by P.S. Uryson which measure how X spreads in dimension $k+1$.

(A) Euclidean recollection. Consider two subsets X and A in $\mathbb{R}^n$ and say that X is ε-close to A if

$$\mathrm{dist}(x,A) \le \varepsilon \text{ for all } x \in X,$$

where

$$\mathrm{dist}(x,A) \underset{\mathrm{def}}{=} \inf_{a \in A} |x-a|$$

for the Euclidean distance $|x-a| = |x-a|_{\mathbb{R}^n}$ between x and a.

The 1-codimensional width $\mathrm{Wid}_{n-1} X$ is defined as the smallest $\varepsilon \ge 0$, such that X is $(\frac{\varepsilon}{2})$-close to some hyperplane A^{n-1} in $\mathbb{R}^n$. Similarly $\mathrm{Wid}_k X$ is the smallest ε such that X is ε-close to some affine subspace $A^k \subset \mathbb{R}^n$. Observe that

$$0 = \mathrm{Wid}_n X \le \mathrm{Wid}_{n-1} X \le \dots \le \mathrm{Wid}_1 X \le \mathrm{Wid}_0 X,$$

and that

$$\frac{1}{2}\, \mathrm{Diam}\, X \le \mathrm{Wid}_0 X \le \mathrm{Diam}\, X,$$

where

$$\mathrm{Diam}\, X \underset{\mathrm{def}}{=} \sup_{x,y \in X} |x-y|.$$

(In fact $\mathrm{Wid}_0 \le \sqrt{\frac{n}{2(n+1)}}\, \mathrm{Diam}$ by Yung theorem, see [B-Z]).

Examples (A_1) Let $X \subset \mathbb{R}^n$ be an ellipsoid with principal axes $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then

$$\mathrm{Wid}_k\ X = \lambda_{k+1} \text{ for all } k = 0,1,\ldots,n-1,$$

according to the minmax principle for λ_k .

(A_1') Let X be the rectangular solid,

$$X = [0,\ell_1] \times [0,\ell_2] \times \ldots \times [0,\ell_n] \subset \mathbb{R}^n ,$$

where the intervals $[0,\ell_i] \subset \mathbb{R}$ satisfy $\ell_1 \geq \ell_2 \geq \ldots \geq \ell_n$. Then $\mathrm{Wid}_k\ X = D_k$, where

$$D_k = \mathrm{Diam}\ [0,\ell_{k+1}] \times \ldots \times [0,\ell_n] = (\ell_{k+1}^2 + \ldots + \ell_n^2)^{\frac{1}{2}} .$$

Proof. The solid X is $(\frac{1}{2} D_k)$-close to the k-plane through the center of X parallel to $[0,\ell_1] \times \ldots \times [0,\ell_k]$. Thus $\mathrm{Wid}_k \leq D_k$. To prove that $\mathrm{Wid}_k \geq D_k$ we take an arbitrary k-plane $A \subset \mathbb{R}^n$ and consider the normal $(n-k)$-plane $A^\perp \subset \mathbb{R}^n$ through the center of X . This $A^\perp$ necessarily meets some k-face of X , say at $x \in X$. Take the point x' symmetric to x in the center of X and observe that $|x-x'| \geq D_k$. Hence $\mathrm{Diam}\ A^\perp \cap X \geq D_k$ and the inequality $\mathrm{Wid}_k \geq D_k$ follows.

(A_1'') Corollary. The width $\mathrm{Wid}_k\ X$ is comparable to ℓ_{k+1} . Namely

$$\ell_{k+1} \leq \mathrm{Wid}_k\ X \leq \sqrt{n}\ \ell_{k+1} \text{ for all } k .$$

(A_2) Approximation of convex subsets in $\mathbb{R}^n$ by simplices and ellipsoids. Let X be a compact convex subset in $\mathbb{R}^n$ with non-empty interior and Δ be an n-simplex of maximal volume in X . Then the vertices $x_0,x_1,\ldots,x_n$ of Δ lie on the boundary of X . Moreover the hyperplane H_i through x_i parallel to the opposite face of Δ does not meet the interior of X by the maximality of Δ . Thus the symplex Δ^* bounded by these hyperplanes contains X . If the baricenter of Δ equals the origin of $\mathbb{R}^n$, then $\Delta^* = \lambda\Delta$ for $\lambda = -n$, where

$$\lambda\Delta \underset{\mathrm{def}}{=} \{\lambda y \mid y \in \Delta\}$$

So we can write

$$\Delta \subset X \subset -n\Delta$$

for all convex subsets $X \subset \mathbb{R}^n$.

(A_2') Proposition. Let X and Y be compact convex subsets in $\mathbb{R}^n$ with non-empty interiors. Then there exists a parallel translate X' of X and an affine transform Y' of Y such that

$$Y' \subset X' \subset \lambda Y'$$

for $\lambda = n^2$.

