M. Gromov

ASYMPTOTIC GEOMETRY OF HOMOGENEOUS SPACES.

Let X be a homogeneous Riemannian manifold. If X is non-compact, then "asymptotic" refers to properties of X "near infinity". For example, if X is a symmetric space of non-compact type, then essential asymptotic features of X can be described in terms of the *ideal boundary* ∂X of X, whose points represent asymptotic classes of geodesic rays in X. This boundary plays a crucial role in the work of Mostow on rigidity of symmetric spaces. Mostow's starting points consist of a study of quasi-isometric maps $f : X_1 \to X_2$, which means distortion $< \infty$, where

$$\text{distor } f \overset{\text{def}}{=} \sup_{x \neq y} |\log(\text{dist}_1(x, y)/\text{dist}_2(f(x), f(y)))| .$$

He shows, at least for $\text{rank}_{\mathbb{R}} X_i = 1, i = 1, 2,$ that such an f extends to a continuous map $\partial X_1 \to \partial X_2$, and that this extension is *quasi-conformal* in a suitable sense. Thus he proves (among many other results) that two rank one spaces are quasi-isometric if and only if they are isometric up to the choice of normalizing constants.

This approach applies also to more general homogeneous spaces, but one does not know yet if quasi-isometric homogeneous spaces are necessarily isomorphic. Furthermore, one can sharpen Mostow's result for rank one symmetric spaces by estimating from below a natural distance between different spaces. For example, if a (non-homogeneous) manifold Y has sectional curvature pinched between -1 and $-1 + (2n)^{-1}$, $n = \dim Y$, and if Y is quasi-isometric to a symmetric space X, then X has *a constant* negative curvature.

Now let X be compact homogeneous space. To be specific, let X be a compact Lie group. Then, left invariant Riemannian metrics on X are given by positive definite symmetric forms on $\mathbb{R}^n \approx T_e(X)$, $n = \dim X$. These constitute a convex cone, say $C_+ \subset \mathbb{R}^{\frac{n(n+1)}{2}}$, and asymptotic phenomena appear if a family of metrics $g_t \in C_+$ on X approaches the boundary ∂C_+ for $t \to \infty$. For example, an appropriate family of invariant metrics on $SU(2)$ approaches a non-Riemannian (*Carnot's,* see Pansu's talk in these proceedings) metric on $SU(2)$ whose Hausdorff's dimension equals $4 = 1 + \dim_{\text{top}} SU(2)$. In fact, this $SU(2)$ with the limit metric appears as the ideal boundary of the complex hyperbolic plane CH^2.

Given two metrics g_1 and g_2 in C_+ one seeks a map $f: (X, g_1) \to (X, g_2)$ with a minimal distortion and then defines a metric on C_+ by $\text{DIST}(g_1, g_2) = \text{distor} f$. Thus, one is led to the study of the asymptotic of the metric DIST on C_+.

Even in the simplest case of $X = SU(2)$ one does not know kow this DIST looks like. However, some non-trivial estimates can be obtained either with Pansu's isoperimetric inequality (see these proceedings) or with the L_p-cohomology of CH^2.

REFERENCES

A detailed exposition will appear in a (yet unwritten) paper by M. Gromov: "Asymptotic Geometry of Riemannian Manifolds".

For the background, see:

G. MOSTOW: "Strong Rigidity of Locally Symmetric Spaces", Ann. Math, Studies, 78 (1973), Princeton.

M. GROMOV: "Infinite Groups as Geometric Objects", to appear in the Proc. of the ICM Warsaw 1983.