A.D. Alexandrov spaces with curvature bounded below

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A.D. Alexandrov spaces with curvature bounded below[(1)]

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CONTENTS

§1. Introduction

1.1. In this paper we develop the theory of (basically finite-dimensional) metric spaces with curvature (in the sense of Alexandrov) bounded below [1], [2]. We are talking, roughly speaking, about spaces with an intrinsic metric, for which the conclusion of Toponogov's angle comparison theorem is true (although only in the small); for precise definitions see §2. These spaces are defined axiomatically by their local geometric properties, without the techniques of analysis. They may have metric and topological singularities, in particular, they may not be manifolds. The class considered includes all limit spaces of sequences of complete Riemannian manifolds with sectional curvature uniformly bounded below. Alexandrov spaces arise naturally if Riemannian manifolds are considered from the viewpoint of synthetic geometry and one avoids the excessive assumptions of smoothness connected

(1) We have used this spelling of Alexandrov, rather than Aleksandrov, because of the title of the paper (in English) mentioned on p. 4. (Ed.)

with the use of analytic technique. Spaces with singularities may appear as limits of sequences of ordinary Riemannian manifolds, and therefore the first are necessary for studying the latter.

1.2. We recall that the class of Riemannian manifolds which have dimension and diameter bounded above, and sectional curvatures bounded below, by fixed numbers, is precompact in the Hausdorff topology (see [17], [19], [20]), but the limit spaces may not be Riemannian.

There are two reasons for the appearance of singularities in passing to the limit. The first is that the curvature may not be bounded above; the second is the appearance of so-called collapsing, when the dimension decreases on passing to the limit. If the curvature is bounded on both sides and there is no collapsing, then the limit space is a Riemannian manifold (of class $C^{2-\varepsilon}$ for any $\varepsilon > 0$), see [19], [28], [29], [15]. (Note that the $C^{2-\varepsilon}$-smoothness of the limit manifolds, proved in [28], [29], [15], is also implied by earlier results of Nikolaev [26], see also [8].)

The local structure of the limit space has been thoroughly studied [13] when collapsing occurs with two-sided bounds for the curvature: the limit in this case is the quotient space of a Riemannian manifold (of class $C^{2-\varepsilon}$) with respect to the isometric action of a Lie group, whose identity component is nilpotent.

The simplest singularities that occur when the curvature is not bounded above can be seen on non-smooth (for example polyhedral) convex hypersurfaces in $\mathbb{R}^n$. When collapsing occurs the limit space may have much more complicated singularities.

The topology and geometry of spaces that are limits of sequences of Riemannian manifolds with curvature bounded only below were studied (not as the basic object) in [23] and [22].

We emphasize that the synthetic approach used in this paper enables us to study the structure of a space without depending on whether it is obtained as the result of collapsing, or is the limit of Riemannian manifolds of the same dimension, or has some other origin. Grove has, independently of the authors, also found it necessary to consider spaces with curvature bounded below irrespective of their origin. We thank Professor K. Grove for letting us see his manuscript containing the plan of research and some results about spaces with curvature bounded below.

1.3. Spaces with curvature bounded below, along with spaces with curvature bounded above, were introduced by Alexandrov [1], [2] as generalized Riemannian manifolds as long ago as the fifties. In his research Alexandrov paid much attention, in essence, to the axiomatic question of describing such spaces in terms of the excesses of triangles (the excess is the amount by which the sum of the angles of a triangle exceeds π). In subsequent papers by Alexandrov, Reshetnyak, Nikolaev and others (see the surveys [4], [8]) spaces with curvature bounded above were investigated, mostly by assuming, a priori, that the geodesic between any two points in the domain being considered was unique. We note that, without additional assumptions, the condition that the curvature is bounded above is not, generally speaking, inherited when passing to the limit. Berestovskii and Nikolaev proved that a space with a two-sided bound for the curvature (and without boundary) is a Riemannian manifold (of smoothness $C^{2-\varepsilon}$ for any $\varepsilon > 0$); for more details see [8].

1.4. We describe briefly the contents of the paper. In §2 we give the basic definitions and those known facts about the spaces being considered that we shall need later on. §4 contains a list of standard constructions leading to spaces with curvature bounded below; in particular, it extends the collection of examples of such spaces given in §2. Some are given separately in §3. There it is proved, roughly speaking, that for spaces with curvature bounded below Toponogov's comparison theorem remains true. In other words, it is proved that the condition that the curvature is bounded below, which in defining the class under consideration enters, a priori, as a local condition, must in fact be satisfied globally, that is, for "arbitrarily large" objects.

The presentation in §3 may seem somewhat more difficult than in the next three sections. However, the reader who is interested only in questions of local structure may omit §3 without loss of understanding. In the rest of the paper one can simply start from stronger axioms, assuming that the condition that the curvature is bounded below is satisfied, not in the small, but globally. However, the basis for certain examples will be made more difficult. Possibly the aesthetic feelings of the reader will suffer.

In §§5 – 7 we establish the foundations for the theory of finite-dimensional spaces with curvature bounded below. We bear in mind that we have to create the apparatus we need and the first stages for studying the local structure of such spaces X.

The starting point is the concept of an explosion of the space X at a point p that is characterized by the presence of a collection of geodesics going out from p, which to some extent calls to mind the axes of a Cartesian coordinate system. From this basic concept it is proved that the set X_0, which is to a certain extent the burst points of X, is open and everywhere dense in X. Later on it is established that for any $\varepsilon > 0$ the set X_0 can be chosen so that a certain neighbourhood of every point of X_0 is ε-almost isometric to a domain in $\mathbb{R}^m$, where m is the Hausdorff dimension of X.

The tangent cone C_p to the space X at a point p plays an important role in studying the structure of X in the neighbourhood of a given point p. In the case of a Riemannian manifold C_p is simply the tangent space at the point p, but for general convex hypersurfaces the cone C_p coincides with the usual tangent cone. The tangent cone C_p exists at any point $p \in X$ and to a first approximation it describes the geometry of X near p. The cone C_p has the same dimension as X and is a cone over a compact space of curvature $\geqslant 1$.

Basically some specific local questions are solved in the next two sections. Thus in §8 we get the following natural generalization of the precompactness theorem mentioned in 1.2: for any numbers n, k, D the metric space (with Hausdorff metric), consisting of all spaces of curvature $\geqslant k$ that have Hausdorff dimension not greater than n and diameter not exceeding D, is compact.

The subject of §9 is those situations when spaces near to a given space, in the Hausdorff metric, turn out to be also near to the given space in the Lipschitz metric.

In §10 we study the Hausdorff measure on spaces with curvature bounded below. The basic results are an estimate of the Hausdorff dimension of the singularities and a theorem on the weak convergence of Hausdorff measures for non-collapsing, (Hausdorff) convergent sequences of Alexandrov spaces.

In the next two sections we develop the apparatus of functions with directional derivatives, which are a good substitute for smooth functions on the spaces under consideration. The basic result of §12 is that there are no singularities of codimension 1 other than the boundary.

Everywhere, except in the last section, assertions are given with complete proofs, but there are more details in the first half of the paper and not so many details in the rest. The exception is in the last section. It contains a resumé of subsequent results obtained recently by the authors and also some unsolved questions and conjectures. A full presentation of these results is contained in the preprint: G. Perel'man, "A.D. Alexandrov's spaces with curvatures bounded from below. II".

1.5. This paper is noticeably different from its preliminary version, whch was distributed as a preprint with the same name. Apart from editorial corrections we have made changes of a more fundamental nature. Thus §§11, 14, 15, 16 in the preliminary version were not included in this new edition; §10 about the Hausdorff measure has been added, and so also have the more general theorems on almost isometry in §9; we give a completely new proof of the "generalized theorem of Toponogov".

§2. Basic concepts

In this section we present the known [5], [1], [2] facts on spaces with curvature bounded below in a form convenient for us.

2.1. The metric space $(M, |\cdot,\cdot|)$ is called an *intrinsic metric space* if for any $x, y \in M$, $\delta > 0$ there is a finite sequence of points $z_0 = x, z_1, \ldots, z_k = y$ such that $|z_i z_{i+1}| < \delta$ $(0 \leqslant i \leqslant k-1)$ and $\sum_{i=0}^{k-1} |z_i z_{i+1}| < |xy| + \delta$. A subset in the space M with intrinsic metric is said to be (metrically) *convex* if the metric induced on it from M is intrinsic. A *geodesic* is a curve whose length is equal to the distance between its ends. In a locally compact complete space

with intrinsic metric any two points can be joined by a geodesic, possibly not unique (without completeness this property is only satisfied locally). A limit of geodesics is again a geodesic and moreover with the limit length.
A collection of three points $p, q, r \in M$ and three geodesics pq, pr, qr is called a *triangle* in M and is denoted by Δpqr.

2.2. Let us fix the real number k. A *k-plane* will be a two-dimensional complete simply-connected Riemannian manifold of curvature k, that is, a Euclidean plane, a sphere, or a Lobachevskii plane. With a triad of points p, q, r in a space M with intrinsic metric we associate a triangle $\tilde{\Delta} pqr$ on the k-plane with vertices $\tilde{p}, \tilde{q}, \tilde{r}$ and sides of lengths $|\tilde{p}\,\tilde{q}| = |pq|$, $|\tilde{p}\,\tilde{r}| = |pr|$, $|\tilde{q}\,\tilde{r}| = |qr|$. For $k \leqslant 0$ the triangle $\tilde{\Delta} pqr$ always exists and is unique up to a rigid motion, and for $k > 0$ it exists only with the additional assumption that the perimeter of Δpqr is less than $2\pi/\sqrt{k}$. We let $\tilde{\measuredangle} pqr$ denote the angle at the vertex $\tilde{q}$ of the triangle $\tilde{\Delta} pqr$.

2.3. *Definition.* A locally complete space M with intrinsic metric is called a *space with curvature* $\geqslant k$ if in some neighbourhood U_x of each point $x \in M$ the following condition is satisfied:

(D) for any four (distinct) points $(a;\, b,\, c,\, d)$ in U_x we have the inequality $\tilde{\measuredangle} bac + \tilde{\measuredangle} bad + \tilde{\measuredangle} cad \leqslant 2\pi$.

If the space M is a one-dimensional manifold and $k > 0$, then we require in addition that diam M does not exceed $\pi/\sqrt{k}$.

Remark. If the space M satisfies all the conditions of the definition, apart from local completeness, then it may be converted into a space with curvature $\geqslant k$ by local completion. We use the a priori condition of local completeness only in §§5, 6.

2.4. Berestovskii [5] has shown that condition (D) in Definition 2.3 can be replaced by the following condition:

(E) any four points in U_x admit an isometric embedding into some k'-plane, $k' \geqslant k$, where the number k' depends on the four points chosen.

We shall not use this version of the definition.

2.5. For spaces in which, locally, any two points are joined by a geodesic, in particular for locally compact spaces, the condition (D) in Definition 2.3 can be replaced by the condition:

(A) for any triangle Δpqr with vertices in U_x and any point s on the side qr the inequality $|ps| \geqslant |\tilde{p}\tilde{s}|$ is satisfied, where $\tilde{s}$ is the point on the side $\tilde{q}\tilde{r}$ of the triangle $\tilde{\Delta} pqr$ corresponding to s, that is, such that $|qs| = |\tilde{q}\,\tilde{s}|$, $|rs| = |\tilde{r}\,\tilde{s}|$.

To prove the implication (D) $\Rightarrow$ (A) it is sufficient to substitute the four points $(s;\, p,\, q,\, r)$ in condition (D) and use the following elementary observation.

Lemma. *Let triangles pqs, prs be given on a k-plane, which are joined to each other in an exterior way along the side ps, and let there also be given a triangle bcd, where* $|bc| = |pq|$, $|bd| = |pr|$, $|cd| = |qs|+|sr|$, *and* $|bc|+|bd|+|cd| < 2\pi/\sqrt{k}$ *in the case* $k > 0$. *Then* $\measuredangle psq + \measuredangle psr \leqslant \pi$ $[\geqslant \pi]$ *if and only if* $\measuredangle pqs \geqslant \measuredangle bcd$ *and* $\measuredangle prs \geqslant \measuredangle bdc$ *[respectively* $\measuredangle pqs \leqslant \measuredangle bcd$ *and* $\measuredangle prs \geqslant \measuredangle bdc$*].*

We give below the proof of the converse implication (A) $\Rightarrow$ (D), see 2.8.2.

2.6. Let γ, σ be geodesics with origin p, and let q, r be points on these geodesics at distances $x = |pq|$, $y = |pr|$ from the point p. The angle $\tilde{\measuredangle} qpr$ is denoted by $\omega_k(x, y)$. Obvioously condition (A) may be replaced by the following condition:

(B) for any geodesics γ, σ in U_x the function $\omega_k(x, y)$ is non-increasing with respect to each of the variables x, y.

2.7. The limit $\lim\limits_{x,y\to 0} \omega_k(x, y)$ is called the *angle between* γ, σ; clearly this limit does not depend on k. If (B) is satisfied, then the angle between the geodesics exists. It is easily verified that the angles between three geodesics with common origin satisfy the triangle inequality.

The following assertion is easily deduced from (A) and (B).

(C) For any triangle Δpqr contained in U_x none of its angles is less than the corresponding angle of the triangle $\tilde{\Delta} pqr$ on the k-plane.

(C_1) The sum of adjacent angles is equal to π, that is, if r is an interior point of the geodesic pq, then for any geodesic rs we have $\measuredangle prs + \measuredangle qrs = \pi$.

Conversely (A) and (B) are easily deduced from (C) and (C_1).

A consequence of the condition (C_1) is that geodesics do not branch. This means that if two geodesics have two common points, then either they are continuations of a third geodesic, their intersection, beyond its distinct ends, or these common points are the ends of both geodesics. In particular, if a geodesic is extendable, then it is extendable in a unique way.

2.8. We list a few other known properties of spaces of curvature bounded below which follow easily from (C) and (C_1).

2.8.1. If the geodesics $p_i q_i$ converge to pq, while the geodesics $p_i r_i$ converge to pr, then $\measuredangle qpr \leqslant \lim\limits_{i\to\infty} \inf \measuredangle q_i p_i r_i$.

This follows easily from (C).

2.8.2. If pa, pb, pc are geodesics, then $\measuredangle apb + \measuredangle bpc + \measuredangle cpa \leqslant 2\pi$.

In fact, if $d \in ap$, then by property (C_1)

$$\measuredangle adb + \measuredangle adc + \measuredangle bdc \leqslant (\measuredangle adb + \measuredangle bdp) + (\measuredangle adc + \measuredangle cdp) = 2\pi.$$

All that remains is to use property 2.8.1 when $d \to p$ if the geodesics pb, pc are unique. Otherwise the points b and c can be replaced by points in pb and pc and the argument repeated.

Now condition (D) follows immediately from the property just proved and (C).

2.8.3. ***Lemma on the limit angle.*** *If the points a_i of a geodesic pa converge to p and $b \notin pa$, then* $\limsup_{i\to\infty} \measuredangle aa_ib \leqslant \measuredangle apb$ *for any geodesics pb, a_ib.*

This property, unlike the preceding ones, is of a non-local character. However, it is easily proved from the local property (C) in exactly the same way as in the case of Riemannian manifolds, see for example [16].

2.9. Examples of spaces of curvature $\geqslant k$.

(1) Riemannian manifolds without boundary or with locally convex boundary, whose sectional curvatures are not less than k.

(2) Convex subsets in such Riemannian manifolds.

(3) Convex hypersurfaces in such Riemannian manifolds, see [9].

(4) Hilbert space ($k = 0$).

(5) Quotient space $\mathbb{R}^3/\sim$, where $x \sim -x$, that is, the cone over $\mathbb{R}P^2$ is a space of curvature $\geqslant 0$. This example will be justified below.

(6) Locally finite simplicial spaces glued (along all the faces) from simplexes of a space of constant curvature k and of fixed dimension, so that every face of codimension 1 belongs to precisely two simplexes and the sum of the dihedral angles for each face of codimension 2 does not exceed 2π.

The last example will be justified in 4.5. There will also be described in §4 natural constructions which enable more complicated examples to be constructed.

§3. Globalization theorem

3.1. Spaces with curvature bounded below were defned in §2 using local conditions. In general, the analogues of these conditions "in the large" may not be satisfied. It is sufficient to consider, for example, a plane from which a closed disc has been removed. However, for complete spaces the global conditions may be deduced from the corresponding local ones. In dimension 2 this was first discovered by Alexandrov. For Riemannian manifolds of arbitrary dimension the corresponding theorem was proved by Toponogov. The arguments in §2 proving that conditions (A), (B), (C, C_1) and (D) are equivalent for locally compact spaces are still completely valid if these conditions are replaced by the corresponding assertions "in the large". Therefore it is sufficient to deduce just one such global condition from the local ones.

We give the proof for condition (D), which is related to a more general situation.

3.2. ***Globalization theorem.*** *Let M be a complete space with curvature $\geqslant k$. Then for any quadruple of points $(a;\ b,\ c,\ d)$ we have the inequality* $\widetilde{\measuredangle} bac + \widetilde{\measuredangle} bad + \widetilde{\measuredangle} cad \leqslant 2\pi$.

3.3. *Remarks.* a) The theorem has to be formulated more precisely when $k > 0$. In this case we assume a priori that the perimeters of the triangles *bac, bad, cad* are less than $2\pi/\sqrt{k}$. In 3.6−3.7 we shall show that in fact under the conditions of the theorem the perimeter of any triangle in M does not exceed $2\pi/\sqrt{k}$. The proof of the theorem when $k > 0$ requires, in comparison with the case $k \leqslant 0$, some additional arguments which will be enclosed in square brackets.

b) Throughout the proof of the theorem and also in 3.7 we assume that there exist geodesics between any two points of M. This is done only to simplify the arguments, as it enables us to use phrases of the type "we choose a point r on the geodesic pq so that ..." . In the general case, it is necessary in such a situation to choose a point r from a sequence constructed from the points p, q and a sufficiently small $\delta > 0$ in accordance with Definition 2.1 of an intrinsic metric.

3.4. Proof of the globalization theorem.

3.4.1. The *size* of the quadruple $(a;\ b,\ c,\ d)$ is $S(a;\ b,\ c,\ d)$, where this denotes the greatest of the perimeters of the triangles Δbac, Δbad, Δcad, while the *excess* $E(a;\ b,\ c,\ d)$ is $\max\{\tilde{\measuredangle} bac + \tilde{\measuredangle} cad + \tilde{\measuredangle} bad - 2\pi,\ 0\}$.

The following is the key to the proof of the theorem.

Lemma on the subadditiveness of the excess. *Let p, q, r, s be points in a space with intrinsic metric, and let t lie on the geodesic pq. Then*

(1) $S(p;\ q,\ r,\ s) \geqslant \max\{S(p;\ t,\ r,\ s),\ S(t;\ p,\ q,\ s),\ S(t;\ p,\ q,\ r)\}$.

(2) $E(p;\ q,\ r,\ s) \leqslant E(p;\ t,\ r,\ s) + E(t;\ p,\ q,\ s) + E(t;\ p,\ q,\ r)$.

[If $k > 0$, then additional assumptions are necessary for assertion (2) to be justified. It is sufficient to assume that

$$\max\{|pq|,\ |pr|,\ |ps|\} + 10|pt| < \pi/2\sqrt{k}.]$$

Assertion (1) is obviously implied by the triangle inequality. To prove assertion (2) we need a numerical refinement of Lemma 2.5.

Technical lemma. *Let triangles pqr, prx, qry be given on a k-plane such that $|rx| = |ry|$, $|pq| = |px| + |qy|$, $\measuredangle qyr + \measuredangle pxr \geqslant \pi$. Then $\measuredangle qpr - \measuredangle xpr \leqslant \measuredangle qyr + \measuredangle pxr - \pi$. [If $k > 0$, then in addition we assume that* $\max\{|pr|,\ |pq|\} + 10|px| < \pi/2\sqrt{k}$.]

Proof. We may suppose that the triangles *prx, qry* lie in the triangle *pqr* and do not overlap. Let z be a point on the side pq such that $|qz| = |qy|$, $|pz| = |px|$. We calculate the sum of the complete angles for x, y, z. We have $5\pi = (\measuredangle qyr + \measuredangle pxr) + (\pi + \delta(\Delta ryx) - \measuredangle yrx) + (\pi + \delta(\Delta xyz)) + (\pi + \delta(\Delta qyz) - \measuredangle yqz) + (\pi + \measuredangle xpr - \measuredangle qpr + \delta(\Delta xpz))$, where $\delta(\Delta)$ denotes the excess of the triangle Δ, that is, the sum of its angles in excess of π.

Now in the case $k \leqslant 0$ the assertion of the technical lemma is obvious, since all excesses $\delta(\Delta)$ are non-posiitve.

[In the case $k > 0$ the condition $\max\{|pr|, |pq|\} + 10|px| < \pi/2\sqrt{k}$ guarantees that the total area of the triangles Δryx, Δqyz, Δxyz, Δxpz is less than half the total area of the spherical lunes with angles $\measuredangle yrx$ and $\measuredangle yqz$, that is, the sum of the excess is less than $\measuredangle yrx + \measuredangle yqz$, and this is just sufficient to make our deduction. This proves the technical lemma. ■]

We turn to the proof of (2). It is now sufficient for us to add together the inequalities arising in the technical lemma

$$E(t; q, r, p) \geqslant \tilde{\measuredangle}\, qpr - \tilde{\measuredangle}\, tpr,$$
$$E(t; q, s, p) \geqslant \tilde{\measuredangle}\, qps - \tilde{\measuredangle}\, tps$$

and use the definitions

$$E(p;\ t,\ r,\ s) = \tilde{\measuredangle}\, tpr + \tilde{\measuredangle}\, tps + \tilde{\measuredangle}\, rps - 2\pi,$$
$$\tilde{\measuredangle}\, qpr + \tilde{\measuredangle}\, qps + \tilde{\measuredangle}\, rps - 2\pi = E(p;\ q,\ r,\ s).$$

The lemma on subadditiveness is proved. ■

3.4.2. We now turn to the proof of Theorem 3.2 itself. Let us assume that the assertion of the theorem is false. Then there exist a point $p \in M$ and a number $l > 0$ [$l < 2\pi/\sqrt{k}$ if $k > 0$] such that:

(a) The excess of any quadruple of size $\leqslant (0.99)\, l$ lying in the ball $B_p(100l)$ is zero.

(b) There is a quadruple of size $\leqslant l$ lying in the ball $B_p(10\, l)$ and having a positive excess.

In fact, otherwise we could find a sequence of points p_i and numbers l_i satisfying condition (b) and such that $l_{i+1} \leqslant (0.99)\, l_i$, $|p_i p_{i+1}| \leqslant 1000 l_i$. But then in view of the fact that the space is complete, the sequence p_i would have a limit point in an arbitrary small neighbourhood of which the condition (D) is violated.

3.4.3. The basic stage in the proof of the theorem is to check that a quadruple $(a;\ b,\ c,\ d)$ of size $\leqslant l$ lying in the ball $B_p(20\, l)$ has zero excess if

$$(1) \qquad |ab| \leqslant 0.01l, \quad |cd| \geqslant 0.1l + \max\{|ac|,\ |ad|\}.$$

[If $k > 0$, then we include, in addition, the condition $\max\{|ac|, |ad|\} < \pi/2\sqrt{k}$.] We assume that there is a quadruple $(a;\ b,\ c,\ d)$ for which this is not so and let $\delta = E(a;\ b,\ c,\ d) > 0$. We take the point x on the geodesic ac at a distance $\varepsilon \leqslant (0.01)\, l$ from a.

