Chopping Riemannian Manifolds

by

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O. Introduction

Let M^n be a complete riemannian manifold. For $X \subset M^n$ a closed subset, let G_x denote the isometry group of X. Let $T_r(X)$ denote the set of points at distance $\leq r$ from X. Finally, if $Y \subset M^n$ is a smooth submanifold, let II_Y denote the second fundamental form of Y.

The main result of this note is the following

Theorem 0.1 : Let M^n have bounded sectional curvature, $|K| \leq 1$. Given $X \subset M^n$, $0 < r \leq 1$, there is a submanifold, U^n, with smooth boundary, ∂U^n, such that

(0.2) $$X \subset U \subset T_r(X),$$

(0.3) $$\mathrm{Vol}(\partial V) \leq c(n)\, \mathrm{Vol}(T_r(X) \setminus T_{r/2}(X)) r^{-1},$$

(0.4) $$\|II_{\partial U}\| \leq c(n)\, r^{-1}$$

Moreover, U can be chosen such that $G_{T_r(X)} \subset G_U$.

Let H be a compact group of isometries acting on M^n. By applying Theorem 0.1 (with say $r = \frac{1}{4}$) to an exhaustion, $M = \bigcup_{j=1}^{\infty} H(B_j(p))$, by orbits of metric balls of radius j, we get

Theorem 0.5 : (Good choppings) Let M^n be complete, $|K| \leq 1$, $\mathrm{Vol}(M^n) < \infty$. Then M^n admits an exhaustion $M = \bigcup_{j=1}^{\infty} M_j^n$, by manifolds with smooth boundary, such that

(0.6) $$\lim_{j\to\infty} \mathrm{Vol}\left(M_j^n\right) = 0,$$

(0.7) $$\left\|\Pi_{M_j^n}\right\| \leq c(n)\,.$$

Moreover, M_j^n can be chosen invariant under H.

Let $P_\chi(\Omega)$ denote the Chern-Gauss-Bonnet form of M^n. By applying the Chern-Gauss-Bonnet formula to the sets, M_j^n, in Theorem 0.5 and noting that as $j \to\infty$, the boundary term goes to zero, we obtain

Theorem 0.8 : Let M^n be complete, $|K| \leq 1$, $\mathrm{Vol}(M^n) < \infty$. Then given any exhaustion, $M^n = \bigcup_{j=1}^{\infty} M_j^n$, as in Theorem 0.5, for j sufficiently large,

(0.9) $$\int_{M^n} P_\chi(\Omega) = \chi\left(M_j^n\right)$$

In particular, the integral in (0.9) is an integer.

Even if the manifold M^n in Theorems 0.5 and 0.8 has finite topological type (which need not be the case) we can not guarantee that the sets, M_j^n, are homeomorphic to M^n. In fact, there exist metrics on R^4 of the above type, for which the integral in (0.9) vanishes ; see [CG_1] for further details and examples.

Another application pertains to the notion of pure F-structure ; see [CG_3].

Theorem 0.10 : Let M^{4k} be a closed oriented manifold, which admits a pure F-structure of positive rank, whose holonomy group is amenable. Then all the Pontrjagin numbers of M^{4k} vanish.

Recall that there exist closed oriented manifolds, M^{4k}, which admit pure F-structures of positive rank, for which the conclusion of Theorem 0.10 fails ; see [CG_3], Example 1.9.

Finally, as explained in the introduction to $[CG_1]$, the main result of the first part of that paper (Theorem 1.1) is a very easy consequence of Theorem 0.1. The technically rather complicated argument given there was intended to avoid an even more technically complicated proof of Theorem 0.1 which we envisaged at that time.

An earlier result along the lines of Theorem 0.1 appears in $[CG_1]$, (Theorem 2.1). There however, we were forced to assume a lower bound on the injectivity radius and we could not guarentee that $G_{T_r(X)} \subset G_X$. These two defects are actually related for the following reason.