Proof. Move X , such that the maximal symplex $\Delta(X')$ has baricenter in the origin and transform Y , such that $\Delta^*(Y') = \Delta(X')$. Then

$$Y' \subset \Delta^*(Y') \subset X' \subset \Delta^*(X') = n^2\ \Delta(Y') \subset n^2\ Y' .$$

(A_2'') Corollary. There exists an ellipsoid $E = E(X)$, such that some translate X' of X satisfies

$$E \subset X' \subset n^2\ E .$$

(A_3) A width criterion for $X \leq Y$. Say that $X \leq Y$ if Y contains a congruent copy of X . Clearly,

$$X \leq Y \Rightarrow \mathrm{Wid}_k\ X \leq \mathrm{Wid}_k\ Y \quad \text{for all} \quad k .$$

(A_3') Proposition . Let X be a compact convex body and Y be an ellipsoid. If $\mathrm{Wid}_k\ X \geq n^2\ \mathrm{Wid}_k\ Y$ for $k = 0,1,\dots,n-1$, then $X \geq Y$. Similarly, $\mathrm{Wid}_k\ Y \geq n^2\ \mathrm{Wid}_k\ X$ implies that $Y \geq X$.

Proof. The ellipsoid $E \subset X'$ from (A_2'') has $\mathrm{Wid}_k\ E \geq n^{-2}\ \mathrm{Wid}_k\ X' = n^{-2}\ \mathrm{Wid}_k\ X \geq \mathrm{Wid}_k\ Y$. Since E and Y are ellipsoids, the inequalities $\mathrm{Wid}_k\ E \geq \mathrm{Wid}_k\ Y$ imply that $E \geq Y$. As $E \leq X$, we obtain the inequality $Y \leq X$. The second inequality follows by a similar argument.Q.E.D.

(A_3'') Corollary . If convex bodies X and X_1 in $\mathbb{R}^n$ satisfy $\mathrm{Wid}_k\ X \geq n^4\ \mathrm{Wid}_k\ X_1$ for $k = 0,1,\dots,n-1$, then $X \geq X_1$.

Proof. Apply (A_3') to an intermediate ellipsoid Y , such that

$$\mathrm{Wid}_k\ X \geq n^2\ \mathrm{Wid}_k\ Y \geq n^4\ \mathrm{Wid}_k\ X_1 .$$

(A_3''') Remark. This corollary shows that the numbers $\mathrm{Wid}_k X$ characterize X up to a multiplicative constant. For example, the n-dimensional volume of X can be estimated by $\mathrm{Wid}_k X$ as follows,

$$\lambda_n^{-1} \prod_{k=o}^{n-1} \mathrm{Wid}_k X \leq \mathrm{Vol}\, X \leq \lambda_n \prod_{k=o}^{n-1} \mathrm{Wid}_k X$$

for some positive $\lambda_n \leq n^{4n}$. In fact, the previous discussion allows a slightly better bound on λ_n. (See [B-Z] and [T] for various generalizations and refinements of these results).

(B) Intermediate diameters of metric spaces. For a metric space X we denote by $|x-y|$ or $|x-y|_X$ the distance between x and y in X. We say that X is ε-close to a topological space A if there exists a continuous map $p : X \to A$, such that the fibers $X_a = p^{-1}(a) \subset X$ satisfy $\mathrm{Diam}\, X_a \leq 2\varepsilon$ for all $a \in A$.

(B_1) Definition. The codimension k diameter of a compact metric space X is the infimum of those $\delta > 0$, such that X is $(\delta/2)$-close to some metrizable space A of dimension k.

Remarks (B_2). If X is locally compact rather than compact, then one should modify the definition by replacing $\mathrm{Diam}\, X_a$ by $\inf_U \mathrm{Diam}\, p^{-1}(U)$ where $U \subset A$ are the neighborhoods of $a \in A$.

(B_2') Since the image $p(X) \subset A$ is a compact space of dimension k, it admits an approximation by finite polyhedra of dimension k. Namely, for every metric in $p(X)$ and every $\varepsilon > 0$, there exists a k-dimensional polyhedron A_ε, such that $p(X)$ is ε-close to A_ε. In fact, the dimension $\dim p(X)$ can be defined as the minimal integer k, such that $p(X)$ admits an approximation by k-dimensional polyhedra (see [G-W]).

By composing $p : X \to p(X) \subset A$ with the implied maps $p(X) \to A_\varepsilon$ one obtains continuous maps $p_\varepsilon : X \to A_\varepsilon$, such that

$$\sup_{a \in A_\varepsilon} p_\varepsilon^{-1}(a) \to \sup_{a \in A} p^{-1}(a) \quad \text{for } \varepsilon \to 0 .$$

Hence, one can use k-dimensional polyhedra A instead of general metrizable spaces in the definition of Diam_k.