[In the case $k > 0$ in addition we assume that $10\varepsilon + \max\{|ac|, |ad|\} < \pi/2\sqrt{k}$.]

By applying the lemma on subadditiveness we get $\delta = E(a;\ b,\ c,\ d) \leqslant E(x;\ a,\ c,\ d)$, since the sizes of the quadruples $(x;\ a,\ b,\ c)$ and $(a;\ x,\ b,\ d)$ are less than $(0.99)\, l$, this follows from (1). We now take the point y on the geodesic xd at a distance ε from x and, again applying the lemma on subadditiveness, we get $E(y;\ x,\ c,\ d) \geqslant E(x;\ a,\ c,\ d) \geqslant \delta$. In addition,

$S(y; x, c, d) \leqslant l$, and this quadruple satisfies the inequalities analogous to (1). Finally

$$(2) \qquad |yc|+|yd| \leqslant |ac|+|ad|-\delta^2\varepsilon/2.$$

In fact, by adding the inequalities

$$\tilde{\measuredangle}\, xab \geqslant \tilde{\measuredangle}\, cab \qquad \text{(by Lemma 2.15, since } E(x; a, b, c) = 0\text{)},$$
$$2\pi \geqslant \tilde{\measuredangle}\, xab + \tilde{\measuredangle}\, dab + \tilde{\measuredangle}\, xad \quad \text{(since } E(a; x, d, b) = 0\text{)},$$
$$\tilde{\measuredangle}\, dac + \tilde{\measuredangle}\, dab + \tilde{\measuredangle}\, cab = 2\pi + \delta,$$
$$\pi \geqslant \tilde{\measuredangle}\, dac,$$

we get $\tilde{\measuredangle} xad \leqslant \pi - \delta$. Similarly $\tilde{\measuredangle} yxc \leqslant \pi - \delta$. Thus

$$|dx| \leqslant |da| + (1-\delta^2/4)\varepsilon,$$
$$|cx| = |ca| - \varepsilon,$$
$$|dy| = |dx| - \varepsilon,$$
$$|cy| \leqslant |cx| + (1-\delta^2/4)\varepsilon,$$

from which we get (2).

Thus from the given quadruple we have constructed another quadruple that satisfies the same sort of bounds but "essentially less" than those expressed by inequality (2). After not more than $2l/\delta^2\varepsilon$ such steps we get a contradiction.

3.4.4. We now reduce the general case to the particular case just considered. Let $(a; b, c, d)$ be a quadruple of size $\leqslant l$ in the ball $B_p(10l)$, with positive excess. We take the point x on the geodesic ab at a distance $\varepsilon \leqslant (0.001)l$ from a [if $k > 0$, we assume further that $10\varepsilon < 2\pi/\sqrt{k} - l$] and consider the quadruples $(a; x, c, d)$, $(x; a, b, c)$, $(x; a, b, d)$. By the lemma on subadditiveness at least one of these quadruples has positive excess, and they all have size $\leqslant l$. So we have reduced the general case to the study of quadruples $(a; b, c, d)$ such that $|ab| = \varepsilon$.

Let $(a; b, c, d)$ be such a quadruple. Our immediate aim is to reduce everything to a consideration of a quadruple $(x; y, z, t)$ of size $\leqslant l$ for which $|xy| \leqslant 2\varepsilon$, $|xz| \leqslant 2\varepsilon$. We shall divide the geodesic ac by the points $a_0 = a, a_1, \dots, a_n = c$ into segments of length $\leqslant \varepsilon$. The lemma on subadditiveness implies that at least one of the quadruples $(a_0; a_1, b, d)$, $(a_1; a_0, b, c)$, $(a_1; a_0, c, d)$ has positive excess. If this is one of the first two quadruples, then our aim is achieved. Otherwise one of the quadruples $(a_1; a_2, a_0, d)$, $(a_2; a_1, a_0, c)$, $(a_2; a_1, c, d)$ has positive excess. At this stage also, if it is one of the first two, our aim is achieved, otherwise we continue "thinning out" the quadruples with positive excess until we get a quadruple of the desired kind.

Finally let $(a; b, c, d)$ be a quadruple of size $\leqslant l$ with positive excess for which $|ab| \leqslant 2\varepsilon$, $|ac| \leqslant 2\varepsilon$. We choose a point x on the geodesic ad such that $|cx| = |dx|$. Clearly in this case $|cx| \leqslant l/4+2\varepsilon$. The lemma on subadditiveness enables us to assert that at least one of the quadruples $(a;\, x, b, c)$, $(x;\, a, b, d)$, $(x;\, a, c, d)$ has positive excess. But the first of these has size $\leqslant (0.99)\,l$, consequently its excess is zero. Without loss of generality we may suppose that just the last quadruple has positive excess. We choose a point y on the geodesic ax at a distance ε from x. By the lemma on subadditiveness one of the quadruples $(x;\, y, c, d)$, $(y;\, x, a, c)$, $(y;\, x, a, d)$ must have positive excess. But the last quadruple has excess 0, since $\tilde{\measuredangle}\, dyx = 0$, the second quadruple also has zero excess, since its size is small, and the first quadruple corresponds to the particular case that was considered in section 3.4.3. The theorem is proved. ■

3.5. *Remark.* If M is not complete, then the proof given above enables us to assert that condition (D) is satisfied for those quadruples that are at a distance which considerably exceeds their size from the points of the completion of M.

3.6. ***Theorem.*** *Let M be a complete space with curvature $\geqslant k$, $k > 0$. Then* diam $M \leqslant \pi/\sqrt{k}$.

3.6.1. *Beginning of the proof.* We first consider the exceptional case when the neighbourhood of some point $p \in M$ is a section of a geodesic. Let q, r be points of this geodesic such that $|pq| = |pr| = |qr|/2 < \pi/2\sqrt{k}$. Then all the points $s \in M$ can be separated into two (perhaps intersecting) classes: those for which $\tilde{\measuredangle}\, spq = 0$, that is, $|qs| = ||ps|-|pq||$, and those for which $\tilde{\measuredangle}\, spr = 0$, that is, $|rs| = ||pr|-|ps||$. Moreover in each class there is at most one point s with a given distance $|ps| < \pi/\sqrt{k}$, and the set of values $|ps|$, realizable by the points s in a single class, is an interval—this follows from condition (D) and the fact that M is complete. By isometrically moving the points q, p, r and repeating the arguments it is not difficult to establish that M contains an open convex subset which is a one-dimensional manifold. Hence clearly M is also this manifold and according to our Definition 2.3 diam $M \leqslant \pi/\sqrt{k}$.

3.6.2. If the space M is locally compact, then the proof can now be completed fairly simply. Let p be the midpoint of the geodesic qr for which $\pi/\sqrt{k} < |qr| < (1.1)\pi/\sqrt{k}$. We may suppose that there are points $s \in M$ near p that do not lie on qr. Let s' be the point of qr nearest to s. A simple calculation, based on condition (C), shows that for s sufficiently near to p, $|qs| < |qs'|$, $|rs| < |rs'|$, which contradicts the fact that qr is a geodesic.

For completeness we give a formal proof without local compactness.

3.6.3. We consider the general case. We need the following observation. Let the points a, b be such that $\pi/\sqrt{k} < |ab| < (1.01)\pi/\sqrt{k}$. We consider sequences of points x_i, $y_i \in M$ such that

$$|ay_i|+|by_i| \to |ab|, \quad |x_iy_i|+\max\{|ay_i|,\ |by_i|\} < 0.99\cdot\pi\sqrt{k}.$$

Then $|ax_i|+|bx_i| \to |ab|$.

In fact we choose another sequence $z_i \in M$ such that $|y_i z_i| \to 0$, $\tilde{\measuredangle} y_i z_i b \to \pi$, $\tilde{\measuredangle} z_i y_i a \to \pi$. Then by applying condition (D) to the sets $(y_i; z_i, x_i, a)$ and $(z_i; y_i, x_i, b)$ we get $\overline{\lim}(\tilde{\measuredangle} a y_i x_i + \tilde{\measuredangle} z_i y_i x_i) \leqslant \pi$, $\overline{\lim}(\tilde{\measuredangle} b z_i x_i + \tilde{\measuredangle} y_i z_i x_i) \leqslant \pi$. On the other hand, $\lim(\tilde{\measuredangle} y_i z_i x_i + \tilde{\measuredangle} z_i y_i x_i) = \pi$ if $|x_i y_i|$ is not zero. Therefore $\overline{\lim}(\tilde{\measuredangle} a y_i x_i + \tilde{\measuredangle} b y_i x_i) \leqslant \pi$ and this means, in view of $|ay_i| + |by_i| > \pi/\sqrt{k}$, that $|ax_i| + |bx_i| \to |ab|$.

3.6.4. We now show that, for any $l \in [(0.9)\pi/2\sqrt{k}, (0.99)\pi/2\sqrt{k}]$, any sequence of points $s_i \in M$ such that $|as_i| \to l$ and $|as_i| + |bs_i| \to |ab|$ has limit points and moreover at most two. In this way we reduce the theorem to the exceptional case already considered. So, let s_i be a sequence of points with the properties indicated. If our assertion is false, then we can choose three subsequences s_i^1, s_i^2, s_i^3 such that the distances $|s_i^1, s_i^2|$, $|s_i^1, s_i^3|$, $|s_i^2, s_i^3|$ are not zero. By applying condition (D) to the quadruples $(a; s_i^1, s_i^2, s_i^3)$ we may, without loss of generality, conclude that $\tilde{\measuredangle} s_i^1 a s_i^2 \leqslant 2\pi/3$ for all i. If now the x_i lie half way between s_i^1 and s_i^2, then the sequences x_i, $y_i = s_i^1$ satisfy the conditions of 3.6.3 and consequently $|ax_i| + |bx_i| \to |ab|$. But on the other hand if we estimate $|ax_i|$ and $|bx_i|$, by considering the quadruples $(x_i; s_i^1, s_i^2, a)$ and $(x_i; s_i^1, s_i^2, b)$ respectively (in essence we are using condition (A) here), then it is not difficult to show that $|ax_i| + |bx_i|$ must be distinct from $|ab|$, which gives a contradiction. ■

3.7. *Theorem.* *Let M be a complete space with curvature $\geqslant k$, $k > 0$. Then for any three points $a, b, c \in M$ we have $|ab| + |ac| + |bc| \leqslant 2\pi/\sqrt{k}$ and condition* (D) *is satisfied for all quadruples of size $2\pi/\sqrt{k}$.* (Here we suppose that, if $|ab| = \pi/\sqrt{k}$, $|ac| + |bc| = \pi/\sqrt{k}$, then $\tilde{\measuredangle} bac = \tilde{\measuredangle} abc = 0$, $\tilde{\measuredangle} acb = \pi$.)

Proof. We first check the second assertion of the theorem for a quadruple $(a; b, c, d)$ of size $2\pi/\sqrt{k}$. We may suppose that $\max\{|ab|, |ac|, |ad|\} < \pi/\sqrt{k}$. Let b_α denote a variable point of the geodesic ab. Then for b_α near to a the perimeters of the triangles $ab_\alpha c$, $ab_\alpha d$ are less than $2\pi/\sqrt{k}$. In addition, if the perimeter of the triangle abc is less than $2\pi/\sqrt{k}$, then by applying condition (D) to the quadruple $(b_\alpha; a, b, c)$ and using Lemma 2.5 we get $\tilde{\measuredangle} b_\alpha ac \geqslant \tilde{\measuredangle} bac$. Finally, if the perimeter of $ab_\alpha d$ is near to $2\pi/\sqrt{k}$, then $\tilde{\measuredangle} b_\alpha ad$ is near to π. Thus, if the excess of the quadruple $(a; b, c, d)$ were positive, we could construct a quadruple $(a; b_\alpha, c, d)$ with positive excess where the perimeters of the triangles $ab_\alpha c$, $ab_\alpha d$ will be less than $2\pi/\sqrt{k}$. Similarly one can get rid of the triangle acd if its perimeter is equal to $2\pi/\sqrt{k}$.

We now return to the first assertion of the theorem and assume that the perimeter of the triangle abc is greater than $2\pi/\sqrt{k}$. It may be supposed that $\max\{|ab|, |ac|, |bc|\} < \pi/\sqrt{k}$. We choose a point x on the geodesic bc such that $2\pi/\sqrt{k} < |ac| + |ax| + |cx| < |ab| + |ac| + |bc|$. Let y be a point of the geodesic xa such that the perimeter of the triangle cxy is equal to $2\pi/\sqrt{k}$.

Clearly $|xb| < |xy|$. Since $|xc| < \pi/\sqrt{k}$ and $|xy| < \pi/\sqrt{k}$, then $\tilde{\measuredangle}cxy = \pi$. By now applying condition (D) to the quadruple $(x;\ c,\ y,\ b)$ and taking into account that $|xb| < |xy|$ we get $|xy| = |xb| + |by|$. But then $|xa| = |xb| + |ba|$, which contradicts the choice of x. The theorem is proved. ■

§4. Natural constructions

4.1. Direct product.
Proposition. *The direct product* (with the metric determined by Pythagoras' theorem) *of a finite number of spaces with curvature* $\geqslant 0$ *is a space with curvature* $\geqslant 0$.

This assertion will become almost obvious if we use the following (equivalent) modification of condition (D).

(D′) For any quadruple $(a;\ b,\ c,\ d)$ lying in U_x it is possible [impossible] to construct a quadruple $(\bar{a};\ \bar{b},\ \bar{c},\ \bar{d})$ on the k-plane such that the segments $\bar{a}\bar{b}$, $\bar{a}\bar{c}$, $\bar{a}\bar{d}$ divide the complete angle at the point $\bar{a}$ into three angles each $\leqslant \pi$, where $|\bar{a}\bar{b}| = |ab|$, $|\bar{a}\bar{c}| = |ac|$, $|\bar{a}\bar{d}| = |ad|$ and $|\bar{b}\bar{c}| \geqslant |bc|$, $|\bar{b}\bar{d}| \geqslant |bd|$, $|\bar{c}\bar{d}| \geqslant |cd|$ [respectively $|\bar{b}\bar{c}| < |bc|$, $|\bar{c}\bar{d}| < |cd|$, $|\bar{b}\bar{d}| < |bd|$].

4.2. The cone.
4.2.1. Let X be a metric space. The *cone over* X with vertex A is the quotient space $C_A(X) = X \times [0, \infty]/\sim$, where $(x_1, a_1) \sim (x_2, a_2) \sim A \Leftrightarrow$ $\Leftrightarrow a_1 = a_2 = 0$. Let $\Pi : C_A(X) \setminus A \to X$ be the natural projection. The metric of the cone is defined from the cosine formula $|\bar{x}_1\bar{x}_2|^2 =$ $= a_1^2 + a_2^2 - 2a_1a_2\cos(\min\{|x_1x_2|, \pi\})$, where $\bar{x}_1 = (x_1, a_1)$, $\bar{x}_2 = (x_2, a_2)$.

4.2.2. It is easily seen that if the metric of X is intrinsic, then the metric of $C_A(X)$ is also intrinsic. The converse assertion, generally speaking, is false but it becomes true if for any two points $x_1, x_2 \in X$ there are arbitrarily nearby points $y_1, y_2 \in X$ such that $|y_1y_2| < \pi$. If γ is a geodesic in X of length not greater than π, then $\Pi^{-1}(\gamma)$ is a convex subset in $C_A(X)$ isometric to the plane sector with angle equal to the length of γ. Conversely, if $\bar{\gamma}$ is a geodesic in $C_A(X)$ not passing through A, then $\Pi(\bar{\gamma})$ is a geodesic in X.

4.2.3. ***Proposition.*** *Let X be a complete metric space. The following two conditions are equivalent*:

(a) *X is a space with curvature* $\geqslant 1$.

(b) *$C_A(X)$ is not a straight line and is a space with curvature* $\geqslant 0$.

Proof. (a) $\Rightarrow$ (b). The fact that $C_A(X)$ is not a straight line is obvious.

We assume that there is a quadruple $(\bar{a};\ \bar{b},\ \bar{c},\ \bar{d})$ in $C_A(X)$ that violates condition (D). We may suppose that A is not one of these points and consider their projections $a = \Pi(\bar{a})$, ..., $d = \Pi(\bar{d})$. By applying (D′) we can find a corresponding quadruple $(a_0;\ b_0,\ c_0,\ d_0)$ on the sphere S^2 for which $|a_0b_0| = |ab|$, $|a_0c_0| = |ac|$, $|a_0d_0| = |ad|$ but $|b_0c_0| \geqslant |bc|$, $|b_0d_0| \geqslant |bd|$, $|c_0d_0| \geqslant |cd|$.

If we think of Euclidean space E^3 as the cone $C_{A_0}(S^2)$ with projection Π_0, we can find a quadruple $(\bar{a}_0; \bar{b}_0, \bar{c}_0, \bar{d}_0)$ in E^3 such that $|A_0\bar{a}_0| = |A\bar{a}|, \ldots, \ldots, |A_0\bar{d}_0| = |A\bar{d}|$ and $\Pi_0(\bar{a}_0) = a_0, \ldots, \Pi_0(\bar{d}_0) = d_0$. Now it is clear that $|\bar{a}_0\bar{b}_0| = |\bar{a}\bar{b}|$, $|\bar{a}_0\bar{c}_0| = |\bar{a}\bar{c}|$, $|\bar{a}_0\bar{d}_0| = |\bar{a}\bar{d}|$ and $|\bar{b}_0\bar{c}_0| \geqslant |\bar{b}\bar{c}|$, $|\bar{b}_0\bar{d}_0| \geqslant |\bar{b}\bar{d}|$, $|\bar{c}_0\bar{d}_0| \geqslant |\bar{c}\bar{d}|$, consequently the quadruple $(\bar{a}_0; \bar{b}_0, \bar{c}_0, \bar{d}_0)$ violates condition (D)—a contradiction.

(b) $\Rightarrow$ (a). We shall first show that the metric X is intrinsic by verifying the condition formulated in 4.2.2. Let $x_1, x_2 \in X$, $\bar{x}_1 \in \Pi^{-1}(x_1)$, $\bar{x}_2 \in \Pi^{-1}(x_2)$. If $|\bar{x}_1\bar{x}_2| \neq |\bar{x}_1A| + |\bar{x}_2A|$, then $|x_1x_2| < \pi$ and our condition is satisfied. If however $|\bar{x}_1\bar{x}_2| = |\bar{x}_1A| + |\bar{x}_2A|$, then the segments $\bar{x}_1A$, $A\bar{x}_2$ form a geodesic. Since $C_A(X)$ is not a straight line and since geodesics do not branch, we can find a point $\bar{y}_1$, arbitrarily near to $\bar{x}_1$, for which $|\bar{y}_1\bar{x}_2| < |\bar{y}_1A| + |\bar{x}_2A|$. Thus one can find a point $y_1 = \Pi(\bar{y}_1)$, arbitrarily near to x_1, for which $|y_1x_2| < \pi$, which is what we required.

Now we suppose one can find a quadruple $(a; b, c, d)$ in X that violates condition (D′). We may suppose, by using the globalization theorem, that the pairwise distances between the points of the quadruple are all less than $\pi/2$. Let $(a_0; b_0, c_0, d_0)$ be a quadruple on the unit sphere S^2 for which $|a_0b_0| = |ab|$, $|a_0c_0| = |ac|$, $|a_0d_0| = |ad|$, $|b_0c_0| < |bc|$, $|b_0d_0| < |bd|$, $|c_0d_0| < |cd|$, and the segments a_0b_0, a_0c_0, a_0d_0 divide the complete angle at the point a_0 into three angles each $\leqslant\pi$. In the Euclidean space $E^3 = C_{A_0}(S^2)$ we find a quadruple $(\bar{a}_0; \bar{b}_0, \bar{c}_0, \bar{d}_0)$ of points lying in the same plane for which $\Pi_0(\bar{a}_0) = a_0, \ldots, \Pi_0(\bar{d}_0) = d_0$ and we construct the corresponding quadruple $(\bar{a}; \bar{b}, \bar{c}, \bar{d})$ in $C_A(X)$ so that $\Pi(\bar{a}) = a, \ldots, \Pi(\bar{d}) = d$, $|A\bar{a}| = |A_0\bar{a}_0|, \ldots, \ldots, |A\bar{d}| = |A_0\bar{d}_0|$. We have $|\bar{a}_0\bar{b}_0| = |\bar{a}\bar{b}|$, $|\bar{a}_0\bar{c}_0| = |\bar{a}\bar{c}|$, $|\bar{a}_0\bar{d}_0| = |\bar{a}\bar{d}|$, $|\bar{b}\bar{c}| > |\bar{b}_0\bar{c}_0|$, $|\bar{b}\bar{d}| > |\bar{b}_0\bar{d}_0|$, $|\bar{c}\bar{d}| > |\bar{c}_0\bar{d}_0|$. But $\tilde{\measuredangle}\bar{b}_0\bar{a}_0\bar{c}_0 + \tilde{\measuredangle}\bar{b}_0\bar{a}_0\bar{d}_0 + \tilde{\measuredangle}\bar{c}_0\bar{a}_0\bar{d}_0 = 2\pi$, consequently $\tilde{\measuredangle}\bar{b}\bar{a}\bar{c} + \tilde{\measuredangle}\bar{c}\bar{a}\bar{d} + \tilde{\measuredangle}\bar{b}\bar{a}\bar{d} > 2\pi$, which contradicts the condition (D). ■

4.3. The construction of the cone can be modified by using the spherical or hyperbolic cosine formula instead of the Euclidean cosine formula.

4.3.1. Spherical suspension.
Let X be a metric space of diameter $\leqslant\pi$. The *spherical suspension* is the quotient space $S(X) = X \times [0, \pi]/\sim$, where $(x_1, a_1) \sim (x_2, a_2) \Leftrightarrow a_1 = a_2 = 0$ or $a_1 = a_2 = \pi$ with the canonical metric $\cos|\bar{x}_1\bar{x}_2| = \cos a_1 \cos a_2 + \sin a_1 \sin a_2 \cos|x_1x_2|$, where $\bar{x}_1 = (x_1, a_1)$, $\bar{x}_2 = (x_2, a_2)$.

Proposition. *Let X be a complete metric space of diameter $\leqslant\pi$. Then the following two conditions are equivalent.*

(a) *X is a space with curvature $\geqslant 1$.*

(b) *$S(X)$ is not a circle and is a space with curvature $\geqslant 1$.*

4.3.2. Elliptic cone.
Let X be a metric space of diameter $\leqslant\pi$. The *elliptic cone over* X is the quotient space $EC(X) = X\times[0, \infty]/\sim$, where $(x_1, a_1) \sim (x_2, a_2) \Leftrightarrow$ $\Leftrightarrow a_1 = a_2 = 0$ with the canonical metric $\cosh|\bar{x}_1\bar{x}_2| = \cosh a_1 \cosh a_2 - \sinh a_1 \sinh a_2 \cos|x_1x_2|$, where $\bar{x}_1 = (x_1, a_1)$, $\bar{x}_2 = (x_2, a_2)$.

Proposition. *Let X be a complete space with diameter $\leqslant\pi$. Then the following two conditions are equivalent.*
(a) *X is a space with curvature $\geqslant 1$.*
(b) *$EC(X)$ is not a straight line and is a space with curvature $\geqslant -1$.*

4.3.3. Parabolic cone.
Let X be a metric space. The *parabolic cone over* X is the product $PC(X) = X\times\mathbb{R}$ with a canonical metric corresponding to the metric of hyperbolic space represented as the twisted product of a horosphere by a straight line.

Proposition. *Let X be a complete space. Then the following two conditions are equivalent.*
(a) *X is a space with curvature $\geqslant 0$.*
(b) *$PC(X)$ is a space with curvature $\geqslant -1$.*

4.3.4. Hyperbolic cone.
Let X be a metric space. The *hyperbolic cone over* X is the product $HC(X) = X\times\mathbb{R}$ with a canonical metric corresponding to the metric of hyperbolic space represented as the twisted product of a hyperplane by a straight line.

