Large Riemannian Manifolds

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We want to discuss here several unsolved problems concerning metric invariants of a Riemannian manifold $V = (V, g)$ which mediate between the curvature and topology of V.

1. VOLUME OF BALLS $B_v(\rho)$ IN LARGE MANIFOLDS V.

Assume V is complete and define for all $\rho \geq 0$.

$$\sup \mathrm{Vol}(V; \rho) = \sup_{v \in V} \mathrm{Vol}\, B_v(\rho)$$

for the balls $B_v(\rho) \subset V$. If V has bounded geometry (e.g. compact), then the behavior of $\sup \mathrm{Vol}(V; \rho)$ for $\rho \longrightarrow 0$ is controlled by the lower bound of the scalar curvature of V, called

$$\inf S(V) = \inf_{v \in V} S(V, v).$$

On the other hand, the asymptotic behaviour of sup Vol for $\rho \longrightarrow \infty$ has a topological meaning if, for example, V is metrically covers some compact manifold.

1.A. Vague Conjecture.

If V is large compared to $\mathbb{R}^n$ for $n = \dim V$, then

$$\sup \mathrm{Vol}(V; \rho) \geq \sup \mathrm{Vol}(\mathbb{R}^n; \rho) = A_n \rho^n,$$

where A_n is the volume of the unit ball in $\mathbb{R}^n$. Furthermore, $\inf S(V) = 0$ for large manifolds V (compare [GL] and [S]).

To make sense of 1.A, we give several precise notions of largeness.

$\mathcal{L}_1$. <u>Contractible almost homogeneous manifolds</u> (CAH).

This means that V is contractible and that the action of the isometry group $Is(V)$ is cocompact on V. For example the universal coverings of compact aspherical manifolds are CAH.

$\mathcal{L}_2$. <u>Geometrically contractible manifolds</u> (GC).

Define $GC_k(V, \rho)$ for all $\rho \geq 0$ to be the lower bound of the numbers $r \geq \rho \geq 0$, such that the inclusion of the concentric balls in V

$$B_v(\rho) \hookrightarrow B_v(r)$$

is a k-<u>contractible</u> map for all $v \in V$.

Recall, that a continuous map $f\colon X \longrightarrow Y$ is called k-degenerate, if there exist a k-dimensional polyhedron P and continuous maps $f_1\colon X \longrightarrow P$ and $f_2\colon P \longrightarrow Y$, such that $f = f_2 \circ f_1$. Then, f is called k-contractible if it is homotopic to a k-degenerate map.

A manifold V is called GC if $GC_0(V, \rho) < \infty$ for all $\rho \geq 0$.

Obviously, CAH $\Longrightarrow$ GC. (Compare $[G]_2$ P.43.)

$\mathcal{L}_3$. <u>Manifolds with</u> $\mathrm{Diam}_{n-1} = \infty$.

Define $\mathrm{Diam}_k V$ to be the lower bound of those $\delta > 0$ for which there exists a continuous map of V into some k-dimensional polyhedron, say $f\colon V \longrightarrow P$, such that

$$\mathrm{Diam}\, f^{-1}(p) \leq \delta,$$

for all $p \in P$ (compare $[G]_2$ P.127).

It is not hard to prove the following relation between Diam_k and GC for $k + \ell = n - 1 = \dim V - 1$ (compare $[G]_2$ P.143).

<u>There exists a function</u> $\rho_n(\delta)$ <u>for</u> $\delta \geq 0$, <u>such that</u>

$$\mathrm{Diam}_k V \leq \delta \Longrightarrow GC_\ell(V, \rho) = \infty \quad \text{for } \rho \geq \rho_n(\delta).$$

<u>In particular</u>, GC $\Longrightarrow \mathrm{Diam}_{n-1} = \infty$.

$\mathcal{L}_4$. <u>Manifolds with</u> $\mathrm{Cont}_{n-1}\mathrm{Rad} = \infty$.

Imbed V into the space of functions $L_\infty(V)$ by $v \longmapsto \mathrm{dist}(v, *)$. If V is compact, define $\mathrm{Cont}_k\mathrm{Rad}\ V$ to be the lower bound of the numbers $\varepsilon \geq 0$, such that the inclusion map of V into the ε-neighborhood $U_\varepsilon(V) \subset L_\infty(V)$ is k-contractible, where the function space $L_\infty(V)$ is equipped with the L_∞-norm:

$$||f(v)|| = \sup_{v \in V} |f(v)|$$

(compare $[G]_2$ P.P.41, 138).

If V is noncompact, one modifies this definition by restricting to <u>proper</u> k-contracting homotopies which keep pull-backs of bounded subsets in $U_\varepsilon(V)$ bounded in V.

