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CONSTRUCTION OF NONSINGULAR ISOPERIMETRIC FILMS

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Abstract. A mapping $f: M \to \mathbf{R}^N$ of a smooth n-dimensional manifold M with boundary ∂M is said to be *isoperimetric* if $V_n(f) \leqq C_N V_{n-1}(f|_{\partial M})$, where the constant C_N depends only on N, and $V_k(g)$ denotes the k-dimensional volume of the map g. In this paper there is given a necessary and sufficient condition for the existence of an isoperimetric imbedding or immersion $f: M^n \to \mathbf{R}^N$ which extends the given imbedding or immersion $g: \partial M \to \mathbf{R}^N$ of the boundary ∂M of the manifold M. Besides the result just mentioned, the paper proves certain approximation theorems of the following type. Suppose that M is a closed n-dimensional manifold, $k > 0$ and $f: M \to \mathbf{R}^{n+k}$ is a smooth mapping. Then, if there exists an immersion $g: M \to \mathbf{R}^{n+k}$, there also exists a sequence of immersions $f_i: M \to \mathbf{R}^{n+k}$ which approximate the mapping f in the norms of the spaces W_p^l, provided that either $(l-1)p < k$, or $(l-1)p = k$ and $p > 1$.

Bibliography: 9 items.

§1. Introduction

1.1

1.1.1. In the paper of Federer and Fleming [5], it was shown that to an $(n-1)$-dimensional cycle, lying in Euclidean space $\mathbf{R}^q$ and having $(n-1)$-dimensional Hausdorff measure V, may be attached an isoperimetric film; that is, a film whose n-dimensional Hausdorff measure is not greater than $CV^{n/(n-1)}$, where C is a constant depending only on the dimension q of the ambient space. In the same paper an analogous result was proved for a class of objects more general than cycles, specifically the class of rectifiable currents.

A related result appears in [8].

1.1.2. The purpose of the present paper is to show that, if a cycle satisfies certain conditions of regularity (for example, is a smooth manifold), then, if certain necessary conditions, dictated only by topological considerations, are satisfied, one may attach to this cycle an isoperimetric film which satisfies the same regularity conditions.

1.1.3. *Plan of the paper.* In §1.2 we give the fundamental differential-geometric definitions which are employed in the paper. In §1.3 we formulate the fundamental results, which are proved in §§2 and 3. Let us note that the proof given in §2 is very closely related to the corresponding considerations in [5, 8]. In §4 we discuss the effective verification of the topological conditions in the formulation of §§1.3.2.–1.3.4., we establish that the approximation assumption B of §1.3.4 cannot be improved, and we show how the method of this paper may be used for the solution of purely topological problems.

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1.2

1.2.1. A smooth manifold in this paper refers to a manifold of class C^∞ with boundary, and smooth mappings are of class C^∞. For a smooth manifold M, we shall denote by $C^\infty(M, \mathbf{R}^q)$ the space of all smooth mappings $M \to \mathbf{R}^q$ with the usual topology of uniform convergence of all derivatives on compacts.

The mapping $f: M \to \mathbf{R}^q$ may be considered as a collection of functions $f_1, \cdots, f_q: M \to \mathbf{R}^1$, which are called the coordinate functions of the mapping f.

A family of smooth mappings $f_t: M \to \mathbf{R}^q$, $t \in [a, b] \subset \mathbf{R}^1$, is called continuous if the mapping $[a, b] \to C^\infty(M, \mathbf{R}^q)$ which assigns to each $t \in [a, b]$ the mapping $f_t \in C^\infty(M, \mathbf{R}^q)$ is continuous.

1.2.2. In the space $\mathbf{R}^q$ we may identify all of its tangent spaces with $\mathbf{R}^q$ itself. In accordance with this, the differential of a smooth mapping $M \to \mathbf{R}^q$ at a point $m \in M$ is to be considered as a linear mapping of the tangent space of M at the point $m \in M$ into the space $\mathbf{R}^q$.

The rank of a smooth mapping $M \to \mathbf{R}^q$ at the point $m \in M$ is the dimension of the range of the differential at this point, and the corank is the dimension of the kernel of the differential.

1.2.3. A smooth mapping $M \to \mathbf{R}^q$ is called an immersion if its corank is zero at each point.

Two immersions $f, g: M \to \mathbf{R}^q$ are said to be regularly homotopic if there exists a continuous family of immersions $\varphi_t: M \to \mathbf{R}^q$, $t \in [a, b]$, such that $\varphi_a = f$ and $\varphi_b = g$.

1.2.4. A point $m \in M$ is said to be a multiple point of the mapping $f: M \to \mathbf{R}^q$ if there exists a point $m' \in M$, different from m, such that $f(m) = f(m')$.

For a compact manifold M an imbedding $M \to \mathbf{R}^q$ is a smooth mapping $M \to \mathbf{R}^q$ which in the first place is an immersion and, secondly, does not have any multiple points.

Two imbeddings $f, g: M \to \mathbf{R}^q$ are said to be isotopic if there exists a continuous family of imbeddings $\varphi_t: M \to \mathbf{R}^q$, $t \in [a, b]$, such that $\varphi_a = f$ and $\varphi_b = g$.

1.2.5. For each compact manifold M, each integer $l \geqq 0$, and each real number $p \geqq 1$, let us denote by $W^{l,p}(M, \mathbf{R}^q)$ the space of mappings $M \to \mathbf{R}^q$ whose coordinate functions are measurable functions having generalized derivatives up to order l inclusive which are summable to the power p. The space $W^{l,p}(M, \mathbf{R}^q)$, endowed with the natural topology, is a complete normed space (however, there is no canonical norm for this space).

For $f_i \in C^\infty(M, \mathbf{R}^q)$, $i = 1, 2, \cdots$, and $f \in C^\infty(M, \mathbf{R}^q)$, by $f_i \xrightarrow{W^{l,p}} f$ we shall denote convergence in the topology of the space $W^{l,p}(M, \mathbf{R}^q)$.

We shall say that the smooth mapping $f: M \to \mathbf{R}^q$ admits a $W^{l,p}$-approximation by mappings from the set $A \subset C^\infty(M, \mathbf{R}^q)$ if there exists a sequence $f_i \in A$ such that $f_i \xrightarrow{W^{l,p}} f$.

1.2.6. By the n-dimensional volume $V_n(f)$ of a mapping $f: M \to \mathbf{R}^q$, is meant the n-dimensional Hausdorff measure of its image, multiplicity being taken into account. More precisely, the n-dimensional volume $V_n(f)$ is the

smallest of the numbers α such that, for arbitrary pairwise nonintersecting compacts $C_1, \cdots, C_s \subset M$, it is true that $\sum_{j=1}^{s} V_n(\operatorname{rng} f|_{C_j}) \leqq \alpha$, where $V_n(\operatorname{rng} f|_{C_j})$ denotes n-dimensional Hausdorff measure of the image of the compact C_j under the mapping f.

1.3

1.3.1. Let us define the sequence C_q, $q = 0, 1, \cdots$, recursively: $C_0 = C_1 = 1$; $C_q = (1 + C_{q-1})^3, q \geqq 2$.