Proposition. *Let X be a complete space. Then the following two conditions are equivalent.*
(a) *X is a space with curvature $\geqslant -1$.*
(b) *$HC(X)$ is a space with curvature $\geqslant -1$.*

Propositions 4.3.1–4.3.4 are proved in the same way as Propositon 4.2.3.

4.4. Join.
Let X, Y be complete spaces with curvature $\geqslant 1$. Then the direct product of the cones $C_A(X)\times C_A(Y)$ is a space with curvature $\geqslant 0$ and at the same time is the cone over the join $X*Y = X\times Y\times[0, \pi/2]/\sim$, where $(x_1, y_1, a_1)\sim(x_2, y_2, a_2) \Leftrightarrow a_1 = a_2 = 0$ and $x_1 = x_2$ or $a_1 = a_2 = \pi/2$ and $y_1 = y_2$, with metric $\cos|(x_1, y_1, a_1), (x_2, y_2, a_2)| = \cos a_1 \cos a_2 \cos|x_1x_2| + \sin a_1 \sin a_2 \cos|y_1y_2|$. Thus the join $X*Y$ is also a complete space with curvature $\geqslant 1$.

4.5. We can now justify Example 2.9 (6). In fact, a locally finite simplicial space made by gluing simplicial spaces of constant curvature k can be represented locally as a domain in the space obtained from a lower-dimensional finite simplicial space with curvature $\geqslant 1$ by using the constructions 4.2, 4.1 in the case $k = 0$, using 4.3.1 in the case $k > 0$, and using 4.3.2, 4.3.4 in the case $k < 0$. Thus Example 2.9 (6) is justified by induction on the dimension if we take into account the globalization theorem.

4.6. An equidistant fibred space (see [6]).

Let X be a metric space represented as a disjoint union of closed equidistant subsets X_μ, $\mu \in M$. (We say that the subsets X_{μ_1} and X_{μ_2} are equidistant if the distances $|x_2X_{\mu_1}|$ and $|x_1X_{\mu_2}|$ do not depend on $x_i \in X_{\mu_i}$.) In this case there is a natural metric on the space M defined by the formula $|x_1x_2| = \inf\{|\bar{x}_1\bar{x}_2|, \bar{x}_1 \in \Pi^{-1}(x_1), \bar{x}_2 \in \Pi^{-1}(x_2)\}$, where $\Pi : X \to M$ is a natural projection.

Proposition. *If X is a space with curvature $\geqslant k$, then M is a space with curvature $\geqslant k$.*

Proof. The fact that M is a locally complete space with intrinsic metric is obvious. We assume that condition (D) is violated for the quadruple $(a; b, c, d)$ in M. Let the quadruple $(\bar{a}; \bar{b}, \bar{c}, \bar{d})$ in X be such that $\Pi(\bar{a}) = a, \ldots, \Pi(\bar{d}) = d$ and the distances $|\bar{a}\bar{b}|$, $|\bar{a}\bar{c}|$, $|\bar{a}\bar{d}|$ do not differ much from the corresponding distances $|ab|$, $|ac|$, $|ad|$. Then since $|\bar{b}\bar{c}| \geqslant |bc|$, $|\bar{b}\bar{d}| \geqslant |bd|$, $|\bar{c}\bar{d}| \geqslant |cd|$, the quadruple $(\bar{a}; \bar{b}, \bar{c}, \bar{d})$ also violates condition (D)—a contradiction. ■

Corollary. *Let X be a space with curvature $\geqslant k$ and let the group G act isometrically on X. Then the quotient space $\overline{X/G}$, whose points correspond to the closure of the orbits of G, is a space with curvature $\geqslant k$.*

§5. Burst points

5.1. Content of the section.

All the examples considered above of a space with curvature bounded below included "singular" points, which have arbitrarily small neighbourhoods differing greatly, either just metrically or metrically as well as topologically, from a small neighbourhood of a point in a Riemannian manifold. In all these examples such singular points formed a sparse set. We shall see below that this is the situation in any (finite-dimensional) space with curvature bounded below.

In this section a condition is introduced, expressed in terms of distances, which, if satisfied for a certain point, ensures that it is non-singular in the sense that such a point has neighbourhoods that are homeomorphic and even almost isometric to a ball in Euclidean space. The points satisfying this condition form, in a (finite-dimensional) space with curvature bounded below, an open everywhere dense subset which is a topological manifold.

Let us describe one of the principal ideas. If a maximal (depending on dimension) number of pairwise orthogonal (or at least "linearly independent") geodesics pass through a point, then this point is automatically non-singular.

Generally speaking it is not clear whether such points exist. However, this condition can be modified to make it stable: a point turns out to be non-singular if a maximal number of pairs of geodesics emanate from it so that the geodesics of a single pair have "almost opposite" direction, while the geodesics from distinct pairs are "almost perpendicular".

5.1.1. *Remarks.* 1) From this section on we are dealing mostly with the local properties of spaces with curvature bounded below. For local considerations we shall suppose, without especially saying so, that everything takes place in a domain U_x, where conditions (D), (A), (B), (C) are satisfied. In addition, the explicit constants in our estimates will be for the case $k = 0$, since this greatly simplifies the calculations.

2) In this and the next section, to keep the argument simple, we assume that geodesics exist between the points we consider. In the general case one has to make some trivial modifications.

5.2. *Definition.* Let M be a space with curvature not less than k. A point $p \in M$ is said to be an *(n, δ)-burst point* if there are n pairs of points a_i, b_i distinct from p such that

$$\text{(1)} \qquad \tilde{\measuredangle}\, a_ipb_i > \pi - \delta, \quad \tilde{\measuredangle}\, a_ipa_j > \pi/2 - \delta, \quad \tilde{\measuredangle}\, a_ipb_j > \pi/2 - \delta, \quad \tilde{\measuredangle}\, b_ipb_j > \pi/2 - \delta,$$

where $i \neq j$. The collection of these points a_i, b_i will be called an *(n, δ)-explosion* (or simply an *explosion*) at the point p. Condition (1) also implies the upper bounds $\tilde{\measuredangle} a_ipa_j < \pi/2 + 2\delta$, $\tilde{\measuredangle} a_ipb_i < \pi/2 + 2\delta$, $\tilde{\measuredangle} b_ipb_j < \pi/2 + 2\delta$, where $i \neq j$.

Clearly the set of (n, δ)-burst points is open. It is not difficult to see that for (n, δ)-burst points p the explosion $\{a_i, b_i\}$ can be chosen from points arbitrarily near to p. If M is locally compact, then the fact that $p \in M$ is an (n, δ)-burst point is equivalent to the existence of $2n$ geodesics pa_i, pb_i between which the angles satisfy inequalities analogous to (1).

5.3. *Definition.* A map φ from an open set U in the metric space M into the metric space M_1 is said to be *ε-open* if for any points $x \in U$ and $\bar{y} \in M_1$ such that $\{z \in M : |xz| \leqslant \varepsilon^{-1} |\varphi(x)\bar{y}|_1\} \subset U$ one can find a point $y \in U$ such that $|xy| \leqslant \varepsilon^{-1} |\varphi(x)\bar{y}|_1$ and $\varphi(y) = \bar{y}$. (Here $|\cdot, \cdot|$ and $|\cdot, \cdot|_1$ denote the distances in M and M_1 respectively.)

Clearly an ε-open map is open, and a continuous one-to-one ε-open map φ is a homeomorphism, where the map inverse to the restriction of φ to a sufficiently small subset $U_1 \subset U$ is Lipschitz with coefficient ε^{-1}.

5.4. ***Theorem.*** *Let p be a (n, δ)-burst point with explosion (a_i, b_i), where $\delta < 1/2n$. Then the formula $\varphi(q) = (|a_1q|, |a_2q|, \dots, |a_nq|)$ gives a $(1 - 2n\delta)/\sqrt{n}$-open Lipschitz map from a certain neighbourhood of the point p into $\mathbb{R}^n$. If, in addition, there are no $(n+1, 4\delta)$-burst points near p, then the* *map φ is a bi-Lipschitz homeomorphism between a certain neighbourhood of the point p and a domain in $\mathbb{R}^n$.*

5.5. *Remark.* Below (see 9.4) we shall show that under the conditions of the second half of the theorem, if $\delta > 0$ is sufficiently small, then φ is an "almost isometric" map from a certain neighbourhood of the point p onto a domain in $\mathbb{R}^n$. The map φ is called the *explosion map.*

To prove the second assertion of Theorem 5.4 we need the following lemma which will be useful later on.

5.6. *Lemma.* *Let p, q, r, s be points in a space with curvature $\geqslant k$. If $|qs| < \delta \cdot \min\{|pq|, |rq|\}$ and $\tilde{\measuredangle} pqr > \pi - \delta_1$, then*

$$|\tilde{\measuredangle} pqs + \tilde{\measuredangle} rqs - \pi| < 10\delta + 2\delta_1.$$

(In particular, if geodesics exist, then the angles $\tilde{\measuredangle} pqs$, $\tilde{\measuredangle} rqs$ are little different from the corresponding angles $\measuredangle pqs$, $\measuredangle rqs$.)

Proof. The inequality $\tilde{\measuredangle} psq + \tilde{\measuredangle} rqs - \pi < \delta_1$ follows directly from condition (D) for the quadruple $(q; p, s, r)$. To prove the second inequality we first note that $\tilde{\measuredangle} psr > \pi - 4\delta - \delta_1$—this follows from a consideration of the plane triangles $\tilde{\Delta} prs$, $\tilde{\Delta} pqr$.

In addition, it is obvious that $\tilde{\measuredangle} pqs + \tilde{\measuredangle} psq \geqslant \pi - 2\delta$, $\tilde{\measuredangle} rqs + \tilde{\measuredangle} rsq \geqslant \pi - 2\delta$. Therefore the second assertion of the lemma follows from condition (D) for the quadruple $(s; p, q, r)$. ■

5.7. *Corollary.* *Let p, q, r, s, t be points in a space with curvature $\geqslant k$ such that $|qs| < \delta \cdot \min\{|pq|, |rq|\}$, $\tilde{\measuredangle} pqr > \pi - \delta$, and also $||pq| - |ps|| < \delta \cdot |qs|$ and $\tilde{\measuredangle} qts > \pi - \delta$. Then each of the angles $\tilde{\measuredangle} ptq$, $\tilde{\measuredangle} pts$, $\tilde{\measuredangle} rtq$, $\tilde{\measuredangle} rts$ satisfies the inequality*

$$|\tilde{\measuredangle} \ldots - \pi/2| < 100\delta + 2\delta_1.$$

The proof is not complicated and we leave it to the reader.

5.8. *Proof of Theorem* 5.4. To prove the first part it is sufficient to verify that φ is $(1 - 2n\delta)$-open with respect to the distance in $\mathbb{R}^n$ as determined by the norm $\|x\| = \sum_{i=1}^{n} |x^i|$, where $x = (x^1, \ldots, x^n)$. Let the point q be near to p, let $\varphi(q) = \bar{q}$, and let $\bar{r}$ be close to $\bar{q}$. We find the point $r \in \varphi^{-1}(\bar{r})$ as the limit of a sequence constructed inductively by the rule: $q_1 = q$, and if q_{j-1} is already constructed, then we take as q_j the point on the two-link broken line $a_\alpha q_{j-1} b_\alpha$ which is at a distance $\bar{r}^\alpha$ from a_α, where α is the index for which $||a_i q_{j-1}| - \bar{r}^i|$ is maximal.

For definiteness, let q_j lie on the geodesic $q_{j-1} b_\alpha$. Since $\tilde{\measuredangle} a_\alpha q_{j-1} q_j \geqslant$ $\geqslant \tilde{\measuredangle} a_\alpha q_{j-1} b_\alpha > \pi - \delta$, then $||a_\alpha q_{j-1}| - \bar{r}^\alpha| > |q_{j-1} q_j| \cos\delta$. If we take into account that $\tilde{\measuredangle} a_i q_{j-1} q_j \geqslant \tilde{\measuredangle} a_i q_{j-1} b_\alpha > \pi/2 - \delta$ when $i \neq \alpha$, and on the other hand $\tilde{\measuredangle} a_i q_{j-1} q_j \leqslant 2\pi - \tilde{\measuredangle} a_i q_{j-1} a_\alpha - \tilde{\measuredangle} a_\alpha q_{j-1} q_j < \pi/2 + 2\delta$, we get, when

$|q_{j-1}q_j|$ is sufficiently small and $i \neq \alpha$,

$$||a_iq_j|-\bar{r}^i| \leqslant ||a_iq_j|-|a_iq_{j-1}||+||a_iq_{j-1}|-\bar{r}^i| \leqslant$$
$$\leqslant 2\delta\cdot|q_jq_{j-1}|+||a_iq_{j-1}|-\bar{r}^i| \leqslant (2\delta/\cos\delta)\cdot||a_\alpha q_{j-1}|-\bar{r}^\alpha|+||a_iq_{j-1}|-\bar{r}^i|.$$

Hence $\Delta_j = \|\varphi(q_j)-\bar{r}\| = \sum_{i\neq\alpha}||a_iq_j|-\bar{r}^i| \leqslant (n-1)2\delta/\cos\delta\,||a_\alpha q_{j-1}|-\bar{r}^\alpha|+$ $+\Delta_{j-1}-||a_\alpha q_{j-1}|-\bar{r}^\alpha| = \Delta_{j-1}-(1-2(n-1)\delta/\cos\delta)||a_\alpha q_{j-1}|-\bar{r}^\alpha|$. Thus $\Delta_{j-1}-\Delta_j \geqslant (\cos\delta-2(n-1)\delta)|q_{j-1}q_j| > (1-2n\delta)|q_{j-1}q_j|$, and in addition $\leqslant\Delta_{j-1}(1-(1-2(n-1)\delta/\cos\delta)/n) < \Delta_{j-1}\left(1-\dfrac{1-2n\delta}{n}\right)$ according to the choice of α, Δ_j. Therefore, if $\delta < 1/2n$, the sequence q_j converges to some point r, where $\varphi(r) = \bar{r}$ and $|qr| \leqslant (1-2n\delta)|\overline{qr}|$.

To prove the second part of the theorem all that remains is for us to verify that φ is one-to-one in a sufficiently small neighbourhood U of the point p. We choose the neighbourhood U so small that the pairs (a_i, b_i) form an (n,δ)-explosion for every point $q\in U$ and $\operatorname{diam} U < (0.01)\delta \min_{1\leqslant i\leqslant n}\{|pa_i|, |pb_i|\}$. If we had $\varphi(x) = \varphi(y)$ for $x, y \in U$, then, according to Corollary 5.7, the pairs (a_i, b_i), together with the pair (x, y), would form an $(n+1, 4\delta)$-explosion for the midpoint of the geodesic xy, which contradicts the condition. ■

We need the following lemma in order to characterize the dimension of the space with the help of burst points.

5.9. *Lemma.* *Given an arbitrarily small $\delta' > 0$ there are points having (n, δ')-explosions in any neighbourhood of a point having an (n, δ)-explosion with $\delta < 1/8n$.*

Proof. Let the point p have an (n, δ)-explosion (a_i, b_i) with $\delta < 1/8n$. We construct the desired (n, δ')-burst point p' as the last term of the sequence $p_0 = p, \dots, p_n = p'$ in which each point p_j has an $(n, 4\delta)$-explosion (a_i^j, b_i^j) such that $\tilde{\measuredangle} a_i^jp_jb_i^j > \pi-\delta'/4$ for $i \leqslant j$ and $\tilde{\measuredangle} a_i^jp_ja_{i'}^j > \pi/2-\delta'$, $\tilde{\measuredangle} b_i^jp_jb_{i'}^j >$ $> \pi/2-\delta'$, $\tilde{\measuredangle} a_i^jp_jb_{i'}^j > \pi/2-\delta'$ if $i \neq i'$ and $i, i' \leqslant j$.

The sequence is constructed inductively. Let p_j be already constructed; then we take as p_{j+1} the midpoint of the geodesic p_jq_j, where q_j is a point very near to p_j for which $|p_ja_i^j| = |q_ja_i^j|$ when $i \neq j+1$, but $|p_ja_{j+1}^j| \neq |q_ja_{j+1}^j|$; the construction of this point is ensured by the first part of the proof of Theorem 5.4. We take as points of the explosion $a_i^{j+1} = a_i^j$ and $b_i^{j+1} = b_i^j$ when $i \neq j+1$, $a_{j+1}^{j+1} = p_j$, $b_{j+1}^{j+1} = q_j$. To verify this it is obvious that we only need $\tilde{\measuredangle} p_jp_{j+1}a_i^j > \pi/2-\delta'$ for $i < j+1$ and analogous inequalities with b_i^j instead of a_i^j and/or q_j instead of p_j and/or δ instead of δ' for $i > j+1$. These inequalities are ensured by Corollary 5.7 and the induction assumption. ■

§6. Dimension

6.1. In a space with curvature bounded below, the number n is called the *burst index* near p if, for an arbitrarily small $\delta > 0$, there are (n, δ)-burst points in any neighbourhood of this point but the analogous condition with n replaced by $n+1$ is not satisfied (n is a natural number or 0). If there is no such n, then we suppose the burst index to be ∞. We show that in such a space the burst indices near the various points are equal and coincide with its Hausdorff dimension. In fact, in what follows (up to §10) the Hausdorff measure and Hausdorff dimension do not play an essential role; instead we use the rough volume and the rough dimension defined below.

6.2. *Definition. The rough a-dimensional volume* $Vr_a(X)$ of a bounded set X in a metric space is $\limsup_{\varepsilon\to 0} \varepsilon^a\beta_X(\varepsilon)$, where $\beta_X(\varepsilon)$ is the largest possible number of points $a_i \in X$ that are at least ε pairwise distant from each other. The value $\beta_X(\varepsilon) = \infty$ is not excluded a priori.

$\inf\{a : Vr_a(X) = 0\} = \sup\{a : Vr_a(X) = \infty\}$ is called the *rough dimension* $\dim_r X$. Obviously $\dim_r X \geqslant \dim_H X$, where $\dim_H$ is the Hausdorff dimension. We also note that if $f : X \to Y$ is a Lipschitz map, then $\dim_H f(X) \leqslant \dim_H X$ and $\dim_r f(X) \leqslant \dim_r X$; if f is bi-Lipschitz, then $\dim_H f(X) = \dim_H X$, $\dim_r f(X) = \dim_r X$.

6.3. ***Lemma.*** *Let $u, v \in M$ be points in a space with curvature not less than k, and let U and V be their neighbourhoods. Then, if U and V are sufficiently small,* $\dim_r U = \dim_r V$.

Proof. Since u and v may be joined in M by a sequence of near points, then we may suppose that u and v are so near to each other and U and V are so small that conditions (A), (B), (C), (D) are satisfied for triangles with vertices in $U \cup V$.

Let $\dim_r V > a \geqslant 0$. Then $\limsup_{\varepsilon\to 0} \varepsilon^a\beta_V(\varepsilon) = \infty$ and hence for some sequence $\varepsilon_i \to 0$ we have $\varepsilon_i^a\beta_V(\varepsilon_i) \geqslant c > 0$. For every i there is a collection of points $c_1, \dots, c_{N_i} \in V$ which are at least ε_i pairwise distant from each other and such that $\varepsilon_i^a N_i \geqslant c$. We fix in U a ball $B = B_u(R)$ and some geodesics uc_j and we choose points b_j on them so that $|ub_j| = \frac{R}{D}|uc_j|$, where $D = \sup\{|ux| : x \in V\}$. Clearly the points b_j lie in B and by condition (B) are not less than $\varepsilon_i' = \varepsilon R/D$ distant from each other. Now we have

$$(\varepsilon_i')^a\beta_U(\varepsilon_i') \geqslant (R/D)^a\varepsilon_i^a N_i \geqslant c(R/D)^a,$$

consequently $Vr_a(U) > 0$ and in this way $\dim_r U \geqslant \dim_r V$.

If we exchange U and V we get the assertion of the lemma. ∎

6.4. ***Lemma.*** *Let $p \in M$ be a point in a space with curvature bounded below. Then for a sufficiently small neighbourhood $U \ni p$ the burst index of M near p is equal to* $\dim_r U$ *and equal to* $\dim_H U$.

Proof. We assume first that the burst index of M near p is equal to ∞, that is, for an arbitrarily large n and an arbitrarily small $\delta > 0$ there are (n, δ)-burst points in any neighbourhood $U \ni p$. By Theorem 5.4 there is some neighbourhood $U_1 \subset U$ of the (n, δ)-burst point $p_1 \in U$ that admits a Lipschitz open map into $\mathbb{R}^n$, whence we have $\dim_r U \geqslant \dim_r U_1 \geqslant n$ and $\dim_H U \geqslant \dim_H U_1 \geqslant n$. Since this is true for any natural number n, then in this case $\dim_r U = \dim_H U = \infty$.

Now let the burst index of M near p be n and let n be a natural number (the case $n = 0$ is trivial—in this case M is a point). Then by definition of the burst index and by Lemma 5.9 there are no $(n+1, 1/8(n+1))$-burst points in some neighbourhood $U \ni p$. Therefore Theorem 5.4 ensures that there is a bi-Lipschitz homeomorphism from some neighbourhood $U_1 \subset U$ of an $(n, 1/100n)$-burst point $p_1 \in U$ onto a domain in $\mathbb{R}^n$. This immediately implies that $\dim_r U_1 = n$, $\dim_H U_1 = n$. By Lemma 6.3 we get $\dim_r U = \dim_r U_1 = n$ and finally $\dim_H U = n$, since $\dim_H U_1 \leqslant \dim_H U \leqslant \dim_r U$. ■

6.5. *Corollary.* *The burst indices near the various points of a space with curvature bounded below are equal. They coincide with its Hausdorff dimension. If the burst indices are finite, then they coincide with the topological dimension of the space.*

Proof. The first assertion is implied by Lemmas 6.4 and 6.3, the second by Lemma 6.4 (it is easy to verify that a space is separable if the burst indices are finite). To prove the third assertion we note that on the one hand it is known that the topological dimension of any metric space does not exceed its Hausdorff dimension, and on the other hand, if the burst index of the space near a point is equal to n, then by Theorem 5.4 some neighbourhood of an $(n, 1/100n)$-burst point p_1 near p is homeomorphic to $\mathbb{R}^n$ and hence has topological dimension n. ■

6.6. *Remark.* We have not been able to prove that a space with curvature bounded below which has finite topological dimension must have finite burst indices.

By a finite-dimensional space with curvature bounded below (FDSCBB) we shall mean a space with finite burst indices. By Corollary 6.5 one can talk about the dimension of a FDSCBB without danger of ambiguity. Later on we shall be dealing exclusively with FDSCBB.

6.7. *Corollary.* *If M is an n-dimensional* SCBB, *then for any $\delta > 0$ the set of (n, δ)-burst points is everywhere dense in M.*

A point in a FDSCBB is said to be an *n-burst point* if it is an (n, δ)-burst point for a sufficiently small (depending on the context) $\delta > 0$.

6.8. *Corollary.* *In a complete* FDSCBB *any bounded set is precompact.*

In fact an argument analogous to the proof of Lemma 6.3, but using, instead of condition (B), the global version of it that comes from Theorem 3.2, shows that in a complete n-dimensional SCBB any bounded set has a finite n-dimensional rough volume and further has finite ε-nets for all $\varepsilon > 0$.