Suppose that one has a construction which requires a lower bound on the injectivity radius, but which is invariant under isometries and local. Then by working equivariantly in the tangent space, one obtains a construction for which the lower bound on the injectivity radius is no longer required. This principle was used in $[CG_2]$ (see Lemma 5.3) in order to smooth functions of bounded gradient. More recently, it was used by Abresch, [A], to obtain a local equivariant smoothing of the riemannian metric itself.

It turns out that Theorem 0.1 follows easily by combining the above two smoothing results with a quantitative version of the Morse-Sard Theorem. The statement we need can be obtained by a careful inspection of the usual proof of A.P. Morse' Lemma. However, it is very easy to extract from the proof of this lemma due to Yomdin ; see [G] pp. 123-124.

1. Proof of Theorem 0.1

We begin by recalling the result of Abresch, which, for convenience, we will specialize to the case of bounded curvature.

Let M^n be a complete riemanian manifold with metric, g. There is no harm in assuming that g is C^∞, since our concern is with <u>explicit</u> estimates on covariant derivatives of curvature.

Let ∇, R denote the connection and curvature of g. Roughly, we wish to associate to g a second metric, $\tilde{g}$, on M^n, such that $\tilde{g}$ is quasi-isometric to g and such the covariant derivatives, $\tilde{\nabla}\tilde{R}$ have bounded norm.

<u>Theorem 1.1.</u> (Abresch) On the set of complete riemannian manifolds, (M^n, g), with $|K| \leq 1$, there exists for all $\varepsilon > 0$, a smoothing operator, $g \to S_\varepsilon(g) = \tilde{g}$, such that

(1.2) $$(1+\varepsilon^2)^{-1} g \le \tilde{g} \le (1+\varepsilon^2) g,$$

(1.3) $$\left\| \nabla_V W - \tilde{\nabla}_V W \right\| \le c(n,\varepsilon)\, \|V\| . \|W\|,$$

(1.4) $$\left\| \tilde{\nabla}^k R \right\| \le c(n, R, \varepsilon)\ .$$

Moreover, at $p \in M^n$, the value of $\tilde{g}$ depends only on $g|B_{1/4}(p)$. Finally, any isometry of g is also an isometry of $\tilde{g}$.

For an earlier version of Theorem 1.1 in which the smoothing proceedure is local but not equivariant (and hence requires a lower bound on the injectivity radius) see [CG_1] ; Theorem 2.5. In [BMR], one finds a nonlocal smoothing proceedure for metrics on compact manifolds which is equivariant and which does not require a lower bound on the injectivity radius. However, their result (at least as it is stated) can not be applied to yield Theorem 0.1.

We now state the result of Yomdin.

Let $f \in C^{n+1}(C_n)$, where C_n denotes the unit cube in R^n. Put

(1.5) $$\| f \|_{C^k} = \max_{1 \le |j| \le k} \left| \partial^j f(x) \right| \qquad x \in C_n$$

where $j = (j_i, \ldots j_{|j|})$ is a multi-index. Let $\mathcal{C}_\varepsilon(f)$ denote the set of points, $y \in R$, such that for some $x \in f^{-1}(y)$, we have $\|\text{grad } f(x)\| \le \varepsilon$.

Theorem 1.6 : (Yomdin) There is a constant, $c = c(n, \|f\|_{C^{n+1}})$, such that for all $0 < \varepsilon < 1$, the set $\mathcal{C}_\varepsilon(f)$ can be covered by at most $c\varepsilon^{-1}$ intervals of length $\varepsilon^{(n+1)/n}$. In particular, for any interval, J, of length $L[J] = \ell$, there is a subinterval, $I \subset (J \setminus \mathcal{C}_\varepsilon(f))$ of length

$\|f\|_{C^{n+1}}$

(1.7) $$L[I] \ge \frac{\left(\ell - c\,\varepsilon^{1/n}\right)}{c}\, \varepsilon > 0,$$

provided $\varepsilon < \left(\frac{\ell}{c}\right)^n$.