For an n-dimensional manifold M with boundary B, let us call the mapping $f: M \to \mathbf{R}^2$ *isoperimetric* whenever

$$V_n(f) \leqq C_q(V_{n-1}(f|_B))^{n/(n-1)} \tag{1}$$

($f|_B$ is the restriction to B of the mapping f).

1.3.2. *Let M be a compact n-dimensional $(n > 1)$ manifold with boundary B, and let $g: B \to \mathbf{R}^q$ be a smooth mapping. Then there exists a smooth isoperimetric mapping $M \to \mathbf{R}^q$ which coincides with g on B.*

1.3.3. A. *Suppose M is a compact n-dimensional $(n > 1)$ manifold with boundary B, and $g: B \to \mathbf{R}^q$ is an imbedding. Then, for $q > n + 2$, in order that there exist an isoperimetric imbedding $M \to \mathbf{R}^q$ which coincides with g on B, it is necessary and sufficient that there exist an imbedding $M \to \mathbf{R}^q$ whose restriction to B is isotopic to g.*

B. *Suppose M is a compact n-dimensional $(n > 1)$ manifold with boundary B, and $g: B \to \mathbf{R}^q$ is an immersion. Then, for $q > n$, in order that there exist an isoperimetric immersion $M \to \mathbf{R}^q$ coinciding with g on B, it is necessary and sufficient that there exist an immersion $M \to \mathbf{R}^q$ whose restriction to B is regularly homotopic to g.*

The assertions 1.3.3 follow from §3 and 1.3.2 with the aid of approximation theorems of the following type.

1.3.4. A. *Suppose M is a compact n-dimensional manifold which may be imbedded in $\mathbf{R}^q$, and suppose that the integer $l > 0$ and the real number $p \geqq 1$ satisfy one of the following two conditions:* a) $lp < q - n - 1$; b) $lp = q - n - 1$, $p > 1$. *Then, for $q > n + 1$, any smooth map $M \to \mathbf{R}^q$ admits a $W^{l,p}$ approximation by imbeddings $M \to \mathbf{R}^q$.*

B. *Let M be a compact n-dimensional manifold which can be immersed in $\mathbf{R}^q$, and suppose that the integer $l \geqq 0$ and the real number $p \geqq 1$ satisfy one of the two conditions:* a) $lp < q - n$; b) $lp = q - n$, $p > 1$. *Then, for $q > n$, an arbitrary smooth mapping $M \to \mathbf{R}^q$ admits a $W^{l+1,p}$ approximation by immersions $M \to \mathbf{R}^q$.*

§2. Realization of isoperimetric films by smooth mappings

2.1

2.1.1. The boundary B of a smooth manifold M possesses a closed neighborhood $T \subset M$ which may be decomposed into a product $T = B \times [0, 1]$. Here the boundary B corresponds to the manifold $B \times 0 \subset T = B \times [0, 1]$. This

neighborhood T with a fixed decomposition $T = B \times [0,1]$ is called a tubular neighborhood of the boundary.

2.1.2. *Suppose that the n-dimensional manifold M with boundary B is separated by a closed submanifold A, which does not intersect B, into two manifolds $M_1, M_2 \subset M$, i.e. $M_1 \cup M_2 = M$ and $M_1 \cap M_2 = A$. Let $f: M \to \mathbf{R}^q$ be a continuous mapping whose restrictions $f|_{M_1}$ and $f|_{M_2}$ are smooth mappings. Then there exists a smooth mapping $g: M \to \mathbf{R}^q$ which coincides with f on B and for which $V_n(g) = V_n(f)$.*

PROOF. Let us construct a smooth mapping $p: M \to M$ which leaves B fixed and takes some closed neighborhood $\overline{U} \subset M$ of A into A, and which is one-to-one outside $\overline{U}$. The composition $g = fp: M \to \mathbf{R}^q$ is the desired mapping.

2.1.3. A point $m \in M$ is said to be a critical point of the smooth function $f: M \to \mathbf{R}^1$ if its rank at this point equals zero.

For the smooth manifold M with boundary B and the function $f: M \to \mathbf{R}^1$, let us call a point $x \in \mathbf{R}^1$ a typical point whenever the counterimage $f^{-1}(x) \subset M$ does not contain critical points of the function f or of the function $f|_B: B \to \mathbf{R}^1$. Morse's lemma (see [3]), which is well known in a much more general form under the name of Sard's Theorem (see [4]), asserts that the typical points of a smooth function constitute a set of total measure* in $\mathbf{R}^1$.

2.1.4. *If $x \in \mathbf{R}^1$ is a typical point of the function $f: M \to \mathbf{R}^1$, then the set $f^{-1}(x)$ is a smooth manifold with boundary $f^{-1}(x) \cap B$, where B is the boundary of M.*

The proof is obvious, and well known, from the implicit function theorem.

2.1.5. Let us consider a compact manifold M with boundary B, a smooth function $f: M \to \mathbf{R}^1$, two typical points $x, y \in \mathbf{R}^1$, $x \neq y$, of f and the set $M_0 = f^{-1}[x,y]$ with boundary $B_0 = (B \cap f^{-1}[x,y]) \cup (f^{-1}(x) \cup f^{-1}(y)) \subset M_0$. The set M_0 is not a smooth manifold, because $B_0 \subset M_0$ contains the corners $B \cap f^{-1}(x)$ and $B \cap f^{-1}(y)$. However, these corners may be smoothed:

2.1.6. *There exists a sequence of smooth pairwise diffeomorphic manifolds $M_0^1 \subset \cdots \subset M_0^k \subset \cdots \subset M_0 \subset M$, of total dimension, which do not intersect B_0 and whose boundaries B_0^k converge to B_0 in the following sense.*

There exist a smooth manifold C diffeomorphic to the manifolds B_0^k, which is separated into two submanifolds C_1 and C_2 with a common boundary, a continuous mapping $f: C \to M$ and a sequence of smooth mappings $f_k: C \to M$ with the following properties: the mapping f is a homeomorphism $C \to B_0$ whose restriction to C_1 is a diffeomorphism $C_1 \to B \cap f^{-1}[x,y]$, and whose restriction to C_2 is a diffeomorphism $C_2 \to f^{-1}(x) \cup f^{-1}(y)$. The mappings f_k are diffeomorphisms $C \to B_0^k$, and their restrictions $f_k|_{C_1}$ and $f_k|_{C_2}$ converge in the C^∞ topology to $f|_{C_1}$ and $f|_{C_2}$ respectively.

The proof is standard, and therefore will not be given here.

2.1.7. *Suppose that M is a compact n-dimensional manifold with boundary*

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* *Editor's note.* A thick set in the terminology of W. Ambrose [Trans. Amer. Math. Soc. **61** (1947), 106-121, p. 108, Definition 3].