(B_2'') The meaning of ε-closeness is clarified by the following

Proposition. Let $p : X \to A$ be a continuous map, where X is a compact metric space and A is a metrizable space. Then the following two conditions are equivalent for every $\delta > 0$.

(i) $\sup_{a \in A} \operatorname{Diam} p^{-1}(a) < 2\delta$

(ii) There exists a metric space C, an isometric embedding $X \subset C$ and a topological embedding $A \subset C$, such that

$$|x-p(x)|_C < \delta \text{ for all } x \in X .$$

Proof. If x and x' lie in $p^{-1}(a)$ and $\max(|x-p(x)|, |x'-p(x')|) < \delta$, then $|x-x'| \le 2\delta$ by the triangle inequality. Thus (ii) $\Rightarrow$ (i). To prove the converse we take some metric $|\ |_A$ in A and observe that, by compactness of X, there exist $\lambda > 0$ and $\delta' < \delta$, such that

$$|x-y|_X - 2\delta' \le \lambda |p(x)-p(y)|_A \qquad (*)$$

for all x and y in X. Now we take the disjoint union $X \cup A$ for C and let $|\ |_C$ be the upper bound of the metrics μ on C satisfying the following three conditions

(i) $\mu(x,y) \le |x-y|_X$ for all x and y in X;

(ii) $\mu(a,b) \le \lambda |a-b|_A$ for all a and b in A;

(iii) $\mu(x,p(x)) \le \delta'$ for all $x \in X$.

The inequality (*) implies by a simple argument that the inclusion $X \subset C$ is isometric for this maximal metric $|\ |_C$,

$$|x-y|_C = |x-y|_X \text{ for all } x \text{ and } y \text{ in } X ,$$

and $|x-p(x)|_C \le \delta' < \delta$ by (iii). Q.E.D.

(C) Monotonicity and positivity of Diam_k. First, we observe that Diam_k is decreasing in $k = 0,1,\ldots$,

$$\operatorname{Diam}_0 X \ge \operatorname{Diam}_1 X \ge \ldots .$$

Furthermore, if X is connected, then every continuous map of X into a zero-dimensional space is constant. Therefore,

$$\operatorname{Diam}_0 X = \operatorname{Diam} X ,$$

for connected spaces X.

(C_1) Relation $\lambda Y \le X$. This means that there exists a λ-expanding continuous map $f : Y \to X$,

$$|f(y_1) - f(y_2)|_X \geq \lambda |y_1 - y_2|_Y ,$$

for all y_1 and y_2 in Y . Clearly,

$$\lambda Y \leq X \Rightarrow \mathrm{Diam}_k X \geq \lambda \,\mathrm{Diam}_k Y$$

for all $k = 0,1,\ldots$.

(C_2) If $k \geq \dim X$, then $\mathrm{Diam}_k X = 0$ as X is zero-close to itself. A more interesting property is the inequality

$$\mathrm{Diam}_k X > 0 \quad \text{for} \quad k < \dim X ,$$

which follows from the discussion in (B_2') . For example, Diam_k is > 0 for n-dimensional manifolds X^n if $n > k$, as $\dim X^n = n$ by Lebesgue's dimension theorem.

(D) Estimation of Diam_k of compact subsets in $\mathbb{R}^n$. If $X \subset \mathbb{R}^n$ is ε-close to an affine subspace $A \subset \mathbb{R}^n$ in the sense of (A), then the orthogonal projection $p : X \to A$ has

$$\mathrm{Diam}\, p^{-1}(a) \leq 2\varepsilon$$

for all $a \in A$. Therefore

$$\mathrm{Diam}_k X \leq \mathrm{Wid}_k X$$

for all $k = 0,1,\ldots$, and all compact subsets in $\mathbb{R}^n$.

(D_1) Diam_k of the solid $X = [0,\ell_1] \times \ldots \times [0,\ell_n] \subset \mathbb{R}^n$. We agree as earlier that $\ell_1 \geq \ell_2 \geq \ldots \geq \ell_n$. Then we recall the following

(D_1') Lebesgue's Lemma. Let $p : X \to A$ be a continuous map, where $\dim A < n$. Then there exists a pair of opposite $(n-1)$-faces in X , say X' and X'' , and two points $x' \in X'$ and $x'' \in X''$, such that $p(x') = p(x'')$. (See [H-W]).

Lebesgue's lemma shows that $\mathrm{Diam}_k X \geq \ell_{k+1}$. This implies (see (A_1'')) that $\sqrt{n}\, \mathrm{Diam}_k X \geq \mathrm{Wid}_k X$.