It is easy to see that

$$\mathrm{Cont}_k\mathrm{Rad}\ V \leq \frac{1}{2}\ \mathrm{Diam}_k V$$

and that

$$\mathrm{GC} \Longrightarrow \mathrm{Cont}_{n-1}\mathrm{Rad} = \infty .$$

Furthermore, (see $[G]_2$ P.138).

$$\mathrm{Cont}_{n-1}\mathrm{Rad}\ V \leq C_n(\mathrm{Vol}\ V)^{1/n}$$

for some universal constant $C_n > 0$. In particular

$$\mathrm{CG} \Longrightarrow \mathrm{Vol}\ V = \infty.$$

$\mathcal{L}_5$. <u>Manifolds with</u> Fill Rad $= \infty$.

Define Fill Rad V to be the minimal ε for which V is $\mathbb{Z}_2$-homologous to zero in the ε-neighborhood $U_\varepsilon(V) \subset L_\infty(V)$ (compare $[G]_2$ P.41). Clearly, Filling Rad $\leq \mathrm{Cont}_{n-1}\mathrm{Rad}$. Yet Fill Rad > 0 for all manifolds V. (See $[G]_2$ for applications of Fill Rad and [K] for a computation of Fill Rad of some symmetric spaces). It is also clear that

$$GC \Longrightarrow \text{Fill Rad} = \infty.$$

Also notice that Fill Rad decreases under proper distance decreasing maps $V_1 \longrightarrow V_2$ of degree one (mod 2) (see $[G]_2$ P.8).

$\mathcal{L}_6$. Hyperspherical manifolds.

Assume V is oriented and define $\text{HS Rad}_k V$ to be the upper bound of those numbers $R \geq 0$ for which there exists a proper Λ^k-contracting map of V onto the sphere $S^n(R) \subset \mathbb{R}^{n+1}$ of radius R, say

$$f: V \longrightarrow S^n(R),$$

such that $\deg f \neq 0$. Here "proper" means that the complement of some compact subset in V goes to a single point in S^n and "Λ^k-contracting" signifies that f decreases the k-dimensional volumes of all k-dimensional submanifolds in V (compare [GL]). One says that V is HS if $\text{HS Rad}_1 V = \infty$.

Remark. One can modify the definition of HS Rad by restricting to maps f with $\deg f \equiv 1$ (mod 2). Then modified HS clearly implies Fill Rad $= \infty$.

Stable classes $\mathcal{L}_i^+$ and $\mathcal{L}_i^-$.

Given a class $\mathcal{L}$ of n-dimensional manifolds. One defines $V \in \mathcal{L}^+$ iff V admits a proper distance decreasing map of degree one onto some manifold $V' \in \mathcal{L}$. One also defines $V \in \mathcal{L}^-$ iff the existence of a proper distance decreasing map $V' \longrightarrow V$ of degree one implies $V' \in \mathcal{L}$. The stabilization $\mathcal{L}^+$ looks interesting for the classes $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$. Furthermore, it is logical to allow an arbitrary pseudo-manifold V' in the definition of $\mathcal{L}_3^-$ and $\mathcal{L}_4^-$ and to stabilize (in an obvious way) the invariants Diam_k and Cont_kRad in order to match the classes $\mathcal{L}_3^-$ and $\mathcal{L}_4^-$. Following this line of reasoning, one can define $\text{Diam}_k^- h$ and $\text{Cont}_k^- h$ for an arbitrary homology class h in V by representing h by distance decreasing maps $V' \longrightarrow V$ for $\dim V' = \dim h$.

1.B. On the Vague Conjecture.

There is no solid evidence for 1.A for manifolds in the classes $\mathcal{L}_i$ and $\mathcal{L}_i^{\pm}$. One even does not know if

$$\sup \mathrm{Vol}(V;\rho) \geq \pi\rho^2$$

for CAH surfaces. However it is easy to see that

$$\sup \mathrm{Vol}(V;\rho) \geq 3\rho^2$$

for GC surfaces (compare $[G]_2$ P.40). This suggests relaxing 1.A to the inequality

$$\sup \mathrm{Vol}(V;\rho) \geq C_n\rho^n \tag{1}$$

for some universal constant in the interval $0 < C_n < A_n$. In fact, (a quantitative version of) (1) is proven on P.130 in $[G]_2$ for manifolds V with $\mathrm{Diam}_{n-1}V = \infty$, provided $\mathrm{Ricci}\, V \geq -1$.

Finally, a non-sharp version of 1.A is known to be true asymptoticly for $\rho \longrightarrow \infty$ for CAH manifolds. Namely, the polynomial growth theorem for abstract groups reduces the problem to the universal covering $V \longrightarrow T^n$ of the homotopy n-torus T^n and the argument on P.100 in $[G]_2$ yields the bound

$$\liminf_{\rho\to\infty} \rho^{-n}\, \mathrm{Vol}\, B(\rho) \geq C_n > 0$$

for the concentric balls $B(\rho)$ in the universal covering of T^n.