B, $f: M \to \mathbf{R}^q$ is a smooth mapping for which $V_{n-1}(f|_B) \leqq 1$ and $f_1: B \to \mathbf{R}^1$ is the first coordinate function of f. Then there exist points $x_1 < x_2 < \cdots < x_k \in \mathbf{R}^1$ which are typical for f_1 and satisfy the following conditions:

1) $x_i - x_{i-1} \leqq 1$, $i = 2, \cdots, k$.
2) *The image $f_1(M) \subset \mathbf{R}^1$ is contained in the interval $[x_1, x_k]$.*
3) *If $n = 2$, then each intersection $f_1^{-1}(x_i) \cap B$, $i = 1, \cdots, k$, is empty.*
4) *If $n > 2$, then $\sum_{i=1}^{k} V_{n-2}(f|_{B \cap f_1^{-1}(x_i)}|) \leqq 1$.*

Indeed, from the measurability of the function $\varphi(x) = V_{n-2}(f|_{B \cap f_1^{-1}(x)})$, $x \in \mathbf{R}^1$, the inequality $\int_{-\infty}^{+\infty} \varphi(x)dx \leqq V_{n-1}(f|_B)$, the compactness of M, and 2.1.3, the desired assertion follows.

2.2

2.2.1. Let us pass now to the proof of 1.3.2. We shall prove this by induction with respect to q. For $q = 0$ and 1 the assertion is obvious. Suppose that $q > 1$. If $V_{n-1}(g) = 0$, then as the isoperimetric mapping $f: M \to \mathbf{R}^q$ one needs to take a smooth mapping which carries M into the cone formed by the rectilinear segments joining the origin of coordinates in $\mathbf{R}^2$ to the points $f(b)$, $b \in B$. The smoothness at the vertex of the cone may be achieved by means of 2.1.2.

If $V_{n-1}(g) > 0$, without loss of generality we can assume that $V_{n-1}(g) = 1$. Let us consider a smooth mapping $f: M \to \mathbf{R}^q$ which coincides with g on B, and whose first coordinate function $f_1: M \to \mathbf{R}^1$ and the points $x_1, \cdots, x_k \in \mathbf{R}^1$ satisfy the conclusion of Lemma 2.1.7.

In view of 2.1.4 the sets $N_i = f_1^{-1}(x_i)$ are smooth manifolds, whose boundaries $f_1^{-1}(x_i) \cap B$ we shall designate by D_i. The mapping f takes N_i into the hyperplane $\Gamma_i = x_i \times \mathbf{R}^{q-1} \subset \mathbf{R}^1 \times \mathbf{R}^{q-1} = \mathbf{R}^q$.

Let us apply the induction hypothesis to the mapping $f|_{D_i}: D_i \to \Gamma_i$, and let us construct a smooth mapping $f^0: M \to \mathbf{R}^q$ with first coordinate function $f_1^0 = f_1$, which maps each N_i isoperimetrically into Γ_i, and which coincides with g on B.

Note that for $n = 2$ the manifolds N_i are one-dimensional, and that the inductive hypothesis is not applicable to them. However, in view of 2.1.7 the boundaries D_i are empty, and for manifolds with empty boundaries the assertion 1.3.2 is trivially fulfilled for every n.

Let us put $M_i = (f_1^0)^{-1}[x_i, x_{i+1}]$, $i = 1, \cdots, k-1$, and

$$B_i = ((f_1^0)^{-1}[x_i, x_{i+1}] \cap B) \cup ((f_1^0)^{-1}(x_i) \cup (f_1^0)^{-1}(x_{i+1})).$$

From 2.1.6, it follows that for every $\epsilon > 0$ there exist pairwise nonintersecting smooth manifolds $M_i^0 \subset M_i$ with boundaries B_i^0, for which

$$V_{n-1}(f^0|_{B_i^0}) < V_{n-1}(f^0|_{B_i}) + \varepsilon, \tag{2}$$

$$V_n\left(f^0|_M \setminus \bigcup_{i=1}^{k-1} M_i^0\right) < \varepsilon. \tag{3}$$

Let us denote by $T_i = B_i^0 \times [0, 1] \subset M_i^0$ tubular neighborhoods, and let

us define mappings $\varphi_i : T_i \to \mathbf{R}^q$ by means of the equations $\varphi_i(b,t) = (y^1(1-t) + tx_i, y^2, \cdots, q^q)$, where $b \in B_i^0$ and $t \in [0,1]$, and $f^0(b) = (y^1, \cdots, y^q) \in \mathbf{R}^q$, $i = 1, \cdots, k-1$.

The mapping φ_i takes the manifold $B_i^0 \times 1 \subset T_i$ into the hyperplane Γ_i. From the inductive hypothesis it follows that the mapping $\varphi_i|_{B_i^0 \times 1} : B_i^0 \times 1 \to \Gamma_i$ may be extended to a smooth isoperimetric mapping $\psi_i : M_i^0 \setminus B_i^0 \times [0,1) \to \Gamma_i$ (the set $M_i^0 \setminus B_i^0 \times [0,1)$ is a manifold diffeomorphic to M_i^0).

Let us consider the mapping f^1, which coincides with φ_i on T_i, coincides with ψ_i on $M_i^0 \setminus B_i^0 \times [0,1)$, and coincides on the complement of $\bigcup_{i=1}^{k-1} M_i^0 \subset M$ with f^0. The corners of this mapping occur on the manifold $\bigcup_{i=1}^{k-1} (T_i \times 0 \cup T_i \times 1)$, and may be smoothed by employing 2.1.2, thus yielding a smooth mapping f^2 with $V_n(f^2) = V_n(f^1)$.

Let us prove that for sufficiently small ϵ in the formulas (2) and (3) the mapping f^2 which appears is indeed isoperimetric.

As a matter of fact,

$$V_n(f^2) = V_n(f^1) = V_n\left(f^0|_{M \setminus \bigcup_{i=1}^{k-1} M_i^0}\right) + \sum_{i=1}^{k-1} V_n(\varphi_i) + \sum_{i=1}^{k-1} V_n(\psi_i); \tag{4}$$

$$V_n(\varphi_i) \leqslant V_{n-1}(f^0|_{B_i^0}) \leqslant V_{n-1}(f^0|_{B_i}) + \varepsilon; \tag{5}$$

$$V_n(\psi_i) \leqslant C_{q-1}(V_n(\psi_i|_{B_i^0 \times 1}))^{\frac{n}{n-1}} \leqslant C_{q-1}(V_{n-1}(f^0|_{B_i}) + \varepsilon)^{\frac{n}{n-1}}; \tag{6}$$

$$\sum_{i=1}^{k-1} V_{n-1}(f^0|_{B_i}) \leqslant 1 + C_{q-1} \sum_{i=1}^{k} (V_{n-2}(f|_{B_i}))^{\frac{n-1}{n-2}} \leqslant 1 + C_{q-1}. \tag{7}$$

From (3)–(7) we obtain

$$V_n(f^2) \leqq \delta + 1 + C_{q-1} + C_{q-1}(1 + C_{q-1})^{n/(n-1)},$$

where $\delta \to 0$ as $\epsilon \to 0$, and hence for sufficiently small ϵ we obtain the desired inequality: $V_n(f^2) < (1 + C_{q-1})^3$.

2.2.2. *Under the hypothesis of* 1.3.2, *and for* $V_{n-1}(g) > 0$, *there exists a smooth mapping* $f : M \to \mathbf{R}^q$ *which is an extension of* g *and for which the strict inequality* $V_n(f) < C_q(V_{n-1}(g))^{n/(n-1)}$ *holds.*

The proof obviously follows from the preceding considerations.