6.9. *Remark.* The results of §§5, 6 retain part of their strength if the inequality in condition (D) is replaced by a weaker inequality $\tilde{\measuredangle} bac + \tilde{\measuredangle} cad + \tilde{\measuredangle} bad \leqslant \leqslant 2\pi + \varepsilon_m$, where ε_m is a positive number depending only on the upper bound m of the Hausdorff dimensions of the spaces under consideration. (This inequality is, in a certain sense, an integral bound for the negative curvature.) For sufficiently small $\varepsilon_m > 0$ one can determine inductively, starting with the largest n, positive numbers δ_n, $1 \leqslant n \leqslant m+1$, such that Theorem 5.4 retains its strength when $0 < \delta \leqslant \delta_n$ and in the second assertion $(n+1, 4\delta)$ is replacd by $(n+1, \delta_{n+1})$. If now in defining the burst index near a point we only take into account the (n, δ_n)-burst points, then the sets M_n, $0 \leqslant n \leqslant m$, consisting of the points of M near which the burst index is n form a "stratification" of the space M, that is, the union of all M_n with numbers less than l is open in M for any l, and in every M_n the set of (n, δ_n)-burst points (which is a topological n-manifold) is open and dense.

§7. The tangent cone and the space of directions

7.1. Content of the section.
The tangent cone at a point of a FDSCBB, which we define in this section, generalizes the concept of the tangent space at a point of a Riemannian manifold and of the tangent cone at a point of a convex hypersurface. To a first approximation it characterizes the metric in a neighbourhood of the point. The tangent cone to a point of a FDSCBB is a cone (in the sense of 4.2) over a compact space with curvature $\geqslant 1$, whose dimension is 1 less than that of the original space; thus the tangent cone is (by Proposition 4.2.3) a space of non-negative curvature and its dimension is the dimension of the space. As in the case of a convex hypersurface the tangent cone depends semicontinuously on the point of the space. In addition we show that the tangent cone changes continuously when the point moves along the interior of a geodesic.[1]

One natural definition of a tangent cone would be: one has to take the Hausdorff limit of "blown up" small neighbourhoods of the chosen point. For technical reasons we begin with another definition, which is based on the concept of the space of directions.

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[1] In fact the tangent cone here does not change at all. This was recently proved by A. Petrunin.

7.2. Space of directions.
Let $p \in M$ be a point in a FDSCBB. Geodesics with origin p are considered to be equivalent if they form a zero angle with each other; in this case one of them is a part of the other. The set of equivalence classes of geodesics with origin p is endowed with a metric in which the distance is the angle between the geodesics at the point p. This metric space is denoted by Σ_p' and its metric completion is called the *space of directions* at the point p and is denoted by Σ_p.

7.2.1. *Remark.* Usually (see [8]) the space of directions is defined somewhat differently: one considers those curves with origin $p \in M$ that form a zero upper angle with themselves at p ("having direction"); two such curves are considered equivalent if they form a zero upper angle with each other at p. The set of equivalence classes with the natural angle metric is called the *space of directions* Σ_p^*.

It can be shown that in the case of a FDSCBB this definition gives the same space Σ_p as in 7.2. We know a proof of this which is simple but is based on the following Theorem 7.3.

7.3. ***Theorem.*** *The space of directions at any point of a* FDSCBB *is compact.*

Proof. To show that Σ_p is compact it is sufficient to find a finite ε-net in Σ_p' for any $\varepsilon > 0$. Therefore the theorem follows from the particular case $m = 0$ of the following lemma.

7.4. ***Lemma.*** *For any integers $n \geqslant 1$, $0 \leqslant m \leqslant n$, and a real number $\varepsilon > 0$ there exist a real number $\delta = \delta(n, m, \varepsilon) > 0$ and a natural number $N = N(n, m, \varepsilon)$ with the following property. In an n-dimensional* SCBB *there cannot exist points p, a_i, b_i, c_j, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant N$, such that*

$$(1)\quad \tilde{\measuredangle}\, c_j p a_i > \pi/2 - \delta, \quad \tilde{\measuredangle}\, c_j p b_i > \pi/2 - \delta, \quad \tilde{\measuredangle}\, c_j p c_{j'} > \varepsilon \text{ when } j \neq j'$$

and the pairs (a_i, b_i) form an (m, δ)-explosion at p.

To prove Lemma 7.4 we need the following technical assertion, which will be used later on.

7.5. ***Lemma.*** *Let $\{pa_i\}$ be a finite collection of geodesics in a* FDSCBB. *Then for any $\delta > 0$ there is a neighbourhood U of the point p (depending on the collection of geodesics and on δ) such that the angles of all the triangles Δpqr with vertices q, r on the parts of the geodesics pa_i which are in U differ from the corresponding angles of the triangles $\tilde{\Delta} pqr$ by no more than δ.*

Proof. It is sufficient to consider the case of two geodesics pa, pb. Let $R > 0$ be so small that if $a_1 \in pa$, $b_1 \in pb$ are such that $|pa_1| \leqslant R$, $|pb_1| \leqslant R$, then $\measuredangle a_1 p b_1 - \tilde{\measuredangle} a_1 p b_1 < \delta/2$. We now consider any $\Delta p a_1 b_1$ with $a_1 \in pa$, $b_1 \in pb$ and $|pa_1| < (0.1)\delta R$, $|pb_1| < (0.1)\delta R$. Let the point $b_2 \in pb$ be such that $|pb_2| = R$. Clearly, if we add the triangles $\tilde{\Delta} p a_1 b_1$ and $\tilde{\Delta} a_1 b_1 b_2$

on the plane externally along the side $\tilde{a}_1\tilde{b}_1$, then we obtain a convex quadrangle, and by comparing this with the triangle $\tilde{\Delta} a_1 p b_2$ we get

$$\begin{aligned}\tilde{\measuredangle}\, a_1pb_1 + \tilde{\measuredangle}\, a_1b_2b_1 - \tilde{\measuredangle}\, a_1pb_2 - \tilde{\measuredangle}\, a_1b_2p = \\ = (\tilde{\measuredangle}\, pa_1b_2 - \tilde{\measuredangle}\, pa_1b_1 - \tilde{\measuredangle}\, b_1a_1b_2) + (\pi - \tilde{\measuredangle}\, pb_1a_1 - \tilde{\measuredangle}\, b_2b_1a_1) \geqslant \\ \geqslant \pi - \tilde{\measuredangle}\, pb_1a_1 - \tilde{\measuredangle}\, b_2b_1a_1.\end{aligned}$$

Therefore

$$\begin{aligned}\measuredangle\, pb_1a_1 - \tilde{\measuredangle}\, pb_1a_1 \leqslant (\measuredangle\, pb_1a_1 + \measuredangle\, b_2b_1a_1) - (\tilde{\measuredangle}\, pb_1a_1 + \tilde{\measuredangle}\, b_2b_1a_1) \leqslant \\ \leqslant (\tilde{\measuredangle}\, a_1pb_1 - \tilde{\measuredangle}\, a_1pb_2) + \tilde{\measuredangle}\, a_1b_2b_1 - \tilde{\measuredangle}\, a_1b_2p < \delta,\end{aligned}$$

since the difference in the brackets does not exceed $\delta/2$ and $\tilde{\measuredangle} a_1b_2b_1 \leqslant \delta/2$. Similarly we get $\measuredangle pa_1b_1 - \tilde{\measuredangle} pa_1b_1 < \delta$. ■

7.6. *Proof of Lemma* 7.4. We use inverse induction with respect to m for fixed n. In the case $m = n$ one can take $N = 1$, $\delta = 1/100n$ since, if the points p, a_i, b_i, c satisfied conditions (1), then a point p_1 of the geodesic pc that is sufficiently near p would be a $(n+1, 2\delta)$-burst point, which is impossible.

Induction step. We take δ and N such that

$$\delta(m, \varepsilon) < 0.001\varepsilon\delta(m+1, \varepsilon/2), \tag{2}$$

$$\frac{1}{2}\left(N(m,\varepsilon) - N\left(m+1, \frac{\varepsilon}{2}\right)\right) > \frac{1000}{\delta(m,\varepsilon)} N\left(m+1, \frac{\varepsilon}{2}\right). \tag{3}$$

Let the points p, a_i, b_i, c_j satisfy conditions (1) for given m, ε and chosen N, δ. By making a small shift in p along the geodesic pc_N and renaming $a_{m+1} = p$, $b_{m+1} = c_N$ we may suppose that the points p, a_i, b_i, c_j, $1 \leqslant i \leqslant m+1$, $1 \leqslant j \leqslant N-1$, satisfy the conditions analogous to (1) in which δ is replaced by 2δ and the inequalities $\tilde{\measuredangle} a_{m+1}pc_j > \frac{\pi}{2} - \delta$, $\tilde{\measuredangle} b_{m+1}pc_j > \frac{\pi}{2} - \delta$ are excluded. By making a shift in the points a_i, b_i, c_j towards p we may suppose that they are in a zone where Lemma 7.5 applies. By the inductive hypothesis there cannot exist more than $N\left(m+1, \frac{\varepsilon}{2}\right) - 1$ of the points c_j for which $\tilde{\measuredangle} a_{m+1}pc_j > \frac{\pi}{2} - 3\delta$ and $\tilde{\measuredangle} b_{m+1}pc_j > \frac{\pi}{2} - 3\delta$. Therefore in view of (3) we may suppose that among the c_j there are at least $1000\delta^{-1}N\left(m+1, \frac{\varepsilon}{2}\right)$ of them for which $\frac{\varepsilon}{2} \leqslant \tilde{\measuredangle} a_{m+1}pc_j \leqslant \measuredangle a_{m+1}pc_j \leqslant \leqslant \tilde{\measuredangle} a_{m+1}pc_j + \delta \leqslant \frac{\pi}{2} - 2\delta$. In turn one can find at least $N\left(m+1, \frac{\varepsilon}{2}\right)$ of these points such that for any two of them $|\varphi_j - \varphi_{j'}| < (0.01)\delta$, where $\varphi_j = \tilde{\measuredangle} a_{m+1}pc_j$. If we now take points p_1 on pa_{m+1} and $\bar{c}_j$ on pc_j such that $|p\bar{c}_j| = |pp_1|(\cos\varphi_j)^{-1}$, then the points p_1, a_i, b_i, $\bar{c}_j$ provide a contradiction to the inductive hypothesis. In fact the inequality $\tilde{\measuredangle}\bar{c}_j p\bar{c}_{j'} > \frac{\varepsilon}{2}$ follows easily from our construction and the condition $|\varphi_j - \varphi_{j'}| < (0.01)\delta$. The relation between $|pp_1|$ and $|p\bar{c}_j|$ and Lemma 7.5 implies that $\tilde{\measuredangle} a_{m+1}p_1\bar{c}_j > \frac{\pi}{2} - 10\delta$, $\tilde{\measuredangle} b_{m+1}p_1\bar{c}_j > \frac{\pi}{2} - 10\delta$. Finally to prove the inequalities $\tilde{\measuredangle} a_i p_1\bar{c}_j > \frac{\pi}{2} - \delta\left(m+1, \frac{\varepsilon}{2}\right)$, $1 \leqslant i \leqslant m$ (and the analogous inequalities with b_i instead of a_i) it is sufficient to show that $||a_i\bar{c}_j| - |a_ip_1||$ is bounded above by $|p_1\bar{c}_j|$. Since $\varphi_j \geqslant \frac{\varepsilon}{2}$, then $|p_1\bar{c}_j| > (0.1)\varepsilon(|p_1p| + |p_1\bar{c}_j|)$. On the other hand $||a_i\bar{c}_j| - |a_ip_1|| \leqslant ||a_ip| - |a_ip_1|| + ||a_i\bar{c}_j| - |a_ip|| \leqslant 5\delta(|p_1p| + |p\bar{c}_j|)$. This, in view of condition (2), implies the required inequality. ■

7.7. *Definition.* The *tangent cone* C_p at the point p of a FDSCBB M is the cone over the space of directions Σ_p.

The map $\exp_p$ is defined in the standard way with values in M, and with domain of definition a certain subset $C_p' \subset C_p$; this subset, generally speaking, may not contain any neighbourhood of the vertex of the cone C_p. (The inverse map $\exp_p^{-1}$, considered as a multi-valued map, is defined on all M; this map, generally speaking, is not continuous. It becomes single-valued (but is still not continuous) if it is considered as a map from the set of geodesics with origin p in C_p.)

7.8. Let X, Y be a metric space, $\nu > 0$. The map $f: X \to Y$ is called a (*Hausforff*) ν-*approximation* if $||f(x_1)f(x_2)| - |x_1x_2|| < \nu$ for any $x_1, x_2 \in X$ and Y is contained in the ν-neighbourhood $U_\nu(f(X))$. We say that the compact metric spaces X_i converge in the sense of Hausdorff to a compact metric space X if for arbitrarily small $\nu > 0$ and sufficiently large numbers i there exist ν-approximations $f_i: X_i \to X$. There are other equivalent definitions of Hausdorff convergence for compact spaces; Hausdorff convergence can also be defined for non-compact spaces with a base point (see [19]).

7.8.1. *Theorem.* *Let* (M, ρ) *be a* FDSCBB *and let* $p \in M$. *Then the spaces with base point* $(M, p, \lambda\rho)$ *converge in the sense of Hausdorff as* $\lambda \to \infty$ *to the tangent cone* C_p.

Proof. Let $B^\lambda(R)$ denote the ball in M in the metric $\lambda\rho$ with centre p and radius R, and let $B(R)$ be the ball in C_p of radius R with centre at the vertex of the cone C_p. By definition ([19], 3.14) we need to verify that $B^\lambda(R+\lambda^{-1})$ converges to $B(R)$ in the Hausdorff metric as $\lambda \to \infty$. For this, according to [19], 3.5(b), it is sufficient to verify that for any $\varepsilon > 0$ there exists an ε-net in $B(R)$ which is the limit of ε-nets in $B^\lambda(R+\lambda^{-1})$.

As $\overline{B}(R)$ is compact, we can choose there a finite $\varepsilon/2$-net. With a suitable choice of this net and for λ sufficiently large the map $\xi \to \exp_p \lambda^{-1}\xi$ takes this into an ε-net in $B^\lambda(R+\lambda^{-1})$ where the $(\lambda\rho)$-distances between the points are near to the corresponding distances between points of the original net. The reader will not find it difficult to convert this argument into a rigorous proof. ■

7.9. Corollary. *Under the conditions of Theorem* 7.8.1 *the tangent cone C_p is a space of non-negative curvature.*

In fact the metric of C_p is intrinsic, as a Hausdorff limit of the intrinsic metrics $\lambda\rho$ (see [19]). If (M, ρ) is a space with curvature $\geqslant k$, then $(M, \lambda\rho)$ is a space with curvature not less than $k\lambda^{-2}$. It is immediately clear from Definition 2.3 that the limit space C_p is a space with curvature not less than

$$0 = \lim_{\lambda\to\infty} k\lambda^{-2}. \ ■$$

7.10. Corollary. *If, under the conditions of Theorem* 7.8.1, $\dim M > 1$, *then the space of directions Σ_p is a space with curvature not less than* 1.

This is implied directly by Corollary 7.9 and Proposition 4.2.3.

7.11. Corollary. *Under the conditions of Theorem* 7.8.1

$$\dim \Sigma_p = \dim M - 1$$

or, what is equivalent, $\dim C_p = \dim M$.

To prove that $\dim C_p \geqslant \dim M$ it is sufficient to compare the rough volume of the small ball $B_p(R) \subset M$ with the rough volume of the ball $B(R) \subset C_p$. The corresponding inequality is easily obtained by using the map $B_p(R) \to B(R)$ inverse to the exponential map (see 7.7).

To prove the opposite inequality $\dim \Sigma_p \leqslant \dim M - 1$ it is sufficient to note the following. Let the point $q' \in \Sigma_p'$ have an n-explosion (a_i', b_i'). It may be supposed that a_i', b_i' are arbitrarily near to q' $(a_i', b_i' \in \Sigma_p')$, and let q' be represented by the geodesic pz. Let q be an interior point of the geodesic pz sufficiently near to p. Then q has an $(n+1)$-explosion (a_i, b_i), where a_{n+1}, b_{n+1} are points on the geodesic pz and, when $i \leqslant n$, a_i, b_i are points representing a_i', b_i' on the geodesics, such that $|pa_i| = |pb_i| = |pq|$. ■

7.12. Corollary. *A homogeneous finite-dimensional space with curvature bounded below is a Riemannian manifold.*

In fact, according to a paper of Berestovskii [7], for this it is sufficient to prove that the assertion of Theorem 7.8.1 is satisfied at least at one (and hence at every) point of the space and moreover C_p is isometric to $\mathbb{R}^n$; the last fact is implied immediately by 9.5, 6.7.

7.13. Let X, Y be compact metric spaces. By definition $X \leqslant Y$ if there exists a non-contracting (not necessarily continuous) map $f: X \to Y$, that is, such that $|f(x)f(y)| \geqslant |xy|$. It is easy to see that $X \leqslant Y$ if any finite subset of some everywhere dense subspace $\chi \subset X$ admits a non-contracting map into Y. It can also be verified that if $X \geqslant Y \geqslant X$, then X and Y are isometric. We shall write $\liminf\limits_{i\to\infty} X_i \geqslant X$ if for any Hausdorff subsequence $X_{i_k} \xrightarrow{H} Y$ that converges in the metric we have $Y \geqslant X$. Similarly one can give a meaning to the inequality $\limsup\limits_{i\to\infty} X_i \leqslant X$.

7.14. ***Theorem*** (on the semicontinuity of tangent cones). *If q_i, $p \in M$ are points in a* FDSCBB *and $q_i \to p$, then* $\liminf \Sigma_{q_i} \geqslant \Sigma_p$.

Proof. Let $\{p_j\}$ be a subsequence of $\{q_i\}$ such that $\Sigma_{p_j} \xrightarrow{H} \Sigma$. Let us verify that $\Sigma \geqslant \Sigma_p$. We take a finite ε-net $A_\varepsilon = \{a_1', \dots, a_m'\}$ in Σ_p' so that the geodesic pa_i goes out in the direction a_i'. Let $a_{ji}' \in \Sigma_{p_j}$ be the direction of the geodesic $p_j a_i$. We have (see 2.8.1) $\liminf\limits_{j\to\infty} \measuredangle a_i p_j a_{i'} \geqslant \measuredangle a_i p a_{i'}$, that is, $\liminf\limits_{j\to\infty} |a_{ji}' a_{ji'}'| \geqslant |a_i a_{i'}|$. Some point $b_{ji} \in \Sigma$ will correspond to each point a_{ji}' such that $|a_{ji}' a_{ji'}'| - |b_{ji} b_{ji'}| \to 0$, consequently $\liminf\limits_{j\to\infty} |b_{ji} b_{ji'}| \geqslant |a_i a_{i'}|$.

Thus we can define a non-contracting map $A_\varepsilon \to \Sigma$ by considering one of the limit points of the sequence b_{ji} as the image of the point a_i. ■

7.15. Our next aim is to prove that the tangent cone is continuous when a point moves along the interior of a geodesic. In fact we shall prove, after some preliminary considerations, a somewhat stronger assertion.

Let p be an interior point of the geodesic ab in the FDSCBB M. Then the points a', $b' \in \Sigma_p$ that are images of the geodesics pa and pb respectively are at a maximal possible distance π in the space Σ_p with curvature $\geqslant 1$. Hence it is not difficult to deduce from the angle comparison theorem in Σ_p that Σ_p is the spherical suspension (in the sense of 4.3.1) over a space of one less dimension with curvature $\geqslant 1$, which naturally identifies with

$$\Lambda_p = \left\{\xi \in \Sigma_p : |\xi a'| = \frac{\pi}{2}\right\} = \left\{\xi \in \Sigma_p : |\xi b'| = \frac{\pi}{2}\right\}.$$

7.16. ***Theorem.*** *Let p be an interior point of the geodesic ab in the* FDSCBB *M, and let the sequence $\{p_i\}$ of points in ab converge to p. Then Λ_{p_i} converges to Λ_p in the Hausdorff metric.*

Proof. In order to show that $\liminf\limits_{i\to\infty} \Lambda_{p_i} \geqslant \Lambda_p$ it is sufficient to repeat with small alterations the proof of Theorem 7.14, taking into account that if the point $x \in M$ is near to p and $(|px| \leqslant \min\{|pa|, |pb|\})$, then for sufficiently large i the angles $\measuredangle xpa$ and $\measuredangle xp_i a$ are near.

To prove the opposite inequality lim sup $\Lambda_{p_i} \leqslant \Lambda_p$ we consider the following construction. Let the points r, p, q be situated on the geodesic ab so that the order is: a, r, p, q, b. Let the points r_1, r_2 be near to r, and let the points p_1, p_2 lie on the geodesics qr_1, qr_2 so that $\frac{|qp_1|}{|qr_1|} = \frac{|qp_2|}{qr_2|} = \frac{|qp|}{|qr|}$. We assume that the angles $\measuredangle r_1rp$, $\measuredangle r_2rp$ are near to $\pi/2$ and $|r_1r| = |r_2r| \ll |pr| \ll |pq| \leqslant \min\{|pa|, ||pb|\}$. Then, as we shall show below, the angles $\measuredangle p_1pq$, $\measuredangle p_2pq$ are near to $\pi/2$, while $|pp_1|$ and $|pp_2|$ are near to $|rr_1|\cdot\frac{|qp|}{|qr|}$. Therefore, since $|p_1p_2| \geqslant |r_1r_2|\cdot\frac{|qp|}{|qr|}$ by condition (B), then $\measuredangle p_1pp_2$ cannot be substantially less than $\measuredangle r_1rr_2$. Thus the construction described, taking p_i as r, enables us to establish, using standard arguments, that $\limsup\limits_{i\to\infty} \Lambda_{p_i} \leqslant \Lambda_p$ and thus to conclude the proof of the theorem. All that remains is for us to verify the assertions stated here without proof.

7.17. ***Lemma.*** *Let the points a, r, p, q, b, r_1, p_1 be as in the construction described above. Then there exists $\delta > 0$ (not depending on the given points) such that if $|rr_1| < \delta^2|pr|$, $|\measuredangle r_1rp-\pi/2| < \delta$, $|qr| < \delta \min\{|ar|, |bq|\}$, then*

a) $\measuredangle r_1qr < (1 + 2\delta)\dfrac{|rr_1|}{|qr|}$,

b) $\tilde{\measuredangle} r_1rq > \dfrac{\pi}{2} - 2\delta$,

c) $\left|\dfrac{|pp_1|}{|rr_1|}\cdot\dfrac{|qr|}{|qp|} - 1\right| < 1 + 5\delta$,

d) $\left|\measuredangle p_1pq - \dfrac{\pi}{2}\right| < 3\delta$.

Proof. By applying condition (C) and the cosine formula we get

$$|ar_1| \leqslant (|ar|^2 + |rr_1|^2 + 2|ar||rr_1|\cos\measuredangle r_1rp)^{1/2} \leqslant \leqslant |ar| + |rr_1|\cos\measuredangle r_1rp + \frac{|rr_1|^2}{2|ar|},$$

$$|br_1| \leqslant (|qr_1|^2 + |qb|^2 + 2|qr_1||qb|\cos\measuredangle r_1qr)^{1/2} \leqslant \leqslant |qr_1| + |qb| - \frac{|qr_1||qb|}{|qr_1|+|qb|}(1 - \cos\measuredangle r_1qr),$$

$$|qr_1| \leqslant (|qr|^2 + |rr_1|^2 - 2|qr||rr_1|\cos\measuredangle r_1rp)^{1/2} \leqslant \leqslant |qr| - |rr_1|\cos\measuredangle r_1rp + \frac{|rr_1|^2}{2|qr|}.$$

In addition

$$|ar| + |qr| + |qb| = |ab| < |ar_1| + |br_1|.$$

By adding these four inequalities we get

$$\frac{|qr_1||qb|}{|qr_1|+|qb|}(1-\cos \measuredangle\, r_1qr) < \frac{1}{2}|rr_1|^2\left(\frac{1}{|ar|}+\frac{1}{|qr|}\right)$$

or

$$1-\cos \measuredangle\, r_1qr < \frac{|rr_1|^2}{2|qr|^2}\cdot\left(1+\frac{|qr|}{|ar|}\right)\cdot\frac{|qr|}{|qr_1|}\cdot\left(1+\frac{|qr_1|}{|qb|}\right) <$$
$$< (1+3\delta)\frac{|rr_1|^2}{2|qr|^2},$$

hence $\measuredangle\, r_1qr < (1+2\delta)\dfrac{|rr_1|}{|qr|}$.