(D_2) Diam_k of convex subsets in $\mathbb{R}^n$. Since every compact convex subset X in $\mathbb{R}^n$ can be approximated by solids (see (A_2')), we obtain

$$\mathrm{Diam}_k X \leq \mathrm{Wid}_k X \leq n^{\frac{5}{2}}\, \mathrm{Diam}_k X ,$$

for all compact convex subsets in $\mathbb{R}^n$.

(D_2') Exercise. Show that the unit disk $X \subset \mathbb{R}^2$ has $\mathrm{Diam}_1 X = \sqrt{3}$. (Compare [K] and ($D_3$) below).

(D_3) Diam_{n-1} and Inradius. The inradius of an $X \subset \mathbb{R}^n$ is the radius of the maximal ball in X,

$$\mathrm{Inrad}\, X = \sup_{x \in X} \mathrm{dist}(x, \mathbb{R}^n \setminus X).$$

(D_3') Every compact $X \subset \mathbb{R}^n$ satisfies

$$\alpha_n \,\mathrm{Inrad}\, X \leq \mathrm{Diam}_{n-1} X \leq 2\, \mathrm{Inrad}\, X ,$$

where $\alpha_n = \sqrt{\frac{2(n+1)}{n}}$ is the diameter of the regular simplex inscribed into the unit sphere S^{n-1}.

Proof. The lower bound on Diam_{n-1} follows from the following simplicial version of

(D_3'') Lebesgue's Lemma. Let p be a continuous map of the n-simplex Δ into an $(n-1)$-dimensional space. Then there exist points x' and x'' lying in two opposite faces of Δ, such that $p(x') = p(x'')$. (See [H-W]).

This lemma applies to maps of round balls $B \subset \mathbb{R}^n$ to A, where B is identified with Δ via a homeomorphism $\Delta \leftrightarrow B$ which radially projects the boundary of Δ on that of B. Then one sees that

$$\mathrm{Diam}_{n-1} B \geq \alpha_n \,\mathrm{rad}\, B ,$$

which implies

$$\mathrm{Diam}_{n-1} X \geq \alpha_n \mathrm{Inrad}\, X$$

for all $X \subset \mathbb{R}^n$.

To get the upper bound on Diam_{n-1} we approximate X by a compact domain $X^+ \supset X$ with a smooth boundary and project X^+ onto the cut-locus $A \subset X^+$ with respect to the boundary. Recall the definition of this projection $p : X^+ \to A \subset X^+$. Take a point $x \in X^+$, let $B(x)$ be the maximal ball in X^+ with center x and take a maximal ball $B' \subset X^+$ which contains $B(x)$. It is not hard to see that this B' is unique, the map $p : x \mapsto y = \mathrm{center}(B')$ is continuous and $\dim p(X^+) \leq n-1$. With such a p (where $A = p(X^+)$, one sees that

$$\mathrm{Diam}_{n-1} X^+ \leq 2 \,\mathrm{Inrad}\, X^+ ,$$

which implies the same inequality for X.

Exercises. Show that the unit ball B in $\mathbb{R}^n$ has $\mathrm{Diam}_{n-1} = \alpha_n$.

Let X be a compact Riemannian manifold with a boundary. Show that

$$\mathrm{Diam}_{n-1} X \leq 2 \sup_{x \in X} \mathrm{dist}(x, \partial X)$$

for $n = \dim X$.

(D_4) Diam_k of convex hypersurfaces. Let Y be a compact convex hypersurface in $\mathbb{R}^n$ and X be the convex body bounded by Y. There are two natural metrics in Y. The first is just the restriction of the Euclidean metric $|\ |$. The second, denoted $|\ |_Y$, is the induced Riemannian metric where the distance between y_1 and y_2 is the length of a shortest path in Y between y_1 and y_2. Clearly, $|\ | \leq |\ |_Y$. In particular,

$$\mathrm{Diam}_k(Y, |\ |) \leq \mathrm{Diam}_k(Y, |\ |_Y) \quad \text{for all } k .$$

On the other hand, if $\dim Y \geq 1$, then

$$\mathrm{Diam}(Y, |\ |_Y) \leq \pi/2\, \mathrm{Diam}(Y, |\ |). \qquad (*)$$

In fact, if $\dim Y = 1$, then $\mathrm{Diam}(Y, |\ |_Y) = \frac{1}{2}$ length Y and the length of Y equals the average of the lengths of the normal projections of Y to the lines in $\mathbb{R}^2 \supset Y$. This proves (*) for $\dim Y = 1$ and the case $\dim Y > 1$ follows by looking at plane sections of X.