2.3

2.3.1. *Suppose that* B_0 *is a closed* $(n-1)$*-dimensional manifold, and that* $g_0 : B_0 \to \mathbf{R}^q$ *is a smooth mapping. Then an isoperimetric cone may be placed on* $g_0(B_0) \subset \mathbf{R}^q$, *that is, a smooth mapping* $f : B_0 \times [0,1] \to \mathbf{R}^q$ *which coincides with* g_0 *on* $B_0 = B_0 \times 0$ *and which takes* $B_0 \times 1$ *into the origin of coordinates, and satisfies the inequality*

$$V_n(f) \leqq C_q(V_{n-1}(g_0))^{n/(n-1)}.$$

For the proof, we apply 1.3.2 to the mapping $g : B_0 \times 0 \cup B_0 \times 1 \to \mathbf{R}^q$ which

coincides with g_0 on $B_0 \times 0 = B_0$, and which takes $B_0 \times 1$ into the origin of coordinates.

2.3.2. *For each convex set $A \subset \mathbf{R}^q$, the smooth mapping $g: B \to A$ may be extended to an isoperimetric mapping $f: M \to A$.*

The proof follows from 1.3.2 and the following fact (see [**5**]):

The mapping $\mathbf{R}^q \to \overline{A}$ ($\overline{A}$ denoting the closure) which assigns to each point $x \in \mathbf{R}^q \setminus \overline{A}$ the closest point in $\overline{A}$ and leaves $\overline{A}$ fixed is a Lipschitzian mapping, with a Lipschitz constant which does not exceed unity.

2.4

Let us note certain generalizations of Proposition 1.3.2, which may be proved along the same lines as 1.3.2.

2.4.1. *Suppose that M is a smooth manifold, N a submanifold, and $U \subset M$ an open set with compact closure. Suppose that the closure in N of the intersection $U \cap N$ is compact and does not intersect the boundary of N. Suppose that $g: N \to \mathbf{R}^q$ is a smooth mapping, and that the supports of its coordinate functions lie in $U \cap N$. Then there exists a smooth $f: M \to \mathbf{R}^q$, which coincides with g on N, the supports of whose coordinate functions lie in U, and which for each $p > 1$ satisfies the inequality*

$$V_p(f) \leqq C_q(V_{p-1}(g))^{p/(p-1)}.$$

2.4.2. The mapping $M \to \mathbf{R}^q$ is said to be a proper mapping if the counterimage of each compact in $\mathbf{R}^q$ is a compact in M.

Suppose that M is a not necessarily compact n-dimensional ($n > 1$) manifold with boundary B. Then a proper mapping $B \to \mathbf{R}^q$ can be extended to a proper isoperimetric mapping $B \to \mathbf{R}^q$.

This proposition is meaningful only when $V_{n-1}(g) < \infty$. Let us also note that in the case of a compact boundary B the assertion 2.4.2 reduces to 1.3.2.

2.4.3. The assertion 1.3.2 and its generalizations carry over, without essential changes, to the case of a finite degree of smoothness.

§3. Approximation of smooth mappings by imbeddings and immersions

3.1

3.1.1. A pair of numbers l, p, where $l \geqq 0$ is an integer and $p \geqq 1$ is a real number, will be said to be compatible with an integer $k > 0$ if one of the following two conditions holds: a) $lp < k$; b) $lp = k$, $p > 1$.

3.1.2. *There exists a sequence of smooth functions $H_i: [0, \infty) \to \mathbf{R}^1$, $i = 1, 2, \cdots$, satisfying the following three conditions.*

1. *Each function H_i is identically equal to unity in a neighborhood of zero.*
2. *The function H_i equals zero outside the interval $[0, 1/i)$.*
3. *Given the Euclidean space $\mathbf{R}^n$, its subspace $\mathbf{R}^{n-k}$, and a point $x \in \mathbf{R}^n$, denote by $r(x)$ the distance of x from $\mathbf{R}^{n-k} \subset \mathbf{R}^n$. If the pair l, p is compatible with k, then*

A. *for each smooth function $\varphi: \mathbf{R}^n \to \mathbf{R}^1$ with compact support, it is true that $\varphi(x) H_i(r(x)) \xrightarrow{w^{l,p}} 0$ as $i \to \infty$;*

B. *for each smooth function* $\varphi: \mathbf{R}^n \to \mathbf{R}^1$ *with compact support which equals zero on* $\mathbf{R}^{n-k} \subset \mathbf{R}^n$, $\varphi(x) H_i(r(x)) \xrightarrow{W^{l+1,p}} 0$ *as* $i \to \infty$.

PROOF. Suppose that $\alpha, \beta: [0, \infty) \to \mathbf{R}^1$ are standard smooth functions with the following properties:

$$\alpha(x) \geqq 0,\ \beta(x) \geqq 0 \text{ for each } x \in [0, \infty);$$
$$\alpha(x) = 1 \quad \text{for } x \leqq 1/4;$$
$$\alpha(x) = 0 \quad \text{for } x \geqq 1/3;$$
$$\beta(0) = 0;$$
$$\beta(x) = 1 \quad \text{for } x \geqq 1.$$

Let us define a smooth function $\gamma: [0, \infty) \to \mathbf{R}^1$ by means of the formula

$$\gamma(x) = \alpha(x) \ln|\ln x|, \qquad x \in [0, \infty).$$

Define $H_i(x) = \beta(\gamma(ix)/i)$, $i = 1, 2, \cdots$; $x \in [0, \infty)$. The sequence of functions H_i is the desired sequence.

3.1.3. Consider a submanifold N of a smooth manifold M, and a sequence of compacta $C_i \subset N$, $i = 1, 2, \cdots$, such that each C_i is contained in the interior of C_{i+1} and $\bigcup_1^\infty C_i = N$. By $\overline{N \setminus C_i}$ we denote the closure of the set $N \setminus C_i \subset M$, $i = 1, 2, \cdots$.

The intersection $\bigcap_1^\infty \overline{N \setminus C_i}$ is called the limiting set of the submanifold N. Let us recall that the definition of a submanifold assumes that a submanifold N does not intersect its limiting set, so that the limiting set coincides with the set $\overline{N} \setminus N \subset M$.

3.1.4. The codimension $\operatorname{codim} N$ of a submanifold N of M is the difference $\dim M - \dim N$.

3.1.5. *Suppose that M is a compact manifold, N a submanifold of positive codimension, $C \subset M$ a compact which contains the limiting set of N, and l, p a pair of numbers which correspond to $k = \operatorname{codim} N$. Then the following assertions are true.*

A. *Given smooth functions $f, g: M \to \mathbf{R}^1$ which coincide in a neighborhood of the compact $C \subset M$, there exists a sequence of smooth functions $\varphi_i: M \to \mathbf{R}^1$, $i = 1, 2, \cdots$, possessing the following properties: 1) $\varphi_i \xrightarrow{W^{l,p}} f$ as $i \to \infty$; 2) each function φ_i coincides with g in a neighborhood of the set $C \cup N \subset M$.*