Now we can proceed with the proof of Theorem 0.1

Let the number, $(1+\varepsilon^2)$, in Theorem 1.1, be chosen to be $\frac{5}{4}$ and let $\underline{c}(n,1)$ be the corresponding bound on the sectional curvature of the metric, $\tilde{g}$. By scaling the metric g, it suffices to prove Theorem 0.1 for $\underline{r} = \frac{\pi}{8}(\underline{c}(n,1))^{-1/2}$. (We take $\underline{c}(n,1) \geq 1$).

Let $X \subset M^n$ be as in Theorem 0.1. Choose a regularized metric, $\tilde{g}$, as in Theorem 1.1, with $(1+\varepsilon^2) = \frac{5}{4}$.

Let $\tilde{\rho}_X$ be the distance function from X for the metric $\tilde{g}$. We apply the smoothing proceedure of [CG_2], Lemma 5.3 to f_X. Since all covariant derivatives of curvature, $\tilde{\nabla}\tilde{R}$, are bounded in norm, we obtain a function, f_X, such that

$$T_{1/5}(X) \subset f_X^{-1}([0, 1/2]), \tag{1.8}$$

$$f_X^{-1}([0, 2/3]) \subset T(X), \tag{1.9}$$

$$\left\|\tilde{\nabla}^k f_X\right\|_{\tilde{g}} \leq \hat{c}(n,k) \tag{1.10}$$

(where the tubular neighborhoods are with respect to the metric g).

Consider first the case in which X is contained in some metric ball $B_{\underline{r}}(p)$, with $\underline{r} = \frac{\pi}{8}(\underline{c}(n,1))^{-1/2}$, with respect to g. Then $f_X^{-1}([0, 2/3]) \subset \tilde{B}_{2\underline{r}}(p)$. The function $f_X \circ \widetilde{\exp}_p$ has all its derivatives bounded in normal coordinates for $\tilde{g}$ on $\tilde{B}_{2\underline{r}}(p)$.
Thus, by Theorem 1.6, we find an interval, $I \subset [1/2, 2/3]$, of length $L[I] > \delta(n) > 0$, such that on I, we have

$$\|f_X\|_{\tilde{g}} \geq \varepsilon(n) > 0. \tag{1.11}$$

Since the Hessian of f_X is bounded, it follows from (1.11) (by a trivial calculation) that for $y \in I$, the hypersurfaces, $f_X^{-1}(y)$, all have second fundamental form bounded as in (0.4), for $\tilde{g}$, $\tilde{\nabla}$. By (1.2), (1.3) the same holds for the original metric and connection g, ∇.

Since $L[I] \geq \delta(n) > 0$, from (1.11) and the coarea formula, we obtain a bound on $\mathrm{Vol}\left(f_X^{-1}(y)\right)$ for some $y \in I$. In fact, since $\|II_{f_X^{-1}(y)}\|$ is bounded, we can find an interval I', with $y \in I' \subset I$, $L[I'] > \delta'(n) > 0$ such that $\mathrm{Vol}\, f_X^{-1}(y')$ is bounded for all $y' \in I'$.

Thus, if we put $U = f_X^{-1}([0, y'])$, for $y' \in I'$, then (0.3) holds. Moreover, (0.2) follows from (1.8), (1.9).

Finally, since the above constructions are invariant under isometries of $T_r(X)$, we have $G_U \supset G_{T_r(X)}$. So Theorem 0.1 hold in this case. By scaling, it holds for $X \subset B_u(p)$, $0 < u \leq 1$, with constants which are uniform for $0 < r_0 \leq r \leq 1$ (where r is as in Theorem 0.1).

We now consider the general case but we temporarily disregard the condition, $G_U \supset G_{T_r(X)}$. We also take $r = 1$.