To prove (b) we note that if $\widetilde{\measuredangle} r_1rq \leqslant \dfrac{\pi}{2}-2\delta$, then

$$|qr_1| = (|rq|^2+|rr_1|^2-2|rq||rr_1|\cos\widetilde{\measuredangle}\, r_1rq)^{1/2} < |rq| - \frac{3}{2}\delta|rr_1|.$$

On the other hand, by condition (C), taking into account that $\measuredangle arr_1 < \dfrac{\pi}{2}+\delta$, we get

$$|ar_1| \leqslant (|ar|^2+|rr_1|^2-2|ar||rr_1|\cos \measuredangle\, arr_1)^{1/2} < |ar| + \frac{3}{2}\delta|rr_1|.$$

By combining these two inequalities with the triangle inequality $|ar|+|rq| = |aq| < |ar_1|+|qr_1|$ we get a contradiction.

Assertion c) follows from assertion b), conditions (B), (C), and assertion a):

$$(1-2\delta)\frac{|rr_1|}{|qr|} < \widetilde{\measuredangle}\, rqr_1 \leqslant \widetilde{\measuredangle}\, pqp_1 \leqslant \measuredangle\, r_1qr < (1+2\delta)\frac{|rr_1|}{|qr|}.$$

We note that c) and the conditions of the lemma imply that $|pp_1| < 2\delta^2|pq|$. The lower bound $\measuredangle p_1pq \geqslant \widetilde{\measuredangle} p_1pq > \dfrac{\pi}{2}-3\delta$ follows in an elementary way from b) and c). To conclude the proof of the lemma it remains for us to verify that $\measuredangle rpp_1 > \dfrac{\pi}{2}-3\delta$. Let us assume that this is not so. Then

$$|p_1r| \leqslant (|pr|^2+|pp_1|^2-2|pr||pp_1|\cos\measuredangle\, rpp_1)^{1/2} < |pr|-2\delta|pp_1|.$$

On the other hand, condition (A) implies that $|p_1r| \geqslant |\widetilde{p}_1\widetilde{r}|$, where $\widetilde{p}_1$ is a point on the side $\widetilde{q}\,\widetilde{r}_1$ of the triangle $\widetilde{\Delta}\, qrr_1$, dividing it in the same ratio as the point p_1 divides the geodesic qr. Thus

$$|p_1r| \geqslant \left(|pr|^2+\frac{(|rr_1||pq|)^2}{|qr|^2}+2|pr|\frac{|rr_1||pq|}{|rq|}\cos\widetilde{\measuredangle}\, qrr_1\right)^{1/2} >$$
$$> |pr| - \delta\frac{|rr_1||pq|}{|qr|}.$$

These inequalities imply that $2|pp_1| < \dfrac{|rr_1||pq|}{|qr|}$, which contradicts c). ■

7.18. *Remark.* A stronger assertion holds for a convex hypersurface in $\mathbb{R}^n$ and this follows immediately from the results of Milka [25]: the tangent cones at the interior points of geodesics on a convex hypersurface are mutually isometric (but of course not necessary congruent as subsets of $\mathbb{R}^n$). We do not know whether an analogous assertion is true for a FDSCBB (see p. 22).

7.19. We shall separate the points of a FDSCBB into two classes: interior points and boundary points. The boundary of a FDSCBB is defined by induction in the following way.

A one-dimensional SCBB M is a manifold and the boundary of M is by definition the boundary of this manifold. Now let M be an n-dimensional SCBB, where $n > 1$. The point $p \in M$ is considered to be a *boundary point* if the space of directions Σ_p is an $(n-1)$-dimensional SCBB with non-empty boundary. If however Σ_p does not have a boundary, then p is called an *interior* point of M.

We note that all (n, δ)-burst points of M for sufficiently small $\delta > 0$ are interior points (see 9.6). With regard to the boundary see also §12 and 13.3.

Conventions and notation

In what follows we shall, without specifically saying so, use the following notation.

1. If A is a subset of the FDSCBB M and the point $p \in M$ does not belong to A, then $A' \subset \Sigma_p$, as a rule, denotes the set of directions of all possible geodesics pA, that is, geodesics pa such that $a \in A$ and $|pa| = |pA|$. Moreover, within the limits of a single argument, A' may denote subsets of the spaces of directions at various points, say $A' \subset \Sigma_p$ and $A' \subset \Sigma_q$; we remove ambiguity by indicating in every case the point at which we are dealing with the space of directions. In certain especially stipulated cases $A' \in \Sigma_p$ may denote the direction of some single geodesic pA.

2. We let c denote positive "constants". This notation ignores whether such "constants" depend on the dimension of the FDSCBB (and also on the lower bound of the curvature). The "constants" depending on certain parameters a, Π, ξ are denoted by $c(a, \Pi, \xi)$. If we wish to emphasize that the given "constant" does not depend on the parameter a (but may depend on the other parameters), we write $c[\widehat{a}]$.

3. We let $\varkappa$ denote positive functions that are infinitesimally small at zero. Moreover the functions $\varkappa$ may also depend on parameters (which are not necessarily present in the notation), but for any fixed values of parameters the function $\varkappa$ is infinitesimally small in all of its variables. For example, the function $\varkappa[\widehat{a}](\delta, R)$ may depend on certain additional parameters but not on a, and for any fixed values of parameters $\varkappa[\widehat{a}](\delta, R) \to 0$ as $\delta, R \to 0$. As a rule the "constants" c and the functions $\varkappa$ could be explicitly indicated.

Moreover the variables of $\varkappa$ are assumed to be so small that for all $\varkappa$ and c taking part in the argument that have to be compared, we have $\varkappa < c$.

4. We let δ and R (possibly with indices) denote positive real numbers which should be considered to be arbitrarily small. Here, as a rule, the "ceiling" of admissible values of δ could be explicitly indicated, depending on the dimension and the "degree of degeneracy" of the FDSCBB under consideration (see below 8.7 and 12.1), whereas the "ceiling" of admissible values of R depends on random circumstances.

§8. Estimates of rough volume and the compactness theorem

In this section we gather together several assertions which concern estimates of the rough volume of a FDSCBB. These estimates are qualitative in character and do not pretend to be exact. The theorem that the class of FDSCBB with bounded diameter is compact turns out to be a simple consequence of an estimate of the rough volume.

8.1. We introduce a function $\psi(k, D)$ defined as follows: $\psi(k, D) = \max\{|pr|(\measuredangle pqr)^{-1}$ for triangles Δpqr on the k-plane such that $|pq| \leqslant D$, $|qr| \leqslant D$, $|pr| \geqslant 2||pq|-|qr||\}$.

It is easily seen that $\psi(k, D)$ is a finite positive number, $\psi(k, D) \underset{D\to 0}{=} O(D)$ for any fixed k, and for fixed $k > 0$ the function $\psi(k, D)$ is bounded. We shall often omit the variable k in the function ψ.

8.2. ***Lemma.*** *Let $p \in M$ be a point in a complete n-dimensional* SCBB; *suppose that $A \subset M$, $\Gamma \subset \Sigma_p$ are such that for every point $a \in A$ other than p there is a geodesic pa such that $a' \in \Gamma$. Then*

$$Vr_n(A) \leqslant Vr_{n-1}(\Gamma)\cdot 2D_1\psi^{n-1}(D),$$

where $D = \operatorname{diam}(A \cup \{p\})$, $D_1 = \max_{a\in A}|ap| - \min_{a\in A}|ap|$.

Proof. If we take $\beta_A(\varepsilon)$ points in A whose distance from each other is at least ε, there are at least $\beta_A(\varepsilon)\cdot\left(\frac{2D_1}{\varepsilon}+1\right)^{-1}$ of them such that their distances to p differ pairwise by not more than $\varepsilon/2$. Thus by the angle comparison theorem we get $\beta_A(\varepsilon)\cdot\left(\frac{2D_1}{\varepsilon}+1\right)^{-1}$ points in Γ at a pairwise distance of at least $\varepsilon/\psi(D)$. We obtain the inequality $\beta_\Gamma\left(\frac{\varepsilon}{\psi(D)}\right) \geqslant \beta_A(\varepsilon)\left(\frac{2D_1}{\varepsilon}+1\right)^{-1}$ from which we get the assertion of the lemma if we let ε tend to 0. ■

8.3. ***Corollary.*** *For complete n-dimensional spaces with curvature $\geqslant 1$ we have the bound $Vr_n(M) \leqslant c(n)$. In addition $\beta_M(\varepsilon) \leqslant c(n)\varepsilon^{-n}$ for all $\varepsilon > 0$.*

The proof is by an obvious induction taking into account that diam $M \leqslant \pi$.

8.4. ***Corollary.*** *For complete n-dimensional spaces with curvature $\geqslant k$ and with* diam $M \leqslant D$ *we have the bound* $Vr_n(M) \leqslant c(n, k, D)$. *In addition,* $\beta_M(\varepsilon) \leqslant c(n, k, D)\varepsilon^{-n}$ *for all* $\varepsilon > 0$.

8.5. Let $\mathfrak{M}(n, k, D)$ denote the metric space (with Hausdorff metric) consisting of complete spaces with curvature not less than k, dimension not greater than n, and diameter not greater than D. We consider $\mathfrak{M}(n, k, D)$ as a subspace (which is complete in the Hausdorff metric) of the space of all compact metric spaces.

Theorem. *The space* $\mathfrak{M} = \mathfrak{M}(n, k, D)$ *is compact.*

Proof. Let us check first that $\mathfrak{M}$ contains its limit points. In fact, if $M_i \in \mathfrak{M}$, where M is a compact metric space and $M_i \to M$ in the sense of Hausdorff, then obviously diam $M \leqslant D$ and by [19], Proposition 3.8 the metric of M is intrinsic. The fact that this metric has curvature not less than k follows trivially from the definitions. Finally, the condition dim $M \leqslant n$ is satisfied, since the presence in M of a (greater than n)-burst point would imply the presence of such points in M_i. We now check that $\mathfrak{M}$ is precompact. It was shown in [19] (see also [20]) that for this it is sufficient to show that for any $\varepsilon > 0$ there is a natural number $N(\varepsilon)$ such that any space $M \in \mathfrak{M}$ has an ε-net of $N(\varepsilon)$ points. But the existence of such an $N(\varepsilon)$ follows directly from Corollary 8.4. ■

The idea of the proof of the next lemma is due to Cheeger [10].

8.6. ***Lemma.*** *Let* $p_1, \ldots, p_m$ *be points in a compact n-dimensional* SCBB. *We let* p_{ij}' *denote the set of directions of the geodesics* p_ip_j *in* Σ_{p_j}. *We assume that for each number* i, $1 \leqslant i \leqslant m$, *there are directions* $\xi_i^- \in p_{i-1,i}'$ *and* $\xi_i^+ \in p_{i+1,i}'$ *such that* $|\xi_i^-, \xi_i^+| > \pi - \delta$, *where* $i+1 = 1$ *for* $i = m$, $i-1 = m$ *for* $i = 1$. (We shall say that such points p_i almost generate a closed geodesic.) *Let* $D =$ diam M, $\delta_1 = D^{-1} \max|p_ip_{i+1}|$. *Then* $Vr_n(M) \leqslant \varkappa(\delta, \delta_1)D\psi^{n-1}(D)$, *where the function* $\varkappa$ *depends on* m *as a parameter.*

Proof. Let $r \in M$ be such that $|p_ir| > \delta_1^{-1/2} \max\limits_j |p_jp_{j+1}|$ for all i. Then

$$\tilde{\measuredangle} rp_{i+1}p_i + \tilde{\measuredangle} rp_ip_{i+1} > \pi - \varkappa(\delta_1). \tag{1}$$

On the other hand, for the direction $r_i' \in \Sigma_{p_i}$ of any geodesic p_ir we have

$$|r_i'\xi_i^-| + |r_i'\xi_i^+| < \pi + \delta, \tag{2}$$

whence

$$\measuredangle rp_ip_{i-1} + \measuredangle rp_ip_{i+1} < \pi + \delta. \tag{3}$$

If we combine the inequalities (1) and (3) for all i, we may conclude that all the inequalities (1)–(3) are nearly equalities, in particular $|\tilde{\measuredangle} rp_ip_{i-1} + \tilde{\measuredangle} rp_ip_{i+1} - \pi| < \varkappa(\delta, \delta_1)$ and $|\tilde{\measuredangle} rp_ip_{i+1} + \tilde{\measuredangle} rp_{i+1}p_i - \pi| < \varkappa(\delta, \delta_1)$, whence

$|\tilde{\measuredangle} rp_ip_{i-1}+\tilde{\measuredangle} rp_{i+1}p_i| < \varkappa(\delta, \delta_1)$. If i_0 is that number for which $|p_ir|$ is minimal, then $\tilde{\measuredangle} rp_{i_0}p_{i_0+1} > \frac{\pi}{2}-\varkappa(\delta_1)$ and $\tilde{\measuredangle} rp_{i_0}p_{i_0-1} > \frac{\pi}{2}-\varkappa(\delta_1)$. Therefore for any i we get $\left|\measuredangle rp_ip_{i+1}-\frac{\pi}{2}\right| < \varkappa(\delta, \delta_1)$. If we take into account that the inequalities (2) are nearly equalities, we can assert, in particular that $\left||\xi_1^+r_1'|-\frac{\pi}{2}\right| < \varkappa(\delta, \delta_1)$. If we apply Lemma 8.2 and Corollary 8.3 we get $Vr_{n-1}\left(\left\{r_1' \in \Sigma_{p_1} : \left||\xi_1^+r_1'| - \frac{\pi}{2}\right| < \varkappa(\delta, \delta_1)\right\}\right) < \varkappa(\delta, \delta_1)$, and again by Lemma 8.2 $Vr_n(\{r \in M : |p_ir| > (\delta_1^{-1/2} \max|p_jp_{j-1}|$ for all $i\}) < \varkappa(\delta, \delta_1)D\psi^{n-1}(D)$. To finish the proof it is sufficient to note that by Lemma 8.2

$$Vr_n(\{r\in M: |p_1r|<(\delta_1^{-1/2}+m)\max_j|p_jp_{j-1}|\})<\varkappa(\delta_1)D\psi^{n-1}(D). \blacksquare$$

8.7. *Remark.* $Vr_n(M)(\operatorname{diam} M\cdot\psi^{n-1}(\operatorname{diam} M))^{-1}$ can serve as a non-degeneracy characteristic of the dimension for an n-dimensional compact SCBB M. The results of this section imply that the existence of a (positive) lower bound for this ensures a uniform lower bound for $Vr_{n-1}(\Sigma_p)$ over points $p \in M$ and also a lower bound for the "lengths of almost closed geodesics". In the case of non-compact FDSCBB there are similar arguments which show that rough volumes $Vr_{n-1}(\Sigma_p)$ and the "lengths of almost closed geodesics" are uniformly separated from zero on compact subsets.

§9. Theorems on almost isometry

9.1. We shall say that a complete finite-dimensional space M with curvature $\geqslant 1$ has an (m, δ)-explosion (A_i, B_i), $1 \leqslant i \leqslant m$, if $A_i, B_i \subset M$ are compact subsets such that $|A_i, B_i| > \pi-\delta$, $|A_i, B_j| > \frac{\pi}{2}-\delta$, $|A_i, A_j| > \frac{\pi}{2}-\delta$, $|B_i, B_j| > \frac{\pi}{2}-\delta$ when $i \neq j$. Clearly a point p in a FDSCBB has an (m, δ)-explosion if and only if its space of directions Σ_p has an (m, δ)-explosion in the sense of the definition given above. We recall that for sufficiently small $\delta > 0$ an (n, δ)-burst point in an n-dimensional space with curvature bounded below has a neighbourhood that admits a bi-Lipschitz homeomorphism onto a ball in $\mathbb{R}^n$. In this section we show that this homeomorphism is almost isometric for small $\delta > 0$ (that is, the distortion coefficient of the distance is near to 1). Correspondingly an $(n-1)$-dimensional space with curvature $\geqslant 1$ that has an (n, δ)-explosion turns out to be an almost isometric unit sphere S^{n-1} (cf. Otsu, Shiohama, and Yamaguchi [27]).

9.2. The following assertions, which are based directly on the angle comparison theorem and some elementary spherical geometry, are used constantly in what follows.

Lemma. Let M be a complete finite-dimensional space with curvature $\geqslant 1$.

a) *Let the sets A_i, B_i form an (m, δ)-explosion in M; $p \in M$, $|pA_i| > \frac{\pi}{2} - \delta$, $|pB_i| > \frac{\pi}{2} - \delta$ for all i. Then the sets A_i', B_i' form an $(m, \varkappa(\delta))$-explosion in Σ_p.*

b) *Let A, B form a $(1, \delta)$-explosion in M; $p, q \in M$, $|pA| > \frac{\pi}{2} - \delta$, $|pB| > \frac{\pi}{2} - \delta$. Let the points $\widetilde{A}, \widetilde{B}, \widetilde{p}, \widetilde{q}$ be given on the unit sphere S^2 such that $|\widetilde{A}\widetilde{B}| = \pi$, $|\widetilde{A}\widetilde{p}| = |\widetilde{B}\widetilde{p}| = \frac{\pi}{2}$, $||\widetilde{p}\widetilde{q}| - |pq|| < \delta$, and also*

$$(1) \qquad |\measuredangle \widetilde{A}\widetilde{p}\widetilde{q}| - |A'q'|| < \delta, \quad |\measuredangle \widetilde{B}\widetilde{p}\widetilde{q} - |B'q'|| < \delta,$$

where $A', B', q' \subset \Sigma_p$, and q' is the direction of one of the geodesics pq. Then $||Aq| - |\widetilde{A}\widetilde{q}|| < \varkappa(\delta)$ and $||Bq| - |\widetilde{B}\widetilde{q}|| < \varkappa(\delta)$.

c) *Let conditions* b) *be all satisfied except for* (1) *and assume instead that $||Aq| - |\widetilde{A}\widetilde{q}|| < \delta$ and $||Bq| - |\widetilde{B}\widetilde{q}|| < \delta$. Then either for the direction $q' \in \Sigma_p$ of any geodesic pq we have $|\measuredangle \widetilde{A}\widetilde{p}\widetilde{q} - |A'q'|| < \varkappa(\delta)$ and $|\measuredangle \widetilde{B}\widetilde{p}\widetilde{q} - |B'q'|| < \varkappa(\delta)$, or $|pq| < \varkappa(\delta)$, or $|pq| > \pi - \varkappa(\delta)$.*

d) *Let conditions* c) *be satisfied and let there be given points r on the geodesic pq and $\widetilde{r}$ on $\widetilde{p}\widetilde{q}$ so that $||pr| - |\widetilde{p}\widetilde{r}|| < \delta$. Then either $||Ar| - |\widetilde{A}\widetilde{r}|| < \varkappa(\delta)$ and $||Br| - |\widetilde{B}\widetilde{r}|| < \varkappa(\delta)$, or $|Ap| + |Aq| + |pq| > 2\pi - \varkappa(\delta)$, or $|Bp| + |Bq| + |pq| > 2\pi - \varkappa(\delta)$.*

The proofs of these assertions are not complicated and are left to the reader.

9.3. *Lemma. Let a complete $(n-1)$-dimensional space M with curvature $\geqslant 1$ have an (n, δ)-explosion (A_i, B_i). Then for any point $q \in M$ we have*

$$\left|\sum_{i=1}^{n} \cos^2 |A_i q| - 1\right| < \varkappa(\delta).$$

Proof. We use induction with respect to the dimension. If $n = 2$, the assertion is obvious. To prove the induction step we chose an arbitrary point $p \in A_n$ and construct the points $\widetilde{A}_i$, $\widetilde{B}_i$ $(1 \leqslant i \leqslant 1)$, $\widetilde{p}$, $\widetilde{q}$ on the unit sphere S^{n-1} so that $(\widetilde{A}_i, \widetilde{B}_i)$ form an $(n-1, 0)$-explosion, where $|\widetilde{p}\widetilde{A}_i| = |\widetilde{p}\widetilde{B}_i| = \frac{\pi}{2}$ for all i, $|\widetilde{p}\widetilde{q}| = |pq|$, and also $|\measuredangle \widetilde{A}_i\widetilde{p}\widetilde{q} - |A_i'q'|| < \varkappa(\delta)$; here $A_i', q' \subset \Sigma_p$, and q' is the direction of one of the geodesics pq. The last requirement can be satisfied since by the inductive hypothesis $\left|\sum_{i=1}^{n-1} \cos^2 |A_i'q'| - 1\right| < \varkappa(\delta)$, because by Lemma 9.2 a) the sets (A_i', B_i') form an $(n-1, \varkappa(\delta))$-explosion in Σ_p. Since $||A_i'q'| + |B_i'q'| - \pi| < \varkappa(\delta)$, and $\measuredangle \widetilde{A}_i\widetilde{p}\widetilde{q} + \measuredangle \widetilde{B}_i\widetilde{p}\widetilde{q} = \pi$, we get $|\measuredangle \widetilde{B}_i\widetilde{p}\widetilde{q} - |B_i'q'|| < \varkappa(\delta)$. Now to prove the induction step it is sufficient to apply Lemma 9.2 b) and suppose that

$$\sum_{i=1}^{n-1} \cos^2 |\tilde{A}_i \tilde{q}| = 1 - \cos^2 |\tilde{p}\tilde{q}|. \ \blacksquare$$

9.4. ***Theorem.*** *Let M be an n-dimensional* SCBB *and let the point $p \in M$ have an (n, δ)-explosion (a_i, b_i). Then the map $f : M \to \mathbb{R}^n$ given by $f(q) = (|a_1 q|, \ldots, |a_n q|)$ maps a small neighbourhood U of the point p almost isometrically onto a domain in $\mathbb{R}^n$.*

The condition that f is almost isometric means that $||f(q)f(r)| - |qr|| < \varkappa(\delta, \delta_1)|qr|$ for any points $q, r \in U$, where

$$\delta_1 = \max_{1 \leq i \leq n} \{|pa_i|^{-1} \operatorname{diam} U, \quad |pb_i|^{-1} \operatorname{diam} U\}.$$

Proof. In view of Theorem 5.4 we need verify only the condition for an almost isometry. Let $q, r \in U$. By Lemma 5.6 we have $|\tilde{\measuredangle} a_i qr - \measuredangle a_i qr| < \varkappa(\delta, \delta_1)$ for any geodesics qr, qa_i and consequently

$$\left| \frac{|a_i q| - |a_i r|}{|qr|} - \cos \measuredangle a_i qr \right| < \varkappa(\delta, \delta_1). \tag{2}$$

We can apply Lemma 9.3 to the direction $r' \in \Sigma_q$ of the chosen geodesic qr if we suppose that the sets $a_i', b_i' \subset \Sigma_q$ form an $(n, \varkappa(\delta, \delta_1))$-explosion in Σ_q. Thus $\left|\sum_{i=1}^{n} \cos^2 \measuredangle a_i qr - 1\right| < \varkappa(\delta, \delta_1)$ and hence $\left|\sum_{i=1}^{n} \frac{(|a_i q| - |a_i r|)^2}{|qr|^2} - 1\right| < \varkappa(\delta, \delta_1)$, which is what was required. $\blacksquare$