B. *Given smooth functions $f, g: M \to \mathbf{R}^1$ which coincide in a neighborhood of the compact $C \subset M$ and on the manifold $N \subset M$, there exists a sequence of smooth functions $\varphi_i: M \to \mathbf{R}^1$, $i = 1, 2, \cdots$, possessing the following two properties: 1) $\varphi_i \xrightarrow{W^{l+1,p}} f$ as $i \to \infty$; 2) each function φ_i, $i = 1, 2, \cdots$, coincides with g in a neighborhood of the set $C \cup N \subset M$.*

PROOF. Let us introduce an infinitely differentiable Riemann metric on the manifold M, and for each point $x \in M$ let $\rho(x)$ denote the distance from x to $N \subset M$. The sequence of functions $\varphi_i(x) = f(x) + H_i(\rho(x))(g(x) - f(x))$, $x \in M$, where H_i is the sequence of functions possessing properties 1, 2 and 3 of Lemma 3.1.2, is the desired sequence. Indeed, for each point $m \in N \subset M$

there exists a diffeomorphism Φ of the space $\mathbf{R}^n$, $n = \dim M$, into M which takes the origin of coordinates into the point $m \in N$, and the subspace $\mathbf{R}^{n-k} \subset \mathbf{R}^n$ into N, such that $r(x) = \rho(\Phi(x))$, $x \in \mathbf{R}^n$. From this, and from part 3 of Lemma 3.1.2, it follows that $H_i(\rho(x)) \xrightarrow{W^{l,p}} 0$ as $i \to \infty$ in case A of Lemma 3.1.5, and $H_i(\rho(x)) \xrightarrow{W^{l+1,p}} 0$ as $i \to \infty$ in case B; thus Lemma 3.1.5 is proved.

3.1.6. Let us consider a smooth manifold M and a set $A \subset M$. By a vector field X on A we shall understand a continuous function which assigns to each point $a \in A$ a tangent vector X_a of M at a.

We shall say that the smooth function $f: M \to \mathbf{R}^1$ has, in $U \subset M$, a positive derivative along X if for an arbitrary point $m \in U \cap A \subset M$ and vector X_m of X at the point m the derivative $X_m f$ of f in the direction of X_m is positive. We recall that $X_m f = \langle X_m, df \rangle$, where df is the (exterior) differential of f and $\langle X_m, df \rangle$ is the value of the linear form df at X_m.

For a set $A \subset M$ which is a smooth submanifold of M, let us call a vector field X on A transversal to A if at no point $a \in A$ is the vector X_a tangent to A (in particular, $X_a \neq 0$).

The following fact is obvious.

3.1.7. *Let N be a submanifold of a smooth manifold M, $C \subset M$ a closed set containing the limiting points of N, and X a vector field on N which is transversal to N. Then for every function $f: M \to \mathbf{R}^1$ which has a positive derivative on C along X there exists a smooth function which coincides with f in some neighborhood of C and on N, and which has on M a positive derivative along X.*

3.1.8. *Suppose M is a compact manifold; $N_1, \cdots, N_s, \cdots, N_t$ are submanifolds; $C_1, \cdots, C_t$ are compacts such that $C_\nu = C_{\nu-1} \cup N_{\nu-1}$, $\nu = 2, \cdots, t$, and X is a vector field on the set $\bigcup_1^s N_\nu \subset M$ which is transversal to the manifolds N_ν, $\nu = 1, \cdots, s$.*

A. *If the number $k = \min_{1 \leq \nu \leq t} \operatorname{codim} N_\nu$ is positive, then for every pair l, p corresponding to k, and for any smooth functions $f, g: M \to \mathbf{R}^1$ which coincide in a neighborhood of the compact C_1, there exists a sequence of smooth functions $\varphi_i: M \to \mathbf{R}^1$, $i = 1, 2, \cdots$, which satisfy two properties: 1) $\varphi_i \xrightarrow{W^{l,p}} f$; 2) each function φ_i coincides with g in a neighborhood of the set $C_t \cup N_t$.*

B. *If the number $k = \min_{1 < \nu < s} \operatorname{codim} N_\nu$ is positive, then for each pair of numbers l, p corresponding to k and each smooth function $f: M \to \mathbf{R}^1$ which has on C_1 a positive derivative along X, there exists a sequence of smooth functions $\varphi_i: M \to \mathbf{R}^1$, $i = 1, 2, \cdots$, possessing the following two properties: 1) $\varphi_i \xrightarrow{W^{l+1,p}} f$; 2) each function φ_i has on M a positive derivative along X, and coincides with f in a neighborhood of C_1.*

The proof will be achieved by means of induction: in case A, with respect to $\nu = 1, \cdots, t$, and in case B, with respect to $\nu = 1, \cdots, s$.

Inductive Hypotheses A$^\nu$: There exists a sequence of functions $\varphi_i \xrightarrow{W^{l,p}} f$ which coincide with g on $C_\nu = C_{\nu-1} \cup N_{\nu-1} \subset M$.

From Lemma A of 3.1.5 it follows that for each $i = 1, 2, \cdots$ there exists a sequence of functions $\psi_{i,j} \xrightarrow{W^{l,p}} \varphi_i$ as $j \to \infty$ which coincide with g in a neighborhood of $C_\nu \cup N_\nu$. It is easy to see that from the double sequence $\psi_{i,j}$ one

can select a subsequence which satisfies the hypothesis $A^{\nu+1}$. Thus case A is taken care of.

Inductive Hypothesis B^{ν}: There exists a sequence $\varphi_i \xrightarrow{W^{l+1,p}} f$ which coincides with f in a neighborhood of C_1 and which has, on $C_{\nu+1}$, a positive derivative along X.

Applying 3.1.7, let us construct for each $i=1,2,\cdots$ a function ψ_i which coincides with φ_i in a neighborhood of $C_{\nu+1}$ and on $N_{\nu+1}$, and having in $C_{\nu+2}=C_{\nu+1}\cup N_{\nu+1}$ a positive derivative along X. In view of B of 3.1.5 there exists for each $i=1,2,\cdots$ a sequence of functions $\psi_{i,j} \xrightarrow{W^{l+1,p}} \varphi_i$ as $j\to\infty$, coinciding with ψ_i in a neighborhood of the compact $C_{\nu+2}$. From the double sequence $\psi_{i,j}$ one can extract a subsequence which satisfies hypothesis $B^{\nu+1}$. Thus case B is taken care of.

3.2

3.2.1. A smooth mapping $h: M\to \mathbf{R}^{q-1}$ is called typical if there exist smooth submanifolds $N_1,\cdots,N_s,\cdots,N_t$ of M satisfying the following conditions.

1. The sets $\bigcup_{\mu=1}^{\nu} N_\mu \subset M$, $\nu=1,\cdots,t$, are closed.

2A. The set $M\setminus\bigcup_{\nu=1}^{t} N_\nu$ does not contain multiple points of the mapping h.

2B. The restrictions $h|_{M\setminus\bigcup_1^s N_\nu}$ and $h|_{N_\nu}$, $\nu=1,\cdots,s$, are immersions.

3A. $\operatorname{codim} N_\nu \geq q-\dim M-1$, $\nu=1,\cdots,t$.

3B. $\operatorname{codim} N_\nu \geq q-\dim M$, $\nu=1,\cdots,s$.

4B. At points of the sets $\bigcup_1^s N_\nu$ the mapping h has positive corank.