By the covering lemma of [CG_1] (Lemma 2.2) we can cover $T_1(X)$ by sets $W_1, \ldots, W_N$, $N = N(n)$, where each W_i is a union of metric balls, $W_i = \bigcup_j B_{1/2}(p_{i,j})$, with centers at mutual distance $\overline{p_{i,j}\,,\, p_{i,j'}} \geq 4$. Put $V_i = \bigcup_j B_1(p_{i,j})$.

We begin by applying the case of our result established above to each set $X \cap B_{1/2}(p_{i,j})$. Let U_1 be the neighborhood of bounded geometry so obtained. Let $\delta'_1(n)$ be the constant corresponding to ∂U_1, which appeared in the argument above ; that is ∂U_1 has a nice tubular neighorhood of radius $\delta'_1(n)$.

Put $Z_2 = U_1 \cup (X \cap W_2)$. Apply the case of our result established above for subsets of balls, to each set, $Z_2 \cap B_1(p_{2,j})$, choosing the radius of the tube to be say $\delta'_1(n) / 4$.
Let $Q_{2,j} \subset T_{\delta'(n)/4}\left(Z_2 \cap B_1(p_{2,j})\right)$ be the neighborhood of bounded geometry so obtained. We can modify $Q_{2,j}$ on the region $B_{3/4}(p_{2,j}) \setminus B_{2/3}(p_{2,j})$, without essentially changing the bounds on the volume and second fundamental form of its boundary, to produce a set, $R_{2,j} \supset Z_2 \cap B_1(p_{2,j})$, such that $R_{2,j}$ agrees with $U_1 \cap B_1(p_{2,j})$, on $B_1(p_{2,j}) \setminus B_{17/24}(P_{2,j})$. Then we put $U_2 = U_1 \cup R_{2,1} \cup R_{2,2} \cup \ldots$

To see that the above modification is possible, observe that on $B_{3/4}(p_{2,j}) \setminus B_{2/3}(p_{2,j})$, $\partial Q_{2,j}$ is just a slight regularization of ∂U_1, on a scale on which it was already regular. Then the interpolation between ∂U_1 and $\partial Q_{2,j}$ can be accomplished by using the distance function from $p_{2,j}$ (regularized as in Lemma 5.3 of $[CG_2]$) as a parameter.

By proceeding as above, we obtain sets $U_3, \ldots, U_N$ and $U = U_N$ is the required neighborhood of bounded geometry.

To construct a set, U, which is actually invariant under the isometry group of $T_1(X)$, we observe that there is a generalization of the covering lemma of $[CG_1]$ to the case in which a compact group, H, of isometries acts. In this case, the sets $W_1, \ldots, W_N$ are replaced by disjoint unions of tubular neighborhoods of orbits, $H(p_{i,j})$. Apart from obvious modifications, the remainder of the statement and proof of this equivariant covering lemma, are just as in $[CG_1]$, Lemma 2.2. Then the equivariant version of Theorem 0.1 can be proved by an obvious modification of the argument just given.

2. A vanishing theorem for amenable holonomy

Proof of Theorem 0.10. Let M^n be a compact manifold with a pure F-structure, $\mathfrak{F}$, whose holonomy group is amenable. Then $\mathfrak{F}$ lifts to a torus action on the holonomy covering, $\tilde{M}^n$, for $\mathfrak{F}$.

Let $\tilde{M}^n = \bigcup_j \tilde{N}_j$ be a Følner exhaustion for $\tilde{M}^n$. Thus, each $\tilde{N}_j$ is a union of β_j copies of a fundamental domain. Moreover, if α_j is the number of these intersecting $\partial \tilde{N}_j$, then $\lim_{j \to \infty} \alpha_j / \beta_j = 0$.