9.5. ***Theorem.*** *Let M be a complete $(n-1)$-dimensional space with curvature ≥ 1 which has an (n, δ)-explosion (A_i, B_i). Then, for small $\delta > 0$, M is almost isometric to the unit sphere S^{n-1}, that is, there exists a homeomorphism $\tilde{f} : M \to S^{n-1}$ such that $||\tilde{f}(r)\tilde{f}(q)| - |rq|| < \varkappa(\delta)|rq|$ is satisfied for any $q, r \in M$.*

Proof. We consider the cone $C(M)$ with vertex p and the points $a_i, b_i \in C(M)$ such that the directions of the rays pa_i, pb_i lie in A_i and B_i respectively. Clearly (a_i, b_i) form an (n, δ)-explosion at the point p. We can choose a_i, b_i so far from p that for the neighbourhood $U = \{q \in C(M) : |pq| < 2\}$ the assertion of the preceding theorem is satisfied with some function $\varkappa(\delta)$ instead of $\varkappa(\delta, \delta_1)$. Consider the map $\tilde{f} : M \to S^{n-1}$, which is a composite of the map f constructed in the previous theorem and the central projection onto the unit sphere with centre $f(p)$. (We suppose, as usual, that M is embedded in $C(M)$ according to the rule $x \to (x, 1)$.) We show that $\tilde{f}$ is almost isometric. Since f is almost isometric, we have $||f(q)f(p)| - 1| < \varkappa(\delta)$ for any point $q \in M$ and consequently the absolute distortion of the distances with $\tilde{f}$ is small, that is, $||\tilde{f}(q)\tilde{f}(r)| - |qr|| < \varkappa(\delta)$ for any $q, r \in M$. It remains to verify the condition of almost isometry for near points q, r. For this it is sufficient to prove that if $|qr| < \varkappa(\delta)$, then $\left|\measuredangle f(p)f(q)f(r)-\frac{\pi}{2}\right| < \varkappa(\delta)$. Since in our case, as in the previous theorem, the inequalities (2) are satisfied (with $\varkappa(\delta)$ instead of $\varkappa(\delta, \delta_1)$) and the analogous inequalities with p instead of r, it is sufficient to verify that $\left|\sum_{i=1}^{n} \cos \measuredangle a_i qr \cdot \cos \measuredangle a_i qp\right| < \varkappa(\delta)$. This inequality is implied by the following assertion. If we are given an $(n, \varkappa(\delta))$-explosion (a_i', b_i') in a complete $(n-1)$-dimensional space Σ_q with curvature $\geqslant 1$ and the points p', r' are such that $\left||p'r'|-\frac{\pi}{2}\right| < \varkappa(\delta)$, then $\left|\sum_{i=1}^{n}\cos|a_i'r'|\cdot\cos|a_i'p'|\right| < \varkappa(\delta)$. The last assertion is implied by the fact that Σ_q, like M, admits a map into the unit sphere S^{n-1} with small absolute distortion of distances. ■

9.6. *Corollary.* (a) *A complete $(n-1)$-dimensional space with curvature $\geqslant 1$ which has an (n, δ)-explosion with $\delta > 0$ sufficiently small is a space without boundary.*

(b) *For sufficiently small $\delta > 0$ the (n, δ)-burst points in an n-dimensional* SCBB *are interior points.*

Proof. Assertion (a) is easily proved by induction with respect to dimension taking into account Theorem 9.5 since, in a space which is almost isometric to the standard sphere of the same dimension, any point has a maximum possible explosion for the given dimension and consequently the space of directions also has a maximal explosion at any point. Assertion (b) follows immediately from (a). ■

9.7. We say that the (m, δ)-explosion (a_i, b_i) at the point p is R-long if $|a_i p| > \delta^{-1}R$, $|b_i p| > \delta^{-1}R$ for all i. The set of points of the space M that have R-long (m, δ)-explosion is denoted by $M(m, \delta, R)$.

The rest of this section is devoted to proving the following theorem.

9.8. *Theorem.* *Let M, M_1 be compact n-dimensional* SCBB *with the same lower bound for the curvature. Suppose we are given an ν-approximation $h : M_1 \to M$. Then, if the numbers δ, $\nu/\delta^3 R$ are sufficiently small, there exists a map $\bar{h} : M_1(n, \delta, R) \to M$ that is (uniformly) $C\nu$-near to h and such that*

$$\left|1-\frac{|\bar{h}(x)\bar{h}(y)|}{|xy|}\right| < \varkappa(\delta, \nu/\delta^3 R),$$

whenever $|xy| \leqslant \delta^3 R$.

We recall that a ν-approximation was defined in 7.8.

9.9. *Corollary.* *If, under the conditions of the theorem, each point of the space M is an (n, δ)-burst point, then there exists a $\varkappa(\delta, \nu)$-almost isometry $\bar{h} : M_1 \to M$ which is (uniformly) $C\nu$-near to h.*

To prove the corollary it is sufficient to note that in this case $M = MN(n, \delta, R)$ for some $R > 0$, and $M_1 = M_1(n, 10\delta, R/10)$.

The map $\bar{h}$ is glued together from almost isometries on balls of radius δR using a partition of unity and the construction for the centre of gravity described below (see [21]). After studying the properties of the centre of gravity we give for completeness a formal proof of Theorem 9.8.

9.10. Centre of gravity.
Let the point p in an n-dimensional SCBB M have an (n, δ)-explosion a_i, b_i and let $f: M \to \mathbb{R}^n$ be the corresponding explosion map (see 5.5). We fix a small neighbourhood U of the point p such that $f(U)$ is convex in $\mathbb{R}^n$. Then for any finite collection of points $X = (x_1, \ldots, x_l)$, $x_j \in U$, and collection of weights $W = (w_1, \ldots, w_l)$, $w_j \geqslant 0$, $\sum_{j=1}^{l} w_j = 1$, the centre of gravity is defined to be $X_W = f^{-1}\left(\sum_{j=1}^{l} w_j f(x_j)\right)$.

9.11. ***Lemma.*** *In addition to the assumptions of* 9.10 *let the point p have another (n, δ)-explosion (s_i, t_i) and let*

$$\delta_1 = \max\left\{\frac{\operatorname{diam} U}{\min\limits_i \{|pa_i|, |pb_i|\}}, \frac{\max\limits_i \{|pa_i|, |pb_i|\}}{\min\limits_i \{|ps_i|, |pt_i|\}}\right\}$$

be small. We assume that the collections of points in U, $Q = (q, \ldots, q_l)$ and $R = (r_1, \ldots, r_l)$, satisfy

$$\max_j \{|q_j r_j|\} \leqslant (1+\delta) \min_j \{|q_j r_j|\},$$

and for every i, $1 \leqslant i \leqslant n$,

$$|\max_j \tilde{\measuredangle}\, s_i q_j r_j - \min_j \tilde{\measuredangle}\, s_i q_j r_j| \leqslant \delta.$$

Then for any weights W^1 and W^2 such that $\| W^1 - W^2 \| < \delta_1$ the centres of gravity Q_{W^1} and R_{W^2} (with respect to the explosion (a_i, b_i)) satisfy

$$\left|1 - \frac{|q_j r_j|}{|Q_{W^1} R_{W^2}|}\right| < \varkappa(\delta, \delta_1), \quad |\tilde{\measuredangle}\, s_i q_j r_j - \tilde{\measuredangle}\, s_i Q_{W^1} R_{W^2}| < \varkappa(\delta, \delta_1)$$

for all $j = 1, \ldots, l$, $i = 1, \ldots, n$.

Proof. We note that, for any $q, r \in U$, any point z from the collection $\{a_i, b_i, s_i, t_i\}$, and any geodesics qz, qr we have (by Lemma 5.6) $|\tilde{\measuredangle} zqr - \measuredangle zqr| < \varkappa(\delta, \delta_1)$. In addition, for any x from the collection $\{a_i, b_i\}$ and y from the collection $\{s_i, t_i\}$ we have $|\tilde{\measuredangle} xqy - \tilde{\measuredangle} xry| < \varkappa(\delta, \delta_1)$ and for any geodesics qx, qy $|\tilde{\measuredangle} xqy - \measuredangle xqy| < \varkappa(\delta, \delta_1)$. Hence if we take into account that the Σ_{q_j} are almost isometric to S^{n-1}, we deduce that $|\max\limits_j \tilde{\measuredangle} x q_j r_j - \min\limits_j \tilde{\measuredangle} x q_j r_j| < \varkappa(\delta, \delta_1)$ for any such point x. The last estimate easily implies the assertion of the lemma. ■

9.12. *Proof of Theorem* 9.8. We choose, from the cover of the set $\overline{M}_1 = M_1(n, \delta, R)$ by all possible balls of radius δR with centres in $\overline{M}_1$, a finite subcover $\{B_{x_i}(\delta R)\}$, $1 \leqslant i \leqslant N_1$. We can suppose that balls a third as small as before cover $\overline{M}_1$ and the multiplicity of the cover is bounded by a number N which depends only on n. For each centre x_i we choose an R-long (n, δ)-explosion (s_j^i, t_j^i) and a δR-long (n, δ)-explosion (a_j^i, b_j^i), $1 \leqslant j \leqslant n$, and let f_i denote the explosion map for (a_j^i, b_j^i). Clearly for small $\nu > 0$ the pairs $(h((a_j^i), h(b_j^i))$ form an $(n, 2\delta)$-explosion at the point $h(x_i)$ and we can consider the corresponding explosion map g_i.

The composition $h_i = g_i^{-1} \circ f_i$ on $B_{x_i}(\delta R)$ is $\varkappa(\delta, \nu/R\delta)$-almost isometric, $C\nu$-near to h. We introduce neighbourhoods U_i, $h_i(B_{x_i}(\delta R/2)) \subset U_i \subset \subset h_i(B_{x_i}(\delta R))$ such that the $g_i(U_i)$ are convex in $\mathbb{R}^n$ and we define the functions $\varphi_i : M_1 \to \mathbb{R}$ by

$$\varphi_i(x) = \begin{cases} (1 - 2\,|xx_i|/\delta R)^N, & \text{if} \quad x \in B_{x_i}(\delta R/2), \\ 0, & \text{if} \quad x \in M \setminus B_{x_i}(\delta R/2). \end{cases}$$

For an arbitrary point $z \in \overline{M}_1$ let us define the sequence $\{z_i\}$, $1 \leqslant i \leqslant N_1$, inductively:

$z_1 = h_1(z)$ if $z \in B_{x_1}(\delta R)$, and $h(z)$ otherwise,

z_{i+1} is the centre of gravity (with respect to the explosion map g_{i+1}) of the points z_i and $h_{i+1}(z)$ with weights $1 - \dfrac{\varphi_{i+1}(z)}{\Sigma_{i+1}}$ and $\varphi_{i+1}(z)/\Sigma_{i+1}$ respectively, where Σ_{i+1} denotes $\sum_{j=1}^{i+1} \varphi_j(z)$. (If $\Sigma_{i+1} = 0$, we put $z_{i+1} = h(z)$.) We put $\overline{h}(z) = z_{N_1}$. Obviously $|z_i h(z)| \leqslant C\nu$ for all i, consequently $|\overline{h}(z)h(z)| \leqslant C\nu$. We still have to verify the condition of almost isometry for $\overline{h}$ in the small.

We first estimate the local Lipschitz constants of the functions $\varphi_i(x)/\Sigma_i(x)$. Let $\max\limits_{1 \leqslant j \leqslant i} \varphi_j(x) = \varphi_{j_0}(x)$. Then for y sufficiently near to x we have

$$\left| \frac{\varphi_i(x)}{\Sigma_i(x)} - \frac{\varphi_i(y)}{\Sigma_i(y)} \right| \leqslant C \frac{|xy|}{\delta R/2 - |xx_{j_0}|}. \tag{1}$$

In fact

$$\left| \frac{\varphi_i(x)}{\Sigma_i(x)} - \frac{\varphi_i(y)}{\Sigma_i(y)} \right| = \left| \frac{\sum\limits_{j=1}^{i-1} (\varphi_i(x)(\varphi_j(y) - \varphi_j(x)) - \varphi_j(x)(\varphi_i(y) - \varphi_i(x))}{\Sigma_i(x)\Sigma_i(y)} \right| \leqslant$$

$$\leqslant 2N \frac{|\varphi_{j_0}(x) - \varphi_{j_0}(y)|}{\varphi_{j_0}(x)} \leqslant 3N^2 \frac{|xy|}{\delta R/2 - |xx_{j_0}|}.$$

Let $z, y \in \overline{M}_1$ be such that $|zy| < \delta^3 R$ and let $\nu/R < \delta^3$. We find a number i_0, $1 \leqslant i_0 \leqslant N_1$, such that

$$\varphi_i(z) \leqslant R\delta^2 |yz|^{-1} \Sigma_{i-1}(z) \tag{2}$$

for all $i > i_0$, but

(3) $$\varphi_{i_0}(z) \geqslant R\delta^2 |yz|^{-1} \Sigma_{i_0-1}(z).$$

Since $\max_j \varphi_j(z) \geqslant (1/3)^N$, then (2) implies that

(4) $$\varphi_{i_0}(z) \geqslant C(\delta^{-2} R^{-1} |yz|)^{N-1}$$

and consequently

(5) $$\delta R/2 - |zx_{i_0}| \geqslant C\delta^{-1} |yz|.$$

From the obvious inequality $\varphi_i(y) \leqslant 2^N(\varphi_i(z) + (2|yz|/\delta R)^N)$ we get

$$\Sigma_{i_0-1}(y) \leqslant C \cdot \Sigma_{i_0-1}(z) + C \cdot (|yz|/\delta R)^N \overset{(3)}{\leqslant} C\delta^{-2} |yz| R^{-1} \varphi_{i_0}(z) + C \cdot (|yz|/\delta R)^N \overset{(4)}{\leqslant}$$
$$\leqslant C\delta^{-2} |yz| R^{-1} \varphi_{i_0}(z)(1 + \delta^N) \leqslant C\delta^{-2} |yz| R^{-1} \varphi_{i_0}(z) \leqslant C\delta^{-2} |yz| R^{-1} \varphi_{i_0}(y),$$

because (5) implies immediately that $|1 - \varphi_{i_0}(z)/\varphi_{i_0}(y)| \leqslant C\delta$. Thus we have

$$|1 - \varphi_{i_0}(z)/\Sigma_{i_0}(z)| \leqslant C\delta^{-2} |zy| R^{-1}, \quad |1 - \varphi_{i_0}(y)/\Sigma_{i_0}(y)| \leqslant C\delta^{-2} |zy| R^{-1},$$

and consequently, since $\nu < \delta^3 R$,

(6) $$|h_{i_0}(z) z_{i_0}| < C\delta |yz|, \quad |h_{i_0}(y) y_{i_0}| < C\delta |yz|.$$

In addition, (1) and (5) imply that for $i > i_0$

(7) $$\left| \frac{\varphi_i(z)}{\Sigma_i(z)} - \frac{\varphi_i(y)}{\Sigma_i(y)} \right| < C\delta.$$

Now if we take into account (6), (7), we can prove that $\bar{h}$ is a local almost isometry by using an inductive argument and applying Lemma 9.11 to each operation of taking the centre of gravity, beginning with the one that reduces to the construction of y_{i_0+1}, z_{i_0+1} and taking the pairs $(h(s_j^{i_0}), h(t_j^{i_0}))$ as the "long" explosion. It is only necessary to check that for any point $h(s)$ from the collection $h(s_j^{i_0})$ and any $i \geqslant i_0$ for which $\varphi_i(z) > 0$ or $\varphi_i(y) > 0$ we have

$$|\tilde{\measuredangle}\, h(s) h_i(y) h_i(z) - \tilde{\measuredangle}\, h(s) h_{i_0}(y) h_{i_0}(z)| < \varkappa\left(\delta, \frac{\nu}{\delta R}\right),$$
$$|1 - |h_i(y) h_i(z)| / |h_{i_0}(y) h_{i_0}(z)|| < \varkappa\left(\delta, \frac{\nu}{\delta R}\right).$$

The second inequality follows directly from the almost isometry of h_i and h_{i_0} on the corresponding balls. To prove the first inequality it is sufficient to verify that for $j = i$ or i_0

$$|\tilde{\measuredangle}\, h(s) h_j(y) h_j(z) - \tilde{\measuredangle}\, syz| < \varkappa\left(\delta, \frac{\nu}{\delta R}\right).$$

This estimate is established in the same way as in the proof of Lemma 9.11, since for any point u from the collection $\{a_i^j, b_i^j\}$ we have

$$|\ \tilde{\measuredangle}\, h(s)\, h_j(y)\, h_j(u) - \tilde{\measuredangle}\, syu\,| < \varkappa\left(\delta, \frac{\nu}{\delta R}\right),$$

$$|\ \tilde{\measuredangle}\, h(u)\, h_j(y)\, h_j(z) - \tilde{\measuredangle}\, uyz\,| < \varkappa\left(\delta, \frac{\nu}{\delta R}\right)$$

and for any point v from the collection $\{s, a_i^j, b_i^j\}$ we have

$$|\ \tilde{\measuredangle}\, h(v)\, h_j(y)\, h_j(z) - \measuredangle\, h(v)\, h_j(y)\, h_j(z)\,| < \varkappa(\delta),$$
$$|\ \tilde{\measuredangle}\, vyz - \measuredangle\, vyz\,| < \varkappa(\delta). \quad \blacksquare$$

9.13. *Remark.* If under the conditions 9.9 the dimension of the space M_1 is m, where $m > n$, and if each point of it is an (m, δ)-burst point, then the ν-approximation $h : M_1 \to M$ may be approximated by the continuous "almost metric submersion" $\bar{h}$. (In particular, $\bar{h} : M_1 \to M$ is a locally trivial fibration.) This assertion generalizes one of Yamaguchi's results [32]. For the proof it is sufficient to carry out the argument from 9.12 twice—first with respect to the "base" and then with respect to the "fibre". We conjecture that in fact the condition that every point of the space M_1 is an (m, δ)-burst point could be omitted.

§10. Hausdorff measure

10.1. In this section we show that the Hausdorff measure ("volume") on a FDSCBB preserves many properties of the volume on Riemannian manifolds with curvature bounded below. We begin with the volume comparison theorem.

10.2. ***Theorem.*** *Let M be a complete n-dimensional space with curvature $\geqslant k$. Then for any $p \in M$, $R > 0$, the n-dimensional Hausdorff measure of the ball $B_p(R)$ does not exceed the volume of the ball $\tilde{B}_{\bar{p}}(R)$ of radius R in the n-dimensional complete simply-connected space of constant curvature k. In the case of equality these balls are isometric.*

It is clearly sufficient for the proof (for the moment without the case of equality) to construct an injective non-distorting map

$$f_n\colon B_p(R) \to \tilde{B}_{\bar{p}}(R).$$

For any map $\varphi : X \to Y$ of metric spaces we let $(\varphi, \mathrm{id}) : C(X) \to C(Y)$ denote the map of cones over X, Y in accordance with the formula $(\varphi, \mathrm{id})(x, a) = (\varphi(x), a)$. Now by induction we put

$$f_n = \exp_{\bar{p}} \circ (f_{n-1}, \mathrm{id}) \circ \exp_p^{-1},$$

where $\exp_p^{-1}$ is the map defined in 7.7. Clearly f_n is injective and non-distorting. It is not difficult to check that in the case when the volumes are equal the map f_n is isometric. ■

We now wish to estimate the Hausdorff dimension of the set of singular points of M. For this we need two technical lemmas. As a preliminary we stipulate that in the course of this section M denotes a compact n-dimensional SCBB with diameter 1, $N_0(\delta) = \max_X \beta_X(\delta)+1$, where the maximum is taken over all $(n-1)$-dimensional complete spaces with curvature $\geqslant 1$. Since all our arguments are essentially local, we consider the lower bound of the curvature k to be zero.

10.3. *Lemma.* *For any natural number l there exists a number $N_1(l)$ such that from any $N_1(l)$ points in M one can choose $l+1$ points $x_0, \ldots, x_l$ so that $|x_0x_i| > 2|x_0x_{i-1}|$, $2 \leqslant i \leqslant l$.*

Proof. We carry out induction with respect to l. The base $l = 1$ is trivial. Suppose we have $N_1(l+1) = 100lN_1(l)N_0(1/10)$ points in M, and let x be one of them. If the desired collection $x_0, x_1, \ldots, x_{l+1}$ with $x_0 = x$ does not exist, then all the points except x may be split into $100l$ classes such that for points y, z in the same class we have $0.9 \leqslant |xy|/|xz| \leqslant 1.1$. We choose that class in which there are $\geqslant N_1(l)\cdot N_0(1/10)$ points and find in it a subset G from among the $\geqslant N_1(l)$ elements such that for each point $y \in G$ the direction $y' \in \Sigma_x$ of at least one geodesic xy lies in a fixed ball $B_\xi(1/10)$. Thus for any points $y, z \in G$ we have $|yz| < 1/2 \min\{|xy|, |xz|\}$, and consequently if we choose from G by the inductive hypothesis a collection $x_0, \ldots, x_l$, we can add to it the point $x_{l+1} = x$ to obtain the desired collection. ■

10.4. We recall that the (m, δ)-explosion (a_i, b_i) at the point p is said to be R-long if $\min\{|a_ip|, |b_ip|\} > \delta^{-1}R$. The set of points in M that admit an R-long (m, δ)-explosion is denoted by $M(m, \delta, R)$.

10.5. *Lemma.* *Let D_p denote the cylinder $\{x \in B_p(R_1) : ||a_ix|-|a_ip|| \leqslant (0.1)R\}$, where (a_i, b_i) is an R_1-long $(m-1, \delta)$-explosion at p, $R \leqslant \delta R_1$, $\delta < 1/2n$. Then*

$$\beta_{D_p \setminus M(m,\delta,R)}(\delta^{-1}R) \leqslant N_1(N_0(\delta/2)).$$

Proof. Let us assume the contrary and choose in $D_p \setminus M(m, \delta, R)$ a subset of $N_1(N_0(\delta/2))$ points with pairwise distances $\geqslant \delta^{-1}R$. By the preceding lemma we can choose a collection $x_0, x_1, \ldots, x_{N_0(\delta/2)}$ from this subset for which $|x_0x_{i+1}| > 2|x_0x_i|$, $1 \leqslant i < N_0(\delta/2)$. By the definition of the function N_0 there are points x_i, x_j, $0 < i < j < N_0(l/2)$, for which $\tilde{\measuredangle} x_ix_0x_j \leqslant \delta/2$ and consequently $\tilde{\measuredangle} x_0x_ix_j > \pi-\delta$. But then the pairs (a_i, b_i), supplemented by the pair (x_0, x_j), obviously form an R-long (m, δ)-explosion at x_i. This is a contradiction. ■

10.6. Let $M(m, -\delta) = \bigcup_{R>0} M(m, \delta, R)$ denote the set of all (m, δ)-burst points.

Theorem. $\dim_H(M \setminus M(m, \delta)) \leqslant m-1$.

Proof. We may suppose that $\delta < 1/2n$. We carry out the induction with respect to m; the base $m = 0$ is trivial. To prove the induction step it is sufficient to cover $M(m, \delta)$ by a countable collection of balls of the form $B_x(R_x)$, where R_x is so small that x has an R_x-long (m, δ)-explosion, and to verify that for any $x \in M(m, \delta)$ the m-dimensional Hausdorff measure $B_x(R_x) \setminus M(m+1, \delta)$ is finite. The last assertion follows from Lemma 10.5. ■

10.6.1. *Remark.* Theorem 10.6 and Corollary 12.8 proved below imply that the Hausdorff dimension of the interior singular points of an n-dimensional SCBB is no greater than $n-2$.