3.2.2. *The set of all typical maps of a compact manifold M constitutes an open everywhere dense set in the space $C^\infty(M,\mathbf{R}^{q-1})$.*

The proof is obvious, and follows from the results of Thom and Boardman concerning singular Σ^1 (see [**1**, **4**]).

3.2.3. *Let M be a compact n-dimensional manifold with boundary B, and let $f,g: M\to\mathbf{R}^q$ be smooth maps which coincide in a neighborhood of B. Then the following assertions are true.*

A. *If g is an imbedding and $k=q-n-1>0$, then there exists a sequence of smooth functions φ^i (we shall write the index i as a superscript, in order to avoid confusion with the number which is attached to the coordinate functions), $i=1,2,\cdots$, obeying the following three conditions:*

1) *For pairs of numbers l, p corresponding to k it is true that $\varphi^i \xrightarrow{W^{l,p}} f_1$ as $i\to\infty$, where f_1 is the first coordinate function of f.*

2) *The mappings $M\to\mathbf{R}^q$ with coordinate functions $\varphi^i, g_2,\cdots,g_q$ $(i=1,2,\cdots)$ are imbeddings ($g_1,\cdots,g_q$ being the coordinate functions of the imbedding g).*

3) *Each function φ^i coincides with f_1 in a neighborhood of B.*

B. *If g is an immersion and $k=q-n>0$, then there exists a sequence of smooth functions φ^i, $i=1,2,\cdots$, satisfying the following three conditions:*

1) *For a pair l, p which corresponds to k it is true that $\varphi^i \xrightarrow{W^{l+1,p}} f_1$ as $i\to\infty$.*

2) *The mappings $M\to\mathbf{R}^q$ with coordinate functions $\varphi^i, g_2,\cdots,g_q$ $(i=1,\cdots)$ are immersions.*

3) *Each function φ^i coincides with f_1 in a neighborhood of B.*

PROOF. It may be supposed that the mappings f and g coincide in a closed tubular neighborhood T of the boundary B. Let us consider the mapping $h: M \to \mathbf{R}^{q-1}$ with coordinate functions $h_1 = g_2, \cdots, h_{q-1} = g_q$. Employing 3.2.2, we can arrange that the restriction $h|_{M \setminus T}: M \setminus T \to \mathbf{R}^{q-1}$ be a typical map. Let us consider for this purpose mappings of the submanifolds $N_1, \cdots, N_s, \cdots, N_t$ of the manifold $M \setminus T \subset M$ satisfying the conditions which were specified in 3.2.1, together with the compacta $C_1 = T \subset M$, $C_2 = C_1 \cup N_1 \subset M, \cdots, C_t = C_{t-1} \cup N_{t-1} \subset M$. Applying to f and g_1 the result A of 3.1.8, we obtain in case A the sequence φ^i, $i = 1, 2, \cdots$, having the properties 1) – 3).

Let us construct, on the set $\bigcup_1^s N_\nu \subset M \setminus T \subset M$, a vector field X, generated by the vectors X_m, $m \in \bigcup_{\nu=1}^s N_\nu$, for which $X_m g_1 = 1$ and $X_m h_1 = X_m h_2 = \cdots = X_m h_{q-1} = 0$. This construction is well defined, in view of 4.B of 3.2.1, and in view of the fact that $g: M \to \mathbf{R}^q$ is an immersion. Consequently the corank of h does not exceed unity at any point. Applying B of 3.1.8 to the function f_1, we obtain the required sequence φ^i in case B.

3.2.4. *Suppose that M is a compact n-dimensional manifold, $f: M \to \mathbf{R}^q$ is a smooth map, and $\varphi^i: M \to \mathbf{R}^1$, $i = 1, 2, \cdots$, is a sequence of smooth functions. Denote by f^i a sequence of smooth maps $M \to \mathbf{R}^q$ with coordinate functions φ^i, $f_2, \cdots, f_q$. If $\varphi^i \xrightarrow{W^{1,1}} f_1$ as $i \to \infty$, then $V_n(f^i) \to V_n(f)$.*

The proof follows from the fact that the first derivatives of each coordinate function of the smooth map appear linearly in the well-known formula for the computation of the volume of this map.

3.2.5. PROOF OF THE FUNDAMENTAL RESULTS. *Suppose M is a compact n-dimensional manifold, and $f: M \to \mathbf{R}^q$ is a smooth map.*

A. *If $q - n > 1$ and there exists an imbedding $g: M \to \mathbf{R}^q$ which coincides in a neighborhood of the boundary B with the mapping f, and if the integer $l \geqq 0$ and the real number $p \geqq 1$ satisfy one of the following two conditions:* a) $lp < q - n - 1$; b) $lp = q - n - 1$, $p > 1$, *then there exists a sequence of imbeddings $g^i: M \to \mathbf{R}^q$ $(i = 1, 2, \cdots)$ which coincide with f in a neighborhood of B and possess the following properties:* 1) $g^i \xrightarrow{W^{l,p}} f$ *as* $i \to \infty$; 2) *for $q \geqq n + 3$ it is true that $V_n(g^i) \to V_n(f)$.*

B. *If $q - n > 0$ and there exists an immersion $g: M \to \mathbf{R}^q$ which coincides with f in a neighborhood of B, and if the integer $l \geqq 0$ and the real number $p \geqq 1$ satisfy one of the following two relations:* a) $lp < q - n$; b) $lp = q - n$, $p > 1$, *then there exists a sequence of immersions $g^i: M \to \mathbf{R}^q$ $(i = 1, 2, \cdots)$ which coincide with f in a neighborhood of B and possess the following properties:* 1) $g^i \xrightarrow{W^{l+1,p}} f$ *as* $i \to \infty$; 2) $V_n(g^i) \to V_n(f)$.

The proof will be carried out by induction with respect to the number of coordinate functions of the mapping f.

Inductive Hypothesis A^r: There exist sequences of smooth functions $\varphi_1^i, \cdots, \varphi_r^i$ $(i = 1, 2, \cdots)$ with the following three properties:

1) $\varphi_1^i \xrightarrow{W^{l,p}} f_1, \cdots, \varphi_r^i \xrightarrow{W^{l,p}} f_r$.

2) The mappings $\psi^i: M \to \mathbf{R}^q$ with coordinate functions $\varphi_1^i, \cdots, \varphi_r^i, g_{r+1}, \cdots,$

g_q are imbeddings which coincide with f in a neighborhood of the boundary.

3) The mappings h^i with coordinate functions $\varphi_1^i, \cdots, \varphi_r^i, f_{r+1}, \cdots, f_q$ are such that $V_n(h^i) \to V_n(f)$ for $q \geq n+3$.

From A of 3.2.3 it follows that there exists a sequence of functions $\varphi_{r+1}^j \xrightarrow{W^{l,p}} f_{r+1}$ such that the mappings $\psi^{i,j}$ with coordinate functions $\varphi_1^i, \cdots, \varphi_r^i, \varphi_{r+1}^j, g_{r+2}, \cdots, g_q$ are imbeddings which coincide with f in a neighborhood of B. For $q \geq n+3$ the pair $l=1$, $p=1$ satisfies condition a) (that is, the condition $lp < q-n-1$), and, in view of 3.2.4 for the mappings $h^{i,j}$ with coordinate functions $\varphi_1^i, \cdots, \varphi_r^i, \varphi_{r+1}^j, f_{r+2}, \cdots, f_q$, we obtain that $V_n(h^{i,j}) \to V_n(h^i)$ as $j \to \infty$, $i = 1, 2, \cdots$. From the sequences $\varphi_1^i, \cdots, \varphi_r^i, \varphi_{r+1}^j$ we extract subsequences which satisfy the hypothesis A^{r+1}, and the proof is concluded in case A.