Exercise. Show that

$$\mathrm{Diam}_k(Y, |\ |_Y) \leq \pi/2\, \mathrm{Wid}_k X \quad \text{for all } k .$$

Now, let p be a normal projection of Y to a hyperplane $H \subset \mathbb{R}^n$. One can invert this projection on the image $p(Y) = p(X) \subset H$ and thus obtain an expanding embedding $p(X) \to Y$. Hence,

$$\mathrm{Diam}_k(Y, |\ |) \geq \sup_p \mathrm{Diam}_k\, p(X).$$

Finally, we approximate X by an ellipsoid (see (A_2'')), and conclude

$$\mathrm{Diam}_k\,(Y,|\ |_Y) \sim \mathrm{Diam}_k\,(Y,|\ |) \sim$$

$$\sim \mathrm{Diam}_k\, X \sim \mathrm{Wid}_k\, X \quad \text{for} \quad k = 0,1,\ldots,n-2\ ,$$

where the equivalence $\alpha \sim \beta$ signifies the existence of a positive constant $C = C_n$, such that

$$C^{-1}\alpha \leq \beta \leq C\alpha\ .$$

(D_4') Corollary. (Compare (A_3''') and (E_3')). The (n-1)-dimensional volume of Y is of the same order of magnitude as the product of Diam_k,

$$\mathrm{Vol}\ Y \sim \prod_{k=o}^{n-2} \mathrm{Diam}_k\,(Y,|\ |_Y)\ .$$

(D_5) Federerer-Fleming inequality. Let $X \subset \mathbb{R}^n$ be a compact subset of finite k-dimensional Hausdorff measure denoted $\mathrm{Vol}_k\ X$. Then

$$\mathrm{Diam}_{k-1}\, X \leq C_n(\mathrm{Vol}_k\ X)^{\frac{1}{k}} \qquad (*)$$

for $C_n \leq \sqrt{n}(n!(n!/(n-k)!)^{\frac{1}{n}}$.

Idea of the proof. Partition $\mathbb{R}^n$ into cubical cells of diameter $\sim(\mathrm{Vol}_k\ X)^{\frac{1}{k}}$. Then $\mathrm{Vol}_k\ X$ has the order of magnitude of the average number of intersection points of parallel translates of X with the (n-k)-skeleton of this partition. Hence, for a partition into slightly larger cubes, there exists a translate X" of X which misses the (n-k)-skeleton. Then we project X' to the (k-1)-skeleton of the dual partition (see Proposition 3.1.A. in $[G]_4$).

Question. Does (*) hold true with a constant C_k depending only on k ?

(E) Diam_k of Riemannian manifolds. Start with the simplest class of flat manifolds.

(E_1) Split tori. Let X be the product of circles $S_1,S_2,\ldots,S_n$ of lengths $\ell_1 \geq \ell_2 \geq \ldots \ell_n$. The projection of X to $S_1 \times S_2 \times \ldots S_k$ provides the inequality

$$\mathrm{Diam}_k\ X \leq \mathrm{Diam} \prod_{i=k+1}^{n} S_i = \frac{1}{2}\Big(\sum_{i=k+1}^{n} \ell_i^2\Big)^{\frac{1}{2}}\ .$$

On the other hand each S_i contains an isometric copy of $[0,\ell_i/2]$.

Hence, $X \geq \frac{1}{2} X'$ for the solid $[0,\ell_1] \times \ldots \times [0,\ell_n]$, and so (see (A_1'))

$$\text{Diam}_k X \geq \frac{1}{2} \text{Diam}_k X' \geq \frac{1}{2} \ell_{k+1}.$$

Thus $\text{Diam}_k X \sim \ell_{k+1}$.

(E_2) <u>Non-split flat tori</u>. Let X be a flat torus. That is $X = \mathbb{R}^n/L$ for some lattice $L \subset \mathbb{R}^n$. By a classical reduction theory for L (see [C]) there exists a split torus X_s equivalent to X . That is there exists a linear homeomorphism $f : X \to X_s$, such that

$$C^{-1}|x_1-x_2| < |f(x_1)| - f(x_2)| \leq C|x_1-x_2|$$

for all x_1 and x_2 in X , where $C = C_n > 0$ is a universal constant. It follows that, (somewhat sacrifying C) one can take $X_s = \prod_i S_i$, where $\text{length}\, S_i = \text{Diam}_{i-1} X$ for all $i = 1,\ldots,n$.

(E_2') <u>Corollary</u>. <u>The volume of every flat torus</u> X <u>is equivalent to the product of</u> Diam_i ,

$$\text{Vol}\, X \sim \prod_{i=0}^{n-1} \text{Diam}_i X .$$

(E_3) <u>Almost flat manifolds</u>. The reduction theory generalizes (see $[G]_2$ and [B-K]) to ε-<u>flat</u> manifolds X satisfying

$$|K|(\text{Diam}\, X)^2 \leq \varepsilon^2 ,$$

where K denotes the sectional curvature of X and $\varepsilon = \varepsilon_n > 0$ is a universal (small but yet positive) constant (one can take $\varepsilon_n = \exp - n^n$). Using this one can generalize (E_2') to ε-flat manifold X for $\varepsilon \leq \exp - n^n$,

$$C_n^{-1} \text{Vol}\, X \leq \prod_{i=0}^{n-1} \text{diam}_i X \leq C_n \text{Vol}\, X ,$$

where $C_n > 0$ is a universal constant.