Let $T^k(\tilde{N}_j)$ denote the orbit of $\tilde{N}_j$ under the torus action which lefts $\mathfrak{F}$. By applying Theorem 0.1 (r = 1) to each set, $T^k(\tilde{N}_j)$, we obtain an exhaustion, $\tilde{M}^n = \bigcup_j \tilde{M}_j^n$, by sets, $\tilde{M}_j^n$, satisfying $T^k(\tilde{M}_j^n) = \tilde{M}_j^n$. As a consequence of the Følner property for $\tilde{N}_j$, we get

$$(2.1) \qquad \lim_{j \to \infty} \mathrm{Vol}\left(\partial \widetilde{M_j}\right) / \mathrm{Vol}\left(\widetilde{M_j}\right) = 0$$

Moreover, if dim n = 4ℓ, for any characteristic form , $P(\Omega)$

(2.2) $$\int_{M^n} P(\Omega) \lim_{j \to \infty} \frac{1}{\mathrm{Vol}\left(\partial \tilde{M}_j^n\right)} \int_{\tilde{M}_j^n} P(\Omega)$$

Now suppose $\mathfrak{F}$ has positive rank ; equivalently, the orbits of the action of T^k all have positive dimension. Then since the action of T^k preserves $\partial\tilde{M}_j^n$, and $II_{\tilde{M}_j}$ is bounded in norm, the collapsing construction of [CG_3] Section 3, can be applied.

We replace the metric $\tilde{g}$ on $\tilde{M}_j^n$ by a metric $\tilde{g}_{j,\varepsilon}$ which coincides with $\tilde{g}$ near $\partial\tilde{M}_j^n$. The manifolds $(\tilde{M}_j^n , \tilde{g}_{j,\varepsilon})$ satisfy

(2.3) $$|K_{j,\varepsilon}| \leq c ,$$

(2.4) $$\mathrm{Vol}\left(\tilde{M}_j^n , \tilde{g}_{j,\varepsilon}\right) < c \, \mathrm{Vol}\left(\partial\tilde{M}_j^n , \tilde{g}\right) ,$$

for some c independent of j, ε.

If $P(\Omega_\varepsilon)$ denotes the characteristic form for $\tilde{g}_{j,\varepsilon}$, we have

(2.5) $$\int_{\tilde{M}_j^n} P(\Omega) = \int_{\tilde{M}_j^n} P(\Omega_\varepsilon),$$

(2.6) $$\left| \int_{\tilde{M}_j^n} P(\Omega_\varepsilon) \right| \leq c(n) \, \mathrm{Vol}\left(\partial\tilde{M}_j^n\right) .$$

By combining (2.1), (2.2) with (2.5), (2.6) we find

(2.7) $$\int_{M^n} P(\Omega) = 0 .$$

The metrics $\tilde{g}_{j,\varepsilon}$ are constructed by means of the natural <u>mixed</u> polarization associated to the orbit structure of the action of T^k. This, together with the fact that $\tilde{M}^n$ covers a compact manifold M^n accounts for the uniformity in (2.3), (2.4). On a tubular neighborhood of $\partial \tilde{M}_j^n$, we have

$$\tilde{g}_{j,\varepsilon} = dr^2 + e^{-2r}\,\tilde{h}_{j,r} \qquad 0 \le r \le \varepsilon^{-1} \tag{2.8}$$

The boundary ∂M_j^n corresponds to $r = 0$ and r increases as we move in towards the interior of $\tilde{M}_j^n$. As $r \to \infty$, we have Vol $(\partial \tilde{M}_j^n, h_{i,j}) \to 0$, although possibly, dia $(\partial \tilde{M}_j^n, h_{i,j}) \to \infty$, The total volume of the remaining piece of $\tilde{M}_j^n$ (the interior) for the collapsed metric, $\tilde{g}_{j,\varepsilon}$, approaches zero as $\varepsilon \to 0$.

Remark 2.9. T. Januszkiewicz (unpublished) has a totally different proof of Theorem 0.10, at least in dim 4.

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