The proof in case B can be carried out similarly.

3.3

3.3.1. Let us formulate two well-known facts, the first of which was established by Thom [**3**], and the second by Smale [**9**].

A. *Suppose that M is a compact manifold with boundary B, $f: M \to \mathbf{R}^q$ is an imbedding, and $g: B \to \mathbf{R}^q$ is an imbedding which is isotopic to the imbedding $f|_B: B \to \mathbf{R}^q$. Then there exists an imbedding $M \to \mathbf{R}^q$ which coincides with g on B.*

B. *Suppose that M is a compact n-dimensional manifold with boundary B, $f: M \to \mathbf{R}^q$ is an immersion, and $g: B \to \mathbf{R}^q$ is an immersion which is regularly homotopic to the immersion $f|_B: B \to \mathbf{R}^q$. Then for $q > n$ there exists an immersion $M \to \mathbf{R}^q$ which coincides with g on B.*

3.3.2. The proof of 1.3.3 obviously follows from 2.2.2, 3.3.1 and 3.2.5.

3.3.4. The proof of 1.3.4 obviously follows from the following lemmas:

A. *Suppose that M is a compact n-dimensional manifold with boundary, for which there exists an imbedding $g: M \to \mathbf{R}^q$. Then for $q-n > 1$, and for any smooth map $f: M \to \mathbf{R}^q$ and pair of numbers l, p corresponding to $k = q-n-1$, there exists a sequence of functions $\varphi^i \xrightarrow{W^{l,p}} f_1$ such that the mappings $M \to \mathbf{R}^q$ with coordinate functions $\varphi^i, g_2, \cdots, g_q$ $(i = 1, 2, \cdots)$ are imbeddings.*

B. *Suppose that M is a compact n-dimensional manifold for which there exists an immersion $g: M \to \mathbf{R}^q$. Then, for $q > n$, for each smooth map $f: M \to \mathbf{R}^q$ and pair of numbers l, p corresponding to $k = q-n$ there exists a sequence of functions $\varphi^i \xrightarrow{W^{l+1,p}} f_1$ such that the mappings $M \to \mathbf{R}^q$ with coordinate functions $\varphi^i, g_2, \cdots, g_q$ $(i = 1, 2, \cdots)$ are immersions.*

The proof of these lemmas is analogous (and, indeed, somewhat simpler, because here one does not bother with the behavior of the functions on the boundary of the manifold) to the proof of assertion 3.2.3.

§4. Topological addenda

4.1

4.1.1. The following condition is implicitly contained in assertion 1.3.2: there exists a compact manifold M whose boundary is diffeomorphic to a given manifold B.

The problem of the existence of such an M was completely solved by Thom (see [3]).

For each n there exist only a finite number (depending on n) of closed $(n-1)$-dimensional manifolds no two of which are together the boundary of a compact n-dimensional manifold.

4.1.2. A. The conditions of assertions A of 1.3.3 and A of 1.3.4 lead to the question of the existence of an imbedding $M \to \mathbf{R}^q$ satisfying additional conditions concerning the behavior on the boundary. The solution of this kind of question has formed the subject of many papers, but sufficiently effective conditions have not yet been found in the general case. Let us formulate a theorem of Haefliger [6].

An imbedding of an $(n-1)$-dimensional sphere in $\mathbf{R}^q$ for $q \geqq 3n/2+2$ may be extended to an imbedding of the n-dimensional ball in $\mathbf{R}^q$.

B. The problem of the existence of an immersion $M \to \mathbf{R}^q$ with prescribed behavior on the boundary leads, in algebraic topology, to the immersion theory of Smale and Hirsch (see [7, 9]; in this connection, see also §4.3 below).

4.2

Let us show that the requirements made on the numbers l and p in B of 1.3.4 are essential.

4.2.1. *Let D^k be the unit ball of dimension k with boundary $S^{k-1} \subset D^k$ and D^r the unit ball of dimension r; suppose $U \subset \mathbf{R}^N$ is an open set and $f: D^k \times D^r \to \mathbf{R}^N$ is a smooth mapping which takes $S^{k-1} \subset D^k = D^k \times 0 \subset D^k \times D^r$ into U. Assume that for the integer $l \geqq 0$ and the real number $p \geqq 1$ satisfying one of the following two conditions:* a) $lp > k-1$; b) $l = k-1$, $p = 1$, *there exists a sequence of smooth maps $f_i: D^k \times D^r \to U \subset \mathbf{R}^N$ such that $f_i \xrightarrow{W^{l,p}} f$. Then the map $f|_{S^{k-1}}: S^{k-1} \to U$ is homotopic in U to zero.*

Proof. From the imbedding theorems it follows that on some $(k-1)$-dimensional sphere $S \subset D^k \times D^r$ concentric with S^{k-1} and arbitrarily near to it the mappings f_i converge in the uniform C^0-topology to the map $f|_S$. But from this it follows that $f|_S$, and indeed the map $f|_{S^{k-1}}$ which is homotopic to it, is homotopic to zero in U.

4.2.2. Let us introduce the standard coordinates $x_1, \cdots, x_k$, $y_1, \cdots, y_r$ in the product $D^k \times D^r$. Let us compare the smooth mapping $f: D^k \times D^r \to \mathbf{R}^q$ with the map $f_*: D^k \times D^r \to \mathbf{R}^{q(k+r)} = \mathbf{R}^q \times \cdots \times \mathbf{R}^q$ which associates with each point $d \in D^k \times D^r$ the vector $((\partial f/\partial x_1)(d), \cdots, (\partial f/\partial y_r)(d))$ consisting of the partial derivatives of the mapping f (we shall consider $\mathbf{R}^{q(k+r)}$ as the space consisting of collections of vectors $A_1, \cdots, A_{r+k} \in \mathbf{R}^q$). Let us denote by U a subspace of $\mathbf{R}^{q(k+r)}$ whose generators are linearly independent vectors $A_1, \cdots, A_{r+k} \in \mathbf{R}^q$.

4.2.3. *For $q = 2k+r-1$ there exists a smooth mapping $f: D^k \times D^r \to \mathbf{R}^q$ which does not admit $W^{l+1,p}$-approximations by immersions if the numbers l and p satisfy one of the following two conditions:* a) $lp > k-1$; b) $l = k-1$, $p = 1$.