<u>Exercise</u>. Prove the equivalence $\text{Vol}\, X \sim \prod_i \text{Diam}_i X$ for <u>flat</u> Riemannian manifolds.

(E_3') It seems that the collapsing techniques (see [C-G]) should yield a similar result for all (possibly large) $\varepsilon > 0$.

$$C^{-1} \text{Vol}\, X \leq \prod_{1}^{n-1} \text{Diam}_i X \leq C \text{Vol}\, X , \qquad (*)$$

for some constant $C > 0$ depending on n and ε.

Here is a more difficult

Question. Does the equivalence $\mathrm{Vol}\, X \sim \prod_{i=1}^{n-1} \mathrm{Diam}_i\, X$ hold true (with the implied constant $C = C_n$) for manifolds X with non-negative sectional curvature ?

A more illuminating but unprecise question is :

Does every X with $K \geq 0$ look roughly as the solid $[0,\ell_1] \times [0,\ell_2] \times \dots \times [0,\ell_n]$ for $\ell_{i+1} = \mathrm{Diam}_i\, X$?

Both questions remain open for manifolds with a lower bound on the sectional curvature, $K(\mathrm{Diam}\, X)^2 \geq -\varepsilon^2$.

(F) Lower bounds on Diam_k. Lebesgue's Lemmas (see (D_1') and $(D_3''))$ provide a lower bound on $\mathrm{Diam}_k\, X$ if X contains a k-dimensional cube (or simplex) with a controlled geometry. A slightly more general estimate $\mathrm{Diam}_k \geq \varepsilon > 0$ can be obtained by the following

(F_1) Proposition (Compare (D_3') and [K]). If $\mathrm{Diam}_k\, X < \alpha_k$ for $\alpha_k = \sqrt{\frac{2(n+1)}{n}}$, then every distance decreasing map f of X into the unit sphere $S^k \subset \mathbb{R}^{k+1}$ is contractible.

Idea of the proof. Let p be a surjective map of X onto a $(k-1)$-dimensional polyhedron A, such that each fiber $X_a = p^{-1}(a)$ for $a \in A$ has $\mathrm{Diam} < \alpha_k$. Then $f(X_a) \subset S^k$ also has $\mathrm{Diam} < \alpha_k$ and hence is contained in a hemisphere by Young theorem (see [B-Z]). It follows that each set $f(X_a) \subset S^k$ contracts to a single point in S^k, such that this contraction is continuous in $a \in A$. This gives a homotopy of f to a map $f_1 : X \to S^k$ which is a composition of $p: X \to A$ with a continuous map $A \to S^k$ obtained by the above shrinking of the subsets $f(X_a) \subset S^k$ to points. As $\dim A < k$, the map $A \to S^k$ is contractible and so f is contractible. Q.E.D.

(F_1') A generalization. Let the above map f send a compact subset $X_o \subset X$ to a point $s_o \in S^k$. Then the above argument shows that the map of pairs,

$$f : (X, X_o) \to (S^k, s_o),$$

is contractible.

(F_1'') Example. Let X be an orientable n-dimensional manifold with boundary $\partial X = X_o$. If $n = k$, then non-contractible maps $(X, X_o) \to (S^k, s_o)$ are those which have non-zero degree. If $n \geq k$, then one defines a generalizated degree of a smooth map f as the framed cobordism class of the manifold $f^{-1}(s) \subset X$ for a generic $s \in S^k$. Non-vanishing of this degree insures non-contractibility of f.

(F_2) Manifolds with large injectivity radius. The essential property of the sphere S^k in the above discussion is a "canonical contractibility" of "small" subsets in S^k. A similar property is shared by all Riemannian manifolds with large injectivity radius and by more general (locally geometrically contractible, see §4.5. in $[G]_4$) manifolds where the balls of a "not very large radius" are contractible within concentric balls of slightly larger radius. Here are two simple examples (see §4.5. in $[G]_4$, $[G]_5$ and §4.2. in $[G]_6$ for the proofs and a further discussion).