10.7. We pass to the question of the convergence of Hausdorff measures for converging sequences of spaces. We begin with the definition.

Let the compact metric spaces X_i converge (in the sense of Hausdorff) to X. We fix a sequence of numbers $\nu_i \to 0$ and a sequence of ν_i-approximations $f_i : X \to X_i$ (see 7.8). We shall say that the measures μ_i on the spaces X_i *converge weakly* to the measure μ on X if for any set $A \subset X$ such that $\mu(\partial A) = 0$ we have the convergence $\mu_i(U_{\nu_i}(f_i(A))) \to \mu(A)$, where U_ε is the ε-neighbourhood.

10.8. ***Theorem.*** *Let the compact n-dimensional* SCBB M_i, M *have the same lower bound for the curvatures and let* $M_i \to M$ *in the sense of Hausdorff. Then the n-dimensional Hausdorff measures* μ_i *of the spaces* M_i *converge weakly to the n-dimensional Hausdorff measure* μ *of the space* M.

For the proof we need the following technical assertion.

10.9. ***Lemma.*** *Let* M *be a compact n-dimensional* SCBB *with diameter* 1. *Then for sufficiently small* $\delta > 0$ *we have, for any integer* m, $0 \leqslant m \leqslant n$, *and any* $R > 0$, $\nu < R$,

$$\nu^n \beta_{M\setminus M(m,\delta,R)}(\nu) < \varkappa[\delta, \hat{\nu}, \hat{M}](R).$$

Corollary. *Under the conditions of the lemma*

$$\mu(M \setminus M(n, \delta, R)) < \varkappa[\delta, \hat{M}](R).$$

Proof. We carry out induction with respect to m. The base $m = 0$ is trivial. We consider the cover of the set $M(m-1, \delta, R)$ by balls of the form $B_x(R_1)$, where the number $R_1 > \delta^{-1}R$ is chosen later. A finite subcover $B_{x_i}(R_1)$ can be chosen from this cover so that the balls $B_{x_i}(R_1/3)$ do not intersect pairwise. According to 8.4 there are no more than cR_1^{-n} balls in this cover. If we divide the ball $B_x(R_1)$ into cylinders and apply to it Lemma 10.5, we deduce that the set $A = M(m-1, \delta, R_1)\setminus M(m, \delta, R)$ is covered by $c(\delta)R_1^{-n}(R_1/R)^{m-1}$ balls of radius $\delta^{-1}R$, and consequently $\nu^n\beta_A(\nu) \leqslant c(\delta)(R/R_1)^{n-m+1}$.

Since for a suitable choice of R_1 we have $\varkappa(R_1)+(R/R_1)^{n-m+1} = \varkappa(R)$, the lemma is proved. ∎

10.10. *Proof of the theorem.* From the ν_i-approximation $f_i : M \to M_i$ it is not difficult to construct a $10\nu_i$-approximation $g_i : M_i \to M$ so that $|(g_i \circ f_i)(x), x| < 20\nu_i$, and thus $g_i(U_{\nu_i}(f_i(A))) \subset U_{100\nu_i}(A)$. If $\mu(\partial A) = 0$, then $\mu(U_{100\nu_i}(A)) \to \mu(A)$ as $i \to \infty$. It can now be seen that Theorem 10.8 follows from Theorem 9.8 and Lemma 10.9. ∎

10.11. ***Corollary.*** *For any point p in a* FDSCBB *the set of directions Σ_p' in which the geodesics emanate from p has total measure in Σ_p.*

This is implied by 10.8 and the proof of 10.2.

10.12. *Remark.* If we use 10.6 and 9.4 it is not difficult to carry across the proof of the "co-area formula" of Kronrod and Federer (see [13]) to the case of a FDSCBB.

10.13. ***Corollary.*** *Let A be a compact subset in an n-dimensional complete space M with curvature $\geqslant k$. Then the ratio $\mu(U_R(A))/\mu(\widetilde{B}(R))$ is a non-increasing function of R for all $R > 0$.* (Here $\widetilde{B}(R)$ is a ball of radius R in the n-dimensional complete simply-connected space with curvature k.)

The proof is analogous to that in [19], p. 65, taking into account 10.12 and the following observation. If $r, s \in M$, $|Ar| = |As| = R$, and r_1, s_1 are points on the geodesics Ar, As at a distance R_1 from A, $R_1 \leqslant R$, then the ratio $|rs|/|r_1s_1|$ does not exceed R/R_1 (assuming that $k = 0$).

§11. Functions that have directional derivatives, the method of succssive approximations, level surfaces of almost regular maps

11.1. In this section we describe a class of functions on a FDSCBB and a class of surfaces in a FDSCBB, which serve as more or less successful analogues of smooth functions and smooth submanifolds for Riemannian manifolds. One of the applications of the tools prepared here is the investigation carried out in the next section of points with nearly maximal explosions. A more general situation is considered in §13.

11.2. ***Lemma.*** *Let $p \in M$ be a point in a* FDSCBB. *Then for any $\delta > 0$ there is a deleted neighbourhood U of the point p so small that:*

a) *for any point $q \in U$ there is a point $q_1 \in M$ such that $|pq| < \delta|pq_1|$ and the pair (p, q_1) forms a $(1, \delta)$-explosion at q;*

b) $\measuredangle qpr - \widetilde{\measuredangle} qpr < \delta$ *for any triangle Δpqr with $q, r \in U$; further:*

c) *for any triangle Δpqr with $q, r \in U$ each angle of Δpqr differs from the corresponding angle of $\widetilde{\Delta} pqr$ by less than δ;*

d) *if* $q \in U$ *is joined to* p *by more than one geodesic, then the angles in any lune* pqp *are less than* δ.

In addition, if $A \subset M$ *is a compact set not containing* p, *then for any* $\delta > 0$ *there is a deleted neighbourhood* U *of* p *so small that*:

e) $|\tilde{\measuredangle} qpA - |q'A'|| < \delta$, $|\tilde{\measuredangle} q_1 qA - |q'A'|| < \delta$ *is satisfied for any point* $q \in U$ *and* $|\tilde{\measuredangle} q_1 pA - |q'A'|| < \delta$ *for some point* $q_1 \in M$ *such that* $|pq| \lesssim \delta |pq_1|$ *and* (p, q) *forms a* $(1, \delta)$-*explosion at* q. (Here $q', A' \subset \Sigma_p$, and $\tilde{\measuredangle} qpA$ is the angle in the triangle with sides $|qp|$, $|qA|$, $|pA|$ on the k-plane.)

We shall not give detailed proofs. They are based on the fact that, as Σ_p is compact, we can choose an arbitrarily dense finite net of directions ξ_i in Σ_p which are realized by the geodesics pa_i, and moreover on account of the proximity of a_i to p we may suppose, by Lemma 7.5, that the angles of the triangle $\Delta a_i p a_j$ differ little from the corresponding angles of $\tilde{\Delta} a_i p a_j$. In addition, it may be supposed that the angles $\tilde{\measuredangle} a_i pA$ differ little from $|a_i' A'|$. If we now put $U = \{q : 0 < |pq| \ll \delta \min\{|pa_i|, |pA|\}$, then assertions a) $-$ e) are easily verified by elementary geometric arguments. ∎

11.3. *Definition.* A function f on some domain of a FDSCBB is said to have *directional derivatives* if it is Lipschitz and for any point p in this domain there exists a continuous function $f'_{(p)}$ on Σ_p (called the *derivative of* f *at the point* p) such that along each geodesic pa the derivative of f (with respect to arc length) at p is equal to $f'_{(p)}(a')$.

We note that for such a function f the differential ratios $\varphi(R, \xi) = \frac{1}{R}(f(\exp_p R\xi) - f(p))$ satisfy a Lipschitz condition in ξ uniformly along R, so that $\varphi(R, \xi)$ converges to $f'_{(p)}$ uniformly on Σ_p as $R \to 0$ (the fact that $\varphi(R, \xi)$ is not defined for all $\xi \in \Sigma_p$ is immaterial).

11.4. *Examples.* The basic examples of functions that have directional derivatives are the distance functions. If $A \subset M$ is a compact set in a FDSCBB, then the function $f(x) = |Ax|$ on $M \setminus A$ satisfies Definition 11.3 with $f'_{(p)}(\xi) = -\cos|A'\xi|$, where $\xi \in \Sigma_p$. This follows immediately from Lemma 11.2 e).

Other examples can be obtained by using the following assertions.

1) Admissible arithmetic operations on functions that have directional derivatives give functions that have directional derivatives, and the derivatives are calculated by the standard rules.

2) The maximum of a finite number of functions that have directional derivatives is a function that has directional derivatives.

3) If f is a function that has directional derivatives and φ is a (one-sided) differentiable function on $\mathbb{R}$, then $\varphi \circ f$ is a function that has directional derivatives.

11.5. The method of successive approximations.
The following assertion is a formal generalization of the first part of Theorem 5.4 (in the finite-dimensional case).

Lemma. *Let the functions $f_1, \ldots, f_m$ that have directional deriatives be given in the domain $U \subset M$ of an n-dimensional* SCBB. *Suppose that for some δ, ε, $0 < n\delta < \varepsilon$, and for any point $p \in U$ there exist directions $\xi_i^{\pm} \in \Sigma_p$, $1 \leqslant i \leqslant m$, such that $|f'_{j(p)}(\xi_i^{\pm})| < \delta$ when $i \neq j$ and $f'_{i(p)}(\xi_i^{+}) > \varepsilon$, $f'_{j(p)}(\xi_i^{-}) < -\varepsilon$. Then the map $f = \{f_1, \ldots, f_m\} : U \to \mathbb{R}^m$ is $\frac{1}{2}(\varepsilon - n\delta)$-open* (see 5.3) *with respect to the metric in $\mathbb{R}^m$ given by the norm $\|x\| = \sum_{i=1}^{m} |x^i|$ for $x = (x^1, \ldots, x^m)$.*

Proof. Let $p \in U$, $A = \{\bar{a} \in \mathbb{R}^m : B_p((\varepsilon-n\delta)^{-1}\|\bar{a}, f(p)\|) \subset U$, and suppose there is an $a \in U$ such that $f(a) = \bar{a}$, $2\|f(p), \bar{a}\| \geqslant (\varepsilon-n\delta)|ap|\}$. Obviously A is closed, $f(p) \in A$. Suppose the assertion of the lemma is not satisfied, and $\bar{a}_0$ is a point that satisfies the condition $B_p(2(\varepsilon-n\delta)^{-1}\|\bar{a}_0, f(p)\|) \subset U$ but does not lie in A. Let the point $\bar{a}_1 \in A$ be the nearest point in A to $\bar{a}_0$, and let the point $a_1 \in U$ be such that $f(a_1) = \bar{a}_1$ and $2\|a_1, f(p)\| \geqslant (\varepsilon-n\delta)|pa_1|$. We choose the direction ξ in Σ_{α_1} so near to one of the directions $\xi_i^{\pm}$ that $|f'_{i(a_1)}(\xi)| > \varepsilon$, $|f'_{j(a_1)}(\xi)| < \delta$ for $j \neq i$ and $\operatorname{sgn} f'_{i(a_1)}(\xi) = \operatorname{sgn}(\bar{a}_0^i - \bar{a}_1^i)$. Then for sufficiently small $\mu > 0$ we see that $f(\exp_{a_1}(\mu\xi) \in A$ and $\|f(\exp_{a_1}(\mu\xi)), \bar{a}_0\| < \|\bar{a}_1\bar{a}_0\|$, which contradicts the choice of $\bar{a}_1$. ■

11.6. ***Corollary.*** *Let the point $p \in U$ and the number $\varepsilon_1 > 0$ be fixed under the conditions of Lemma* 11.5. *Then for sufficiently small R ($R < c(p, f, \varepsilon_1)$) there is, for every direction $\xi \in \Sigma_p$ such that $f'_{i(p)}(\xi) = 0$ for all i, a point $q \in U$ such that $R < |pq| < R(1+\varepsilon_1)$, $f_i(p) = f_i(q)$ for all i, and $q' \subset \Sigma_p$ is in a ε_1-neighbourhood of ξ.*

In particular, if we are now given functions $g_1, \ldots, g_l$ in U that have directional derivatives, then one can ensure also that

$$\left|\frac{g_j(q)-g_j(p)}{|pq|} - g_{j(p)}(\xi)\right| < \varepsilon_1.$$

Proof. To construct the point q it is necessary to get a first approximation q_0 by shifting the point p along a geodesic with direction $\xi_0 \in \Sigma_p$ very near to ξ and then applying the method of successive approximations (Lemma 11.5) in order to compensate for the alteration in f_i. Since $\|f(p)-f(q_0)\| < \varkappa[\hat{\xi}, |\widehat{pq_0}|](|\xi\xi_0|)\cdot|pq_0|$ and the openness parameter for f depends only on ε, δ, then for sufficiently small $|\xi\xi_0|$ ($|\xi\xi_0| < c[\hat{\xi}, |\widehat{pq_0}|]$) one can construct the point q where $|qq_0| < \varkappa[\hat{\xi}, |\widehat{pq_0}|](|\xi\xi_0|)\cdot|pq_0|$. If $|\xi\xi_0|$ and $|pq_0|$ are sufficiently small ($|\xi\xi_0| < c[\hat{\xi}, |\widehat{pq_0}|]$, $|pq_0| < c[\hat{\xi}]$), then this last inequality implies that $||pq|-|pq_0|| < \frac{1}{4}\varepsilon_1|pq_0|$ and by Lemma 11.2 b) that $q' \in \Sigma_p$ is in an ε_1-neighbourhood of ξ. Finally, if R is sufficiently small ($R < c[\widehat{\xi}]$), then there exists in Σ_p a sufficiently dense net of directions that can be realized by geodesics of length not less than $R(1+\varepsilon_1)$, consequently with a choice of ξ_0 and q_0 subject to the condition indicated above that $|\xi\xi_0|$ and $|pq_0|$ be small it is possible to stipulate that the condition $|pq_0| = R(1+\frac{1}{2}\varepsilon_1)$ be satisfied and consequently $R < |pq| < R(1+\varepsilon_1)$. ■

11.7. Almost regular maps.
The definition of almost regular maps, which we give below, is apparently too narrow. However, it turned out to be sufficient in the context of this paper.

Definition. A function f on a domain $U \subset M$ of a FDSCBB is said to be δ-*almost regular* if f is the distance function of some compact set $A \subset M$, $A \cap U = \varnothing$, and if for every point $p \in U$ there is a compact set $B_{(p)}$ which does not contain p such that $\tilde{\measuredangle} ApB_{(p)} > \pi-\delta$ (we recall that $\tilde{\measuredangle} ApB_{(p)}$ denotes the angle of the triangle on the k-plane with sides $|Ap|$, $|AB_{(p)}|$, $|pB_{(p)}|$).

The map $f = (f_1, \dots, f_m) : U \to \mathbb{R}^m$ is said to be δ-*almost regular* if its coordinate functions f_i are δ-almost regular and if the compact sets A_i, $B_{i(p)}$ figuring in the definition of almost regularity satisfy

$$\tilde{\measuredangle}\, A_i p B_{j(p)} > \frac{\pi}{2} - \delta, \quad \tilde{\measuredangle}\, A_i p A_j > \frac{\pi}{2} - \delta,$$

$$\tilde{\measuredangle}\, B_{i(p)} p B_{j(p)} > \frac{\pi}{2} - \delta \text{ when } i \neq j.$$

(The parameter $\delta > 0$, which we consider to be small, is part of the definition. As a rule our arguments will be independent of δ if δ is less than a well-determined "ceiling", depending on the dimension, and also in §12 on the non-degeneracy characteristic of the FDSCBB under consideration (see 8.7).)

Obviously almost regular maps satisfy the conditions of Lemma 11.5 with $\varepsilon = 1-\delta$.

11.8. We now turn to studying the properties of level surfaces of almost regular maps. We begin with dimension estimates.

Theorem. *A level surface Π of an almost regular map from an n-dimensional* SCBB *M into $\mathbb{R}^m$ has topological dimension no greater than $n-m$. In addition, the set of points in Π that are not m_1-burst points in M have topological dimension no greater than m_1-1-m. In particular, the set of points in M that are not m_1-burst points has dimension no greater than m_1-1.*

The proof of the theorem is by a simple inverse induction with respect to m, in which the induction step is based on the following assertion.

11.9. *Lemma.* *Let Π be a level surface of a $\varkappa(\delta)$-almost regular map $f = (f_1, ..., f_m): M \to \mathbb{R}^m$. We introduce the notation $S_R = \{q \in M, |pq| = R\}$, $\Pi_R = \Pi \cap S_R$. Then for sufficiently small $R > 0$ there is a neighbourhood $U \supset \Pi_R$ in which the map $\bar{f}: M \to \mathbb{R}^{m+1}$, $\bar{f}(q) = (f_1(q), ..., f_m(q), |pq|)$, is $\varkappa(\delta)$-almost regular.* (The case $\Pi_R = \varnothing$ is not excluded.)

This lemma follows from Lemma 11.2 e) and 5.6.

11.10. *Remark.* In the general case the points in an n-dimensional SCBB that are not n-burst points may fill a set of topological dimension $n-1$; this, for example, occurs for manifolds with boundary. However, it will be shown in the next section that for an interior point of an n-dimensional SCBB an $(n-1)$-burst point is by implication an n-burst point, therefore in an n-dimensional SCBB without boundary the points that are not n-burst points fill a set of topological dimension no greater than $n-2$. Cf. 10.6.1.

11.11. Let us now turn to the more refined properties.

Theorem. *On a level surface of an almost regular map on a* FDSCBB *the intrinsic metric, which is equivalent to the extrinsic metric, is induced locally. In particular, such a surface is locally linearly connected.*

More precisely, let $f = (f_1, ..., f_m) : U \to \mathbb{R}^m$ be an almost regular map, let $p \in U$, let A_i and B_i be compact sets that figure in the definition of f being almost regular at the point p, and let $\Pi = \{q : f_i(q) = f_i(p) \text{ for all } i\}$ be a level surface, $q, r \in \Pi$, $\delta_1 = \max\{|pq|, |pr|\}/\min\{|pA_i|, |pB_i|, |p, \Pi \cap \partial U|\}$. If δ (the parameter for f to be almost regular) and δ_1 are sufficiently small (δ, $\delta_1 > c$), then q and r can be connected in Π by a curve of length no greater than $|qr|(1+\varkappa(\delta, \delta_1))$.

Proof. We need to verify that a point $s \in \Pi$ can be found arbitrarily near to r such that $|rs| < (1+\varkappa(\delta, \delta_1))\cdot(|qr|-|qs|)$. After this the proof concludes by standard arguments. To construct such a point s it is sufficient, by Corollary 11.6, to find a direction $\xi \in \Sigma_r$ such that $f'_{i(r)}(\xi) = 0$ for all i and $|q'\xi| < \varkappa(\delta, \delta_1)$, $q' \subset \Sigma_r$. Since the sets A'_i, $B'_i \subset \Sigma_r$ form an $(m, \varkappa(\delta, \delta_1))$-explosion in Σ_r and $|q'A'_i| > \pi/2-\varkappa(\delta, \delta_1)$, $|q'B'_i| > \pi/2-\varkappa(\delta, \delta_1)$ (see Corollary 5.7), then all that remains is for us to prove the following assertion.

11.12. *Lemma.* *Let M be a complete n-dimensional space with curvature $\geqslant 1$, let A_i, B_i form an (m, δ)-explosion in M, and let $Q \subset M$ be a compact set such that $|QA_i| > \pi/2-\delta$, $|QB_i| > \pi/2-\delta$. Then for any point $q \in Q$, if δ is sufficiently small ($\delta < c$), there is a point $\xi \in M$ such that $|\xi A_i| = \pi/2$ for all i and $|\xi q| < \varkappa(\delta)$.*

Proof. Clearly, if $\delta > 0$ is sufficiently small, then the map $f: M \to \mathbb{R}^m$, $f(x) = (|xA_1|, ..., |xA_m|)$, is almost regular, with parameter $100\delta^2 n$, in the domain

$$U = \{x \in M: \min(|xA_i|, |xB_i|) > \pi/2 - 10\sqrt{n}\delta \quad \text{for all} \quad i\}.$$

If furthermore $\delta < 1/100n$, then by Lemma 11.5 the map f will be 1/2-open in U and hence in the $4\sqrt{n}\delta$-neighbourhood of any point $q \in Q$ there is a desired point ξ. ■

11.13. We now define and study the space of directions and the tangent cone to the level surface of an almost regular map at some point of the surface.

Definition. Let γ be a curve in a FDSCBB M, where $\gamma(0) = p$ and $\gamma(t) \neq p$ for sufficiently small $t > 0$. The *direction of the curve at the point p* is the limit in Σ_p of the directions of the geodesics $p\gamma(t)$ as $t \searrow 0$ if this limit exists. The *space of directions* $\Sigma\Pi_p$ of the set $\Pi \subset M$ at the point $p \in \Pi$ is the set of directions at the point $p \in \Pi$ of all possible curves at Π beginning at p and having a direction at p (compare 7.2.1).

11.14. ***Theorem.*** *Let $p \in \Pi$ be a point on a level surface of an almost regular map $f = (f_1, ..., f_m)$. Then the sets $\Lambda := \{\xi \in \Sigma_p, f_i{}'_{(p)}(\xi) = 0$ for all $i\}$ and $\Sigma\Pi_p$ coincide.*

Proof. The inclusion $\Sigma\Pi_p \subset \Lambda$ obviously follows from the fact that all the f_i have directional derivatives. The inverse inclusion easily follows from Theorem 11.11, Lemma 11.2 b), and the following assertion.

11.15. ***Lemma.*** *Under the conditions of Theorem* 11.4 *for any $\varepsilon > 0$ and for sufficiently small $R > 0$ ($R < c(p, f, \varepsilon)$) one can find, for any direction $\xi \in \Lambda$ on the surface $\Pi_R = \{q \in \Pi : |pq| = R\}$, a point q such that $q' \subset \Sigma_p$ is contained in the ε-neighbourhood of ξ.*

Proof. By Corollary 11.6 we can construct for some $R_1 : |R-R_1| < (0.1)\varepsilon R$ a point $q_1 \in \Pi_{R_1}$ for which $q_1' \in \Sigma_p$ is contained in the $(0.1)\varepsilon$-neighbourhood of ξ. If R is sufficiently small, then, if we apply the method of successive approximations (Lemma 11.5) to the map $\bar{f}$ (which is almost regular by Lemma 11.9), where $\bar{f}(x) = (f_1(x), ..., f_m(x), |px|)$ with initial approximation q_1, we get the desired point q. ■

11.16. ***Corollary.*** *Under the conditions of Theorem* 11.14 *the spaces Π_R with "metrics" $\rho(q, r) = \arccos(1-|qr|^2/2R^2)$ converge in the Hausdorff metric to the space of directions $\Sigma\Pi_p$, in which the metric is defined as the restriction of the metric on Σ_p.*

(The word "metric" is given in quotation marks, since the triangle inequality may be violated for the function ρ.)