PROOF. The condition that the mapping f is an immersion is equivalent to the fact that the image of the mapping f_* lies in $U \subset \mathbf{R}^{q(k+r)}$. It is known (see, for example, [2]) that the $(k-1)$-dimensional homotopy group of the set U for $q = 2k + r - 1$ is not zero. From this fact, and from Smale's theorem [9], it follows that there exists a smooth map $f: D^k \times D^r \to \mathbf{R}^q$ whose restriction to some neighborhood of the sphere $S^{k-1} \subset D^k \subset D^k \times D^r$ is an immersion, and for which the mapping $f_*|_{S^{k-1}}: S^{k-1} \to U$ is contractible. The existence of an approximating sequence of immersions $f_i \xrightarrow{W^{l+1,p}} f$ implies the existence of a sequence of smooth mappings $(f_i)_*: D^k \times D^r \to U \subset \mathbf{R}^{q(k+r)}$ for which $(f_i)_* \xrightarrow{W^{l,p}} f_*$, which contradicts 4.2.1.

4.3

4.3.1. A stratification in a smooth manifold M is a finite sequence of pairwise nonintersecting smooth submanifolds $S_0, \cdots, S_i, \cdots, S_t$, such that each set $\bigcup_0^i S_j \subset M$, $i = 0, \cdots, t$, is closed. The manifolds $S_i \subset M$, $i = 0, \cdots, t$, are called the strata of the stratification, and the set $\bigcup_0^t S_i \subset M$ is called the carrier of the stratification.

4.3.2. A point of a smooth manifold M is called a singular point of the linear differential forms $L_1, \cdots, L_q$, given on M, if at this point there is a nonzero tangent vector for which every one of these forms has the value zero.

We shall say that the forms $L_1, \cdots, L_q$ constitute a complete set of forms on M if the set of their singular points is empty.

4.3.3. A stratification in M will be called compatible with the forms $L_1, \cdots, L_i, \cdots, L_q$ on M provided that its carrier coincides with the set of singular points of the forms L_i $(i = 1, \cdots, q)$, and that for each stratum S the forms which are induced on S by the forms $L_1, \cdots, L_q$ constitute a complete set of forms in S.

4.3.4. THEOREM OF HIRSCH. *If on the smooth n-dimensional manifold M there exist q continuous linear differential forms which constitute a complete set of forms in M, then for $q \geqq n+1$ there exists an immersion $M \to \mathbf{R}^q$.*

This theorem is one of the fundamental results of the immersion theory of Smale and Hirsch (see [7, 9]). The proof given in §§4.3.5–4.3.8 is considerably simpler than the first proof given by Hirsch.

4.3.5. *Suppose that $f_1, \cdots, f_i, \cdots, f_p$ are smooth functions, and $L_1, \cdots, L_j, \cdots, L_r$ are smooth linear differential forms on the n-dimensional manifold M. If $p + r \geqq n$, then there exist C^∞-approximations of the f_i by functions $\bar{f}_i$, and of the L_j by forms $\bar{L}_j$, such that there exists a stratification on M compatible with the forms $d\bar{f}_1, \cdots, d\bar{f}_p$, $\bar{L}_1, \cdots, \bar{L}_q$.*

The proof of this intuitively obvious fact will be given in a more general situation in another paper. (The case $p = 0$ reduces to 3.2.2; let us remark also that the case $p > 0$ may easily be proved by the approach of Thom and Boardman (see [1]).)

4.3.6. *Suppose that X is a vector field on the carrier of the stratification $S_0, \cdots, S_i, \cdots, S_t$ in M which is transversal to each stratum S_i $(i = 0, \cdots, t)$.*

Then there exists a smooth function $M \to \mathbf{R}^1$ which has a positive derivative along X (see 3.1.6).

The proof can be carried out by an obvious induction, employing the assertion of 3.1.7.

4.3.7. *Suppose that $f_1, \cdots, f_i, \cdots, f_p$ are smooth functions, and $L_1, \cdots, L_j, \cdots, L_r$ are smooth linear differential forms on an n-dimensional manifold M. If the forms $df_1, \cdots, df_p, L_1, \cdots, L_r$ constitute a complete set of forms in M, then for $p+r \geqq n+1$ there exist functions $\bar{f}_1, \cdots, \bar{f}_i, \cdots, \bar{f}_p, f$ and forms $\overline{L_2}, \cdots, \overline{L_j}, \cdots, \overline{L_r}$ such that the forms $d\bar{f}_1, \cdots, d\bar{f}_p, df, \overline{L_2}, \cdots, \overline{L_r}$ constitute a complete set of forms in M.*

PROOF. With the aid of 4.3.5 we can construct functions $\bar{f}_1, \cdots, \bar{f}_p$ and forms $\overline{L_2}, \cdots, \overline{L_r}$ such that the forms $d\bar{f}_1, \cdots, d\bar{f}_p, L_1, \overline{L_2}, \cdots, \overline{L_r}$ constitute a complete set, and such that there exists a stratification compatible with the forms $d\bar{f}_1, \cdots, d\bar{f}_p, \overline{L_2}, \cdots, \overline{L_r}$. Let us consider, on the carrier of this stratification, the vector field generated by the vectors at which the forms $d\bar{f}_1, \cdots, d\bar{f}_p, \overline{L_2}, \cdots, \overline{L_r}$ are zero, while L_1 is one. Applying 4.3.6, we then obtain a function f on M whose derivative along the vectors of this field of vectors is positive.

The functions $\bar{f}_1, \cdots, \bar{f}_p, f$ and the forms $\overline{L_2}, \cdots, \overline{L_r}$ constructed in this way are the desired ones.

4.3.8. PROOF OF THE THEOREM OF HIRSCH. Without loss of generality we may suppose that on M there exist q smooth forms which constitute a complete set. By means of an obvious induction, employing 4.3.7, we obtain that there exist smooth functions $f_1, \cdots, f_q$ on M whose differentials constitute a complete set of forms. The mapping $f: M \to \mathbf{R}^q$ with coordinate functions $f_1, \cdots, f_q$ is the desired one.

4.3.9. Let us formulate a more general result, which may be obtained by means of analogous considerations.

Given a real vector bundle X over the smooth manifold M, let us denote by $\Gamma(X)$ the set of infinitely differentiable sections of this bundle. We shall say that a set $A \subset \Gamma(X)$ generates $\Gamma(X)$ if for an arbitrary section $\gamma \in \Gamma(X)$ there exist sections $\gamma_1, \cdots, \gamma_n \in \Gamma(X)$ and smooth functions $\varphi_1, \cdots, \varphi_n$ on M such that $\gamma = \sum_1^n \varphi_i \gamma_i$.

Suppose that X and Y are vector bundles over a common base M. The linear differential operator $D: \Gamma(X) \to \Gamma(Y)$ is said to be essential if the set of all sections of the type $D(\gamma)$, $\gamma \in \Gamma(X)$, generates $\Gamma(Y)$. (For example, an elliptic operator is essential.)

4.3.10. *Suppose that Y is a fibering with n-dimensional fiber on which there exists a set containing k sections which generate $\Gamma(Y)$, and that $D: \Gamma(X) \to \Gamma(Y)$ is an essential differential operator. Then for $k > n$ there exist sections $\gamma_1, \cdots, \gamma_k \in \Gamma(X)$ such that the set of sections $D(\gamma_1), \cdots, D(\gamma_k) \in \Gamma(Y)$ generates $\Gamma(Y)$.*

In order to obtain the theorem of Hirsch from this theorem, one needs to take for X a one-dimensional trivial fibering on M, for Y the cotangent bundle of the manifold M, and for D the operator which associates to a smooth function its exterior differential.

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