(F_2') Let V be a complete n-dimensional Riemannian manifold, such that the injectivity radius of V at every point $v \in V$ is $\geq R_o$ and let $X \subset V$ be a ball of radius $2R_o$. Then

$$\mathrm{Diam}_{n-1} X \geq R_o/2(n+2) .$$

(F_2'') Let V be a compact n-dimensional manifold without boundary and $\tilde{V} \to V$ be the universal covering of V with the induced Riemannian metric. Let W be a complete Riemannian manifold which admits a Riemannian submersion $W \to \tilde{V}$. If $\tilde{V}$ is contractible, then the balls $X(R) \subset W$ of radius R satisfy

$$\mathrm{Diam}_{n-1} X(R) \to \infty \quad \text{as} \quad R \to \infty .$$

(G) Upper bounds on Diam_{k-1}. The inequality of Federer-Fleming (see (D_5)) provides a bound on $\mathrm{Diam}_{k-1} X$ of k-dimensional subsets $X \subset \mathbb{R}^n$ in terms of the Hausdorff measure $\mathrm{Vol}_k X$. A similar bound applies to all manifolds $Y \supset X$ of non-negative Ricci curvature as follows

(G_1) Let Y be a complete n-dimensional manifold with Ricci $Y \geq 0$. Then all compact subsets $X \subset Y$ satisfy

$$\mathrm{Diam}_{k-1} X \leq C_n (\mathrm{Vol}_k\ X)^{\frac{1}{k}}$$

<u>for some universal constant</u> $C_n > 0$.

<u>Idea of the proof</u> (Compare p.130 in $[G]_4$ and §3.4. in $[G]_3$). Since Ricci ≥ 0 , there exists a covering of Y by balls of radius R , where $R \sim (\mathrm{Vol}_k\ X)^{\frac{1}{k}}$, such that the multiplicity of the covering by the concentric balls of radius $2R$ is bounded by some constant $M = M_n$. Then the partition of unity on Y associated to this covering maps X into the polyhedron of dimension $\leq M_n - 1$ which is the nerve of the covering. Then the image of X can be pushed to the $(k-1)$-skeleton of this polyhedron.

(G_2) If X is homeomorphic to S^2 , then the bound on $\mathrm{Diam}_1 X$ does not need any ambient space Y ,

$$\mathrm{Diam}_1 X \leq 2 (\mathrm{Vol}_2\ X)^{\frac{1}{2}}$$

<u>for all metric spaces</u> X <u>homeomorphic to</u> S^2 .

<u>Proof</u>. Assume for simplicity's sake that X is Riemannian, fix a point $x_o \in X$ and partition X into the connected components of the spheres $S_o(r) = \{x \in X \mid |x - x_o| = r\}$ for all $r \in \mathbb{R}_+$ .The resulting quotient space is one-dimensional and the components of $S_o(r)$ have $\mathrm{Diam} \leq 2 (\mathrm{Area}\ S_o)^{\frac{1}{2}}$ as a simple argument shows (see p.129 in $[G]_4$).

(G_3) It is unknown (and seems unlikely) that the ratio $\mathrm{Diam}_{k-1} / (\mathrm{Vol}_k)^{\frac{1}{k}}$ is bounded by a universal constant C_k for all spaces X . However, such a bound is known for another invariant, called the <u>contractibility radius</u> of X (see App. 2 in $[G]_4$).

Namely, let X be an n-dimensional polyhedron with a piecewise Riemannian metric. <u>Then there exists a continuous map</u> $p: X \to A$ <u>where</u> A <u>is an</u> $(n-1)$-<u>dimensional polyhedron, and a metric on the cylinder</u> $C = C_p$ <u>of the map</u> p , <u>such that</u> (compare (B_2''))

(i) <u>the canonical embedding</u> $X \to C$ <u>is isometric</u>,

(ii) <u>the distance from each</u> $a \in C$ <u>to</u> $X \subset C$ <u>satisfies</u>

$$\mathrm{dist}(a, X) \leq \mathrm{const}_n (\mathrm{Vol}_n\ X)^{\frac{1}{n}} \qquad (*)$$

for some universal $\text{const}_n > 0$.

Recall that C_p is the quotient space of the disjoint union $(X \times [0,1]) \cup A$ for the relation $(x \times 1) \sim p(x)$ for all $x \in X$.

This is proven in App. 2 of $[G]_4$. Probably, a small modification of the argument in $[G]_4$ will yield a similar result for all metric spaces X .

A simple application of (*) (see §1.2.B. in $[G]_4$) yields the following generalization of Minkovski theorem.

Let V be an n-dimensional contractible manifold with a Finsler (e.g. Riemannian) metric and let Γ be a discrete isometry group of V for which the quotient space X is compact. Then there exists a point $v \in V$ and a non-identity element $\gamma \in \Gamma$, such that

$$|v-\gamma(v)| \leq 6\ \text{const}_n (\text{Vol}_n\ X)^{\frac{1}{n}} .$$

This reduces to the original Minkowski theorem, if $V = \mathbb{R}^n$ with a translation invariant (Minkowski) metric and Γ consists of parallel translations of $\mathbb{R}^n$.