11.17. ***Corollary.*** *Under the conditions of Theorem* 11.14

a) $\Sigma\Pi_p$ *being non-empty is equivalent to* Π_R *being non-empty for sufficiently small $R > 0$.*

b) $\Sigma\Pi_p$ *being linearly connected is equivalent to* Π_R *being linearly connected for sufficiently small $R > 0$. In addition, the numbers of linearly connected components of $\Sigma\Pi_p$ and Π_R are the same.*

Proof. Assertion a) is obvious. To prove b) we note that the spaces Π_R with "metric" ρ from 11.16 are uniformly locally connected by Theorem 11.11, taking into account Lemma 11.9. On the other hand, the space $\Sigma\Pi_p$ is locally linearly connected by Theorem 11.11, since by Theorem 11.14, $\Sigma\Pi_p$ is the level surface of the map $\Sigma_p \to \mathbb{R}^m$ with coordinate functions arc cos $f'_{i(p)}$, and it is almost regular in a neighbourhood of $\Sigma\Pi_p$. Therefore assertion b) follows from the nearness of Π_R and $\Sigma\Pi_p$ in the Hausdorff metric. ■

§12. Level lines of almost regular maps

12.1. Let M be an n-dimensional SCBB, let $f = (f_1, ..., f_m) : M \to \mathbb{R}^m$ be an almost regular map, and let Π be a level surface of f. This tells us immediately that $m \leqslant n$, and if $m = n$, then Π consists of isolated points. In this section we show that if $m = n-1$, then Π is a one-dimensional manifold with boundary, and moreover the boundary points of Π are precisely the points of the boundary of M lying in Π. Throughout this section ε denotes a lower bound of $Vr_{n-1}(\Sigma_p)$ taken over all points p from M (from the compact subset under consideration) (see 8.7). We suppose that the almost regular parameter δ is less than a well-determined "ceiling" depending on ε ($\delta < c(\varepsilon)$). This requirement is apparently connected with the method of proof, and most likely the assertions themselves still remain true under the weaker condition[1] $\delta < c$. Our first aim is to show that when $m < n$ the surface Π cannot have isolated points. For this two lemmas are necessary.

12.2. ***Lemma.*** *Let M be a complete n-dimensional space with curvature $\geqslant 1$ having an (m, δ)-explosion A_i, B_i, $m \leqslant n$. Let the point $\xi \in M$ be such that $\sum_{i=1}^{m} \cos^2 |\xi A_i| < 1-\varepsilon_1$. Then, if δ is sufficiently small ($\delta < c(\varepsilon_1)$), we can find a point ζ in M such that $|A_i\zeta| = \pi/2$ for all i.*

Proof. We first construct a point η such that $||A_i\eta|-\pi/2| < \varkappa(\delta)$ for all i; after this, Lemma 11.12 gives us the construction of the point ζ. To construct the point η we may suppose that A_i and B_i are points. We take points $\widetilde{A}_i$, $\widetilde{B}_i$, $\widetilde{\xi}$ on the unit sphere S^m such that $\widetilde{A}_i$, $\widetilde{B}_i$ form an $(m, 0)$-explosion and $|\widetilde{A}_i\widetilde{\xi}| = |A_i\xi|$ for all i. We construct a sequence $\widetilde{\xi}_0 = \widetilde{\xi}, \widetilde{\xi}_1, \dots, \widetilde{\xi}_m = \widetilde{\eta}$ inductively by the following rule: $\widetilde{\xi}_{i+1}$ lies on the great semicircle $\widetilde{A}_{i+1}\widetilde{\xi}_i\widetilde{B}_{i+1}$ at a distance $\pi/2$ from $\widetilde{A}_{i+1}$ (and $\widetilde{B}_{i+1}$). It is obvious that $\sum_{i=1}^{m} \cos^2|\widetilde{\xi}_{j+1}\widetilde{A}_i| \leqslant \sum_{i=1}^{m} \cos^2|\widetilde{\xi}_j\widetilde{A}_i| < 1-\varepsilon_1$, and consequently the perimeters of the triangles $\Delta\widetilde{A}_j\widetilde{\xi}_{j-1}\widetilde{B}_i$, $\Delta\widetilde{A}_j\widetilde{\xi}_{j-1}\widetilde{A}_i$, $\Delta\widetilde{B}_j\widetilde{\xi}_{j-1}\widetilde{A}_i$, $\Delta\widetilde{B}_j\widetilde{\xi}_{j-1}\widetilde{B}_i$ do not exceed $2\pi - c(\varepsilon_1)$ when $i \neq j$. In addition, it is obvious that $|\widetilde{\xi}_j\widetilde{A}_i| = \pi/2$ when $i \leqslant j$, so that $|\widetilde{\eta}\widetilde{A}_i| = \pi/2$ for all i. We carry this construction across to M.

[1] This follows from a recent result of A. Petrunin.

We construct a sequence $\xi_0 = \xi, \xi_1, \ldots, \xi_m = \eta$ inductively by the following rule: ξ_{i+1} lies on the two-link broken line $A_{i+1}\xi_i B_{i+1}$ at a distance $\pi/2$ from A_{i+1}. Now, if we apply induction with respect to j, it can be proved that $||A_i\xi_j|-|\widetilde{A_i\xi_j}|| < \varkappa(\delta)$, $||B_i\xi_j|-|\widetilde{B_i\xi_j}|| < \varkappa(\delta)$ for all i. In fact, the base of induction (when $j = 0$) follows from the conditions $|\widetilde{A}_i\widetilde{\xi}| = |A_i\xi|$, $|A_iB_i| > \pi-\delta$, $|\widetilde{A}_i\widetilde{B}_i| = \pi$, and the induction step is ensured by Lemma 9.2 d). Thus $||A_i\eta|-\pi/2| < \varkappa(\delta)$ and the lemma is proved. ■

12.3. *Lemma.* *Let M be a complete n-dimensional space with curvature $\geqslant 1$ having an (m, δ)-explosion A_i, B_i, $m \leqslant n$. Then*

$$Vr_n\left\{\xi \in M\colon \sum_{i=1}^{m} \cos^2|\xi A_i| > 1-\delta_1\right\} < \varkappa(\delta, \delta_1).$$

Proof. We use induction; the induction step consists of increasing m and n by 1. If $m = 1$, then the assertion follows from Lemma 8.2 and Corollary 8.3.

We consider the general case. We may suppose that A_m is a point. Let $\xi \in M$ be such that $\sum_{i=1}^{m} \cos^2|\xi A_i| > 1-\delta_1$. We construct points $\widetilde{A}_i$, $\widetilde{B}_i$, $\widetilde{\xi}$ on the unit sphere S^m so that $(\widetilde{A}_i, \widetilde{B}_i)$ form an $(m, 0)$-explosion in S^m and $|\widetilde{A}_i\widetilde{\xi}| = |A_i\xi|$ for all i. Clearly either $|\widetilde{A}_m\widetilde{\xi}| < \varkappa(\delta_1)$, or $|\widetilde{A}_m\widetilde{\xi}| > \pi-\varkappa(\delta_1)$, or $\sum_{i=1}^{m-1}\cos^2|\widetilde{A}'_i\widetilde{\xi}'| > 1-\varkappa(\delta_1)$, where $\widetilde{A}'_i, \widetilde{\xi}' \in \Sigma_{\widetilde{A}_m}$. If we apply Lemma 9.2 c), we deduce that either $|A_m\xi| < \varkappa(\delta, \delta_1)$, or $|A_m\xi| > \pi-\varkappa(\delta, \delta_1)$, or $\sum_{i=1}^{m-1}\cos^2|A'_i\xi'| > 1-\varkappa(\delta, \delta_1)$, where A'_i, $\xi' \in \Sigma_{A_m}$. Now our assertion follows from Lemma 8.2, Corollary 8.3, and the inductive hypothesis applied to Σ_{A_m}. ■

12.4. *Corollary.* *Let M be a complete n-dimensional space with curvature $\geqslant 1$ having an (m, δ)-explosion A_i, B_i, $m \leqslant n$, where $Vr_n(M) \geqslant \varepsilon$. Then if $\delta > 0$ is sufficiently small ($\delta < c(\varepsilon)$), there is a point $\xi \in M$ such that $|\xi A_i| = \pi/2$ for all i.*

This follows from Lemmas 12.2 and 12.3.

12.5. *Corollary.* *When $m < n$ and δ is sufficiently small* ($\delta < c(\varepsilon)$, see 12.1) *a level surface of a δ-almost regular map from an n-dimensional* SCBB *into $\mathbb{R}^m$ does not have isolated points.*

This follows from 12.4 and 11.17 a).

12.6. *Lemma.* *Let M be a complete n-dimensional space with curvature $\geqslant 1$ having an (m, δ)-explosion (A_i, B_i), $m \leqslant n-1$, where $Vr_{n-1}(M) \geqslant \varepsilon$. Then if δ is sufficiently small ($\delta < c(\varepsilon)$), the set $\Lambda = \{\xi \in M : |A_i\xi| = \pi/2$ for all $i\}$ is linearly connected.*

Proof. Λ is locally linearly connected by Theorem 11.11 because it is a level surface of an almost regular map. Let Λ_1 and Λ_2 be the nearest components of Λ, where $q \in \Lambda_1$ and $r \in \Lambda_2$ are their nearest points. We shall show that $|qr| > \pi - \varkappa(\delta)$. In fact, otherwise the sets A_i', B_i' form an $(m, \varkappa(\delta))$-explosion in the space of directions Σ_q and there we would have $|A_i'r'| > \pi/2 - \varkappa(\delta)$, $|B_i'r'| > \pi/2 - \varkappa(\delta)$. This follows from the angle comparison theorem in M.

Therefore by Lemma 11.12 a direction $\xi \in \Sigma_q$ can be found such that $|A_i'\xi| > \pi/2$ for all i and $|r'\xi| < \varkappa(\delta)$, and hence by Corollary 11.6 a point $s \in \Lambda$ can be found near q that is nearer to r than q, which contradicts the choice of q and r. So we deduce that Λ has no more than two components and, if there are two of them Λ_1, Λ_2, then they are at a distance of at least $\pi - \varkappa(\delta)$. But then by adding $A_{m+1} = \Lambda_1$ and $B_{m+1} = \Lambda_2$ to the existing A_i, B_i we get an $(m+1, \varkappa(\delta))$-explosion in M. If δ were chosen sufficiently small, then by Corollary 12.4 a point $\xi \in M$ could be found at a distance $\pi/2$ from all A_i, $1 \leqslant i \leqslant m+1$. But then $\xi \in \Lambda \setminus (\Lambda_1 \cup \Lambda_2)$—a contradiction. ■

12.7. ***Theorem.*** a) *Let m be a complete n-dimensional space with curvature* $\geqslant 1$ *having an* (n, δ)*-explosion* (A_i, B_i), *where* $Vr_n(M) \geqslant \varepsilon$. *Then, if* δ *is sufficiently small* ($\delta < c(\varepsilon)$), *the set* $\Lambda = \{\xi \in M : |A_i\xi| = \pi/2$ *for all* $i\}$ *consists of a single point if M has a boundary and of two points at a distance* $\pi - \varkappa(\delta)$ *if M does not have a boundary.*

b) *Let* Π *be a level surface of an* δ*-almost regular map from an* $(n+1)$*-dimensional* SCBB *into* $\mathbb{R}^n$. *Then, if* δ *is sufficiently small* ($\delta < c(\varepsilon)$, see 12.1), Π *is a one-dimensional manifold, and moreover the boundary points* Π *are precisely those boundary points of the* FDSCBB *that lie in* Π.

Proof. Assertion b) follows from a) when applied to the spaces of directions Σ_p at the points $p \in \Pi$, taking into account Theorem 11.14 and its corollaries. In fact, if p is a boundary point, and consequently $\Lambda \subset \Sigma_p$ is a singleton, the "surfaces" $\Pi_R = \{q \in \Pi : |pq| = R\}$, which consist of isolated points since they are level surfaces of an almost regular map into $\mathbb{R}^{n+1}$ (Lemma 11.9), turn out to be singletons by Corollary 11.17 b); in addition, it is obvious that the point Π_R depends continuously on R. If however p is an interior point and $\Lambda \subset \Sigma_p$ consists of two points at a distance $\pi - \varkappa(\delta)$, then Π_R also consists of two points at a distance $2R(1-\varkappa(\delta))$ (by Corollary 11.16), so that Π turns out, in a neighbourhood of p, to be an arc containing p as an interior point.

To prove assertion a) we use induction with respect to the dimension. The base $n = 1$ is obvious. In the general case, if $q, r \in \Lambda$, then $|qr| > \pi - \varkappa(\delta)$, since otherwise the mid-point of the geodesic qr would turn out to be an $(n+1)$-burst point with explosion A_i, B_i and q, r. Thus Λ contains no more than two points, and if it has two points, then M has an $(n+1, \varkappa(\delta))$-explosion and by Corollary 9.6 a) it does not have a boundary. All that remains is for us to verify that if M has no boundary, then Λ cannot be a singleton. By the inductive hypothesis (more precisely by assertion b) that follows from it) the set $\Lambda_n = \{\xi \in M : |\xi A_i| = \pi/2$ when $i \leqslant n-1\}$ is a one-dimensional manifold without boundary, which by Lemma 12.6 is a circle.

Since Lemma 11.12 implies that $|A_n\Lambda_n| < \varkappa(\delta)$ and $|B_n\Lambda_n| < \varkappa(\delta)$, then obviously there are two points on the circle Λ_n at a distance $\pi/2$ from A_n, which is what was required. ■

12.8. ***Corollary.*** *An $(n-1)$-burst interior point in an n-dimensional* SCBB *is an n-burst point.*

12.9. Further results.

12.9.1. A similar approach enables us to show that a small neighbourhood of an $(n-1)$-burst point p in the boundary of an n-dimensional SCBB admits an almost isometric map onto the cube in $\mathbb{R}^n$, where the part of the boundary occurring in this neighbourhood maps onto one of the hyperfaces of this cube. Here the explosion map (see 5.5) provides all the coordinate functions of the desired almost isometry except one—the one that is constant on the indicated hyperface. A basic observation which is necessary for the proof is as follows. Near p, every level line of the explosion map contains exactly one boundary point, and moreover these points may be characterized as the maximum points of the restriction to the corresponding level line of the distance function from some point q. Any point near p lying on the same level line as p can be taken as q.

12.9.2. Small neighbourhoods of interior $(n-2)$-burst points in an n-dimensional SCBB also admit coordinates of the distance type. In more detail, let the point p have an $(n-2)$-explosion (a_i, b_i). Then the corresponding sets of directions $A_i, B_i \subset \Sigma_p$ form a $(n-2)$-explosion in Σ_p. By Theorem 12.7 b) the level line $\Pi = \{\xi \in \Sigma_p : |A_i\xi| = \pi/2, 1 \leqslant i \leqslant n-2\}$ is a circle, whose diameter, as is easily shown using Lemma 8.6, is not less than $c(\varepsilon)$. We choose four points $\xi_1, \xi_2, \xi_3, \xi_4 \in \Pi$ lying on Π in the cyclic order indicated by their numbers at approximately equal distances from each other, and we construct at equal small distances from p the points q_1, q_2, q_3, q_4 such that $q_j' \subset \Sigma_p$ are very near to ξ_j, $1 \leqslant j \leqslant 4$. Then in a small neighbourhood of p the map $f(q) = (|a_1q|, \dots, |a_{n-2}q|, |q_1q| - |q_3q|, |q_2q| - |q_4q|)$ is a $c(\varepsilon)$-bi-Lipschitz homeomorphism onto a domain in $\mathbb{R}^n$. We know a proof of this assertion which requires very laborious arguments; it is set out in §14 of the original version of this article, which was distributed as a preprint.

12.9.3. The result in 12.9.2 and also 12.8 and 11.2 a) imply that in the two-dimensional case the interior of a SCBB is a two-dimensional manifold with the metric singularities forming a discrete set. A simple additional argument shows that a two-dimensional SCBB in the large is a two-dimensional manifold whose boundary is just the boundary of a FDSCBB in the sense of 7.19. In the three-dimensional case the interior of a SCBB may have non-isolated metric singularites, however 12.9.2 and 11.2 a) imply that the interior topological singularities are isolated. In fact, they have small neighbourhoods which are homeomorphic to the cone over $\mathbb{R}P^2$; see 13.2 b).

§13. Subsequent results and open questions

All the spaces in this section are assumed to be finite-dimensional and complete.

13.1. The theorems formulated in 13.1 – 13.6 are the result of research by the third author. The proofs are published in a separate paper.

If one considers examples one is naturally led to the conjecture that a small spherical neighbourhood of a point in a FDSCBB is homeomorphic to the tangent cone at this point. This conjecture has been verified for points with an explosion near to being maximal by using almost regular maps. However, for $(n-3)$-burst points in an n-dimensional SCBB this method, apparently, cannot give the necessary results. To prove the conjecture in complete generality one has to introduce the more general concept of an admissible map.

A map $f = (f_1, \ldots, f_m) : U \to \mathbb{R}^m$ from a domain U in an n-dimensional SCBB M is said to be (ε, δ)-*admissible* (where ε, δ are fixed and moreover $\varepsilon \gg \delta > 0$) if for all $x \in U$, $1 \leqslant i \leqslant m$ we have $f_i(x) = |A_i x|$, where $A_i \subset M$ are compact sets that do not intersect U, and where for every point $p \in U$ there is a compact set $B_{(p)} \subset M$ such that the inequalities $|A_i' A_j'| > \pi/2 - \delta$, $i \neq j$, $|A_i' B_{(p)}'| > \pi/2 + \varepsilon$ are satisfied in Σ_p.

We note that according to the given definition an admissible function is a distance function that does not have "almost critical" points in U.

It can be shown that an admissible map is open, and when $m = n$ it is locally homeomorphic. The advantage of admissible maps over almost regular maps, which plays a most important role in the proofs, is as follows.

Let $p \in \Pi$ be a point on a level surface of an almost regular map $f : U \to \mathbb{R}^m$, and let P denote the set of points in U near which f cannot be added to another coordinate function to make an almost regular map into $\mathbb{R}^{m+1}$. Then, if $p \in P$, there are no other points from P on the level surface Π near to p. However, there may be many points of P near to p on nearby level surfaces; we can only assert that on each such surface they are isolated.

If almost regularity is everywhere replaced by admissibility, then it can be proved that there exists a spherical neighbourhood $B_p(R)$ of the point p in which on each of the nearby level surfaces of the map f there is no more than one point in P. In addition, if there is such a point on some level surface, then it can be described in a canonical way, because at this point the average distance from points of the set $\partial B_p(R) \cap \{x \in U : f_i(x) \geqslant f_i(p) \text{ for all } i\}$ takes its greatest value among points on this level surface that are near p.

13.2. If admissible maps are used, the following assertion can be proved.

Theorem. a) *A proper admissible map is the projection of a locally trivial fibration.*

b) *A small spherical neighbourhood of a point in a* FDSCBB *is homeomorphic to the tangent cone at this point.*

c) *A* FDSCBB *is homeomorphic to a stratified (into topological manifolds) space.*

d) *For a given compact set in an n-dimensional space M with curvature* $\geqslant k$ *there exists* $\nu > 0$ *such that every compact n-dimensional space* M_1 *with curvature* $\geqslant k$ *which is* ν*-near to M in the sense of Hausdorff must be homeomorphic to M. Moreover for each* ν*-approximation* M_1 *to the space M there exists a* $\varkappa(\nu)$*-near homeomorphism from M onto* M_1.

It has not been established that the homeomorphisms in a)−d) are bi-Lipschitz, although there are apparently no known counterexamples.

The above assertions are all proved in a parallel way by induction. The geometric background of the proof has been given above; the necessary topological technique is borrowed from [31].

13.3. In the previous subsection the boundary points of a FDSCBB were in no way distinguished from other singular points. Nevertheless the approach described above enables one to give a more convincing justification for the definition of the boundary of a FDSCBB given in 7.19 and to prove some natural properties of a FDSCBB with a boundary. Namely, the following assertions are true.

Theorem. a) *The property that a point in a* FDSCBB *is a boundary point is determined by the topology of its small conic neighbourhood; the property that a* FDSCBB *has a boundary is determined by its topology.*

b) *The boundary of a* FDSCBB *is closed and homeomorphic to a space stratified into manifolds.*

c) *If M is a* FDSCBB *with a boundary N and if* $p \in N$, *then* ΣN_ν *coincides with the boundary of* Σ_p. *In addition, an intrinsic metric is induced locally on N.*

d) (Doubling theorem). *Let M be a space with curvature* $\geqslant k$ *and with a boundary N. We let* $\overline{M}$ *denote the result of gluing two copies of M along the common boundary N and furnish this with the natural intrinsic metric. Then* $\overline{M}$ *is a space with curvature* $\geqslant k$ *and without boundary.*

Thus, compact FDSCBB with a boundary are the same as convex compact spaces in FDSCBB without boundary.

The proofs of these assertions modulo Theorem 13.2 are not very complicated.

13.4. We note two open questions. 1) It is not clear whether the components of the boundary of a space with curvature $\geqslant k$ with the induced intrinsic metric are spaces with curvature $\geqslant k$. In other words, are the components of the boundary of a convex set in a space with curvature $\geqslant k$ spaces with curvature $\geqslant k$?

2) Is the gluing theorem generalizing 13.3 true: if the boundaries of two spaces M, M_1 with curvature $\geqslant k$ are component-wise isometric, then is the

result of gluing M and M_1 with respect to this isometry a space with curvature $\geqslant k$?

13.5. Equidistant curves.
Let M be a convex compact subset with non-empty boundary N in a space with curvature $\geqslant 0$. It can be shown that the equidistant curves of the set $M_t = \{x \in M : |xN| \geqslant t\}$ are convex.

From what has been said and from Theorem 13.2 a), b) it follows that the interior of a convex compact set in a space with curvature $\geqslant k > 0$ is homeomorphic to a cone over a compact space with curvature $\geqslant 1$. A similar assertion can be proved also for the whole convex compact set.

The concepts described also open the way to proving an analogue of the well-known theorem of Cheeger and Gromoll [12] for non-regular spaces of non-negative curvature.

13.6. A direct generalization of the theorem of Grove and Shiohama [24] follows from Theorem 13.2 a), b). *A space with curvature $\geqslant 1$ and with diameter $> \pi/2$ is homeomorphic to the suspension over a space with curvature $\geqslant 1$.*

In addition, if we use the results of 13.2, we can get bounds for the Betti numbers of spaces with curvature $\geqslant 0$ by a method analogous to [18].

13.7. The next three open questions present difficulties: these are questions about the approximation of FDSCBB by Riemannian manifolds, about deformation of the metric of a FDSCBB, and about their analytic description. Apparently these questions are interlinked.

13.2 d) implies that FDSCBB which are not topological manifolds do not admit approximations by complete Riemannian manifolds of the same dimension and with sectional curvatures uniformly bounded below. This does not exclude the possibility of approximations of arbitrary compact subsets of a FDSCBB M by Riemannian manifolds of larger dimension that collapse onto M. For example, a flat Klein bottle, as is known, can collapse to a segment. For other examples see [32].

As yet there are no means of deforming the metric of a FDSCBB while preserving (or at least controllably changing) the lower bound of the curvature. Such deformations are useful, for example, in solving extremal problems. In particular it seems probable, but completely unclear how to prove it, that if there is a point p with diam $\Sigma_p < \pi$ in a FDSCBB M, then there exists a deformation of the metric of M which is concentrated about p and which preserves the lower bound for the curvature and decreases the volume.

The metric of a FDSCBB (and even more general so-called manifolds of bounded curvature) may be given by using a "non-regular" linear element which is defined almost everyhwere (see [3], [30]). It is not known whether it is possible to give a more detailed description of a FDSCBB in the case of larger dimension. It is well known [8] that for spaces of two-sided bounded curvature a linear element has fairly high smoothness, at least $C^{1,\alpha}$, $0 < \alpha < 1$. It can be expected that a space with curvature bounded below, in which all the geodesics are locally extendable, can also be represented with the help of a fairly smooth linear element.

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