(G_4) Diam_{n-2} and scalar curvature. Let X be a compact Riemannian manifold without boundary of positive scalar curvature $\geq \sigma^2 > 0$.

Question. Does $\text{Diam}_{n-2} X$ is universally bounded by

$$\text{Diam}_{n-2} X \leq \text{const}_n / \sigma \ ?$$

This is known to be true if X is homeomorphic to S^3 . (see p. 129 in $[G]_4$ and $[G\text{-}L]_2$). This is also known for the metrics obtained by surgery (see $[G\text{-}L]_1$ and [S-Y]).

One also may ask what kind of curvature is responsible for an upper bound on Diam_k for $k < n-2$. For example, let each tangent space $T \subset T(X)$ contain an $(n-k+1)$-dimensional subspace $T' \subset T$, such that the sectional curvatures of the two planes in T' dominate the rest of curvatures,

$$K(\tau') + \alpha K(\tau) \geq \sigma^2 > 0 ,$$

for all 2-planes $\tau' \subset T'$ and $\tau \subset T$, and all α in the interval $[0,\alpha_n]$ for some large constant α_n . Then one asks if the following inequality holds true,

$$\mathrm{Diam}_k\, X \le \mathrm{const}/\sigma .$$

(H) Definition of Diam_k with coverings. Fix a number $\delta > 0$ and let us prove the equivalence of the following three properties of a compact metric space X.

(1) $\mathrm{Diam}_k\, X < \delta$.

(2) X admits a covering of multiplicity $\le k+1$ (i.e. no $k+2$ covering subsets intersect) by compact subsets of diameter $< \delta$.

(3) X can be covered by compact subsets X_i, $i = 0,\dots,k$, such that $\mathrm{Diam}_o\, X_i < \delta$.

Proof. Start with the implication (1) $\Rightarrow$ (3). By definition of Diam_k there exists a continuous map $p : X \to A$, where $\dim A \le k$, such that $\mathrm{Diam}\, p^{-1}(a) < \delta$ for all $a \in A$. By definition of $\dim A$, there exists a covering of A by subsets A_i, $i = 0,\dots,k$, such that each A_i is the union of disjoint compact subsets of arbitrarily small diameter. Then the sets $X_i = p^{-1}(A_i)$ provide the required cover of X.

The implication (3) $\Rightarrow$ (2) is trivial as every X_i, by definition of Diam_o, is the union of disjoint subsets of diameter $< \delta$.

Finally we prove (2) $\Rightarrow$ (1) by taking the nerve of the covering for A and by mapping $X \to A$ with an associated partition of unity.

Corollaries (H_1) Let $X = X_1 \cup X_2$, such that $\mathrm{Diam}_i\, X_1 \le \delta$ and $\mathrm{Diam}_j\, X_2 \le \delta$. Then $\mathrm{Diam}_k \le \delta$ for $k = i+j+1$.

(H_1') Let X admit a continuous map $p : X \to A$, such that $\mathrm{Diam}_i\, p^{-1}(a) \le \delta$ for all $a \in A$. Then $\mathrm{Diam}_k\, X \le \delta$ for $k = (i+1)(\dim A + 1) - 1$.

(H_1'') Example. Let X be a $(2k+1)$-dimensional Riemannian manifold Then for every $\varepsilon > 0$ there exists a smooth map $p : X \to \mathbb{R}$, such that $\mathrm{Diam}_{k+1}\, p^{-1}(a) \le \varepsilon$ for all $a \in \mathbb{R}$.

Proof. Take a sufficiently fine triangulation of X, let X_o be the k-skeleton of this triangulation and X_1 be the k-skeleton of the dual triangulation. Then there is a smooth map $p : X \to [0,1]$, such that $p^{-1}(0) = X_o$, $p^{-1}(1) = X_1$ and $p^{-1}(a)$ for $0 < a < 1$ is the boundary of a small regular ε_a-neighborhood of X_o. This $p^{-1}(a)$ is

ε-close to X_o for all $a < 1$.

This example shows that the bound on k in (H_1') is sharp. This also shows that Diam_{n-k-1} cannot fully serve as a measure of "the (n-k)-dimensional spread" of X. An alternative measure of this spread comes from the (n-k)-volume of the fibers of maps $X \to A$ for $\dim A = k$ (see App. 2 in $[G]_4$).

Concluding remarks. The fundamental fact which insures non-vanishing of Diam_k of n-dimensional manifold for $n > k$ (this makes the definition of Diam_k non-vacuous), is the topological invariance of dimension. One may think that other topological invariants can also be studied quatitatively in the framework of the Riemannian geometry. A geometric quantitative approach to the homology and homotopy theory is indicated in $[G]_2$, $[G]_4$, [G-L-P] and [S], where the reader may find further references.

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