Now, we turn to the inequality $\inf S(V) \leq 0$ for large manifolds V. One is able to prove (see [GL] and $[G]_2$ P.129) that

$$\inf S(V) \leq (\pi 6\sqrt{2})/\mathrm{Diam}_1 V)^2 \tag{2}$$

for complete simply connected 3-manifolds. In particular $\mathrm{Diam}_1 V = \infty$ implies $\inf S(V) \leq 0$ for these V. Next one believes that

$$\inf S(V) \leq C_n(\mathrm{HS\ Rad}_2 V)^{-2}. \tag{3}$$

This is proven for spin manifold V in [GL] and a similar inequality is anounced in [S] for the general case. Yet, one does not know the best constant C_n in (3). For example, let a metric g on S^n satisfy $g \geq g_0$ for the standard metric g_0 on S^n. One does not know if $\inf S(g) \leq S(g_0)$.

Many CAH manifolds V are shown to be HS (see [GL] and references therein) and no counterexample to CAH $\Longrightarrow$ HS is known. More generally, let V' be a closed manifold whose classifying map to the Eilenberg Maclain space $K(\pi, 1)$ for $\pi = \pi_1(V')$ sends the fundamental class $[V]$ (here, V is assumed oriented) to a <u>non-zero</u> class in $H_n(K(\pi, 1); \mathbb{Q})$. Then, one asks if the universal covering V of V' is HS. (The HS property of V does not depend on the metric in V'). If so, the manifold V' admits no metric with $S(V) > 0$ as it follows from (3).

If $V \in \mathcal{L}_i$, $i = 1,\dots,6$, then, clearly, $V \times \mathbb{R}^N \in \mathcal{L}_i$ for all N. In particular, if V is HS then $V \times R^N$ also is HS. The converse is unlikely to be true but no counter example is known. On the other hand, the largeness of $V \times \mathbb{R}^N$ has roughly the same effect on $S(V)$ as that of V itself. Namely,

$$\inf S(V) \leq C'_{n+N}(\text{HS Rad}_2 V \times \mathbb{R}^N)^{-2}, \qquad (3')$$

provided V is spin (compare [S] for non-spin manifolds).

2. MANIFOLDS WITH $K \geq 0$.

Let V be a complete connected manifold with non-negative sectional curvature. Then one can show that the largeness conditions $\mathcal{L}_i$ are equivalent for $i = 3,4,5,6$, and V is $\mathcal{L}_i$-large for $i = 3,\dots,6$ if and only if

$$\sup \text{Vol}(V; \rho) = \sup \text{Vol}(\mathbb{R}^n; \rho) = A_n \rho^n \qquad (4)$$

for all $\rho \geq 0$. Furthermore, if

$$\sup \text{Vol}(V; 1) \leq A' < A_n,$$

then

$$\sup \mathrm{Vol}(V; \rho) \leq C\rho^{n-1} \tag{5}$$

for all $\rho \geq 1$ and for some universal constant $C = C(n, A')$.

If in addition to $K(V) \geq 0$ one assumes $S(V) \geq \sigma^2 > 0$, then one can strengthen (5) by

$$\sup \mathrm{Vol}(V; \rho) \leq C'_{n,\sigma}\rho^{n-2} \tag{5'}$$

and show that

$$\mathrm{Diam}_{n-2}V \leq C''_{n,\sigma}/\sigma. \tag{6}$$

2.A. Open Questions.

(a) It seems likely, that complete hyperspherical manifolds with $K(V) > 0$ are geometrically contractible.
(b) The relating (4), (5) and (5') may generalize to the case Ricci $V \geq 0$. This seems quite realistic if $|K(V)| \leq 1$ and Inj Rad $V \geq 1$.
(c) It is unknown if (6) holds true for all complete manifolds with $S(V) \geq \sigma^2$.

2.B. Idea of the Proof of (4) - (6).

For certain sequences of points $v_i \in V$ the sequences of the pointed metric spaces (V, v_i) converge in the Hausdorff topology to isometric products $\mathbb{R}^d \times V'$ for (possibly singular) spaces V' with $K \geq 0$. If d is the largest possible, then V' with is compact and $\mathrm{Diam}_d V \leq \text{const} \sup_{V'} \mathrm{diam}\, V'$. In particular, if V is large, then (the maximal) $d = n$ and $\lim_{i\to\infty} \mathrm{Vol}\, B_{V_i}(g) = A_n\rho^n$. This proves (4); the inequalities (5), (5') and (6) follow by a similar argument.

2.C.

To grasp the geometric meaning of the invariants $\mathrm{diam}_k V$, consider the Euclidean solid

$$V' = \{(x_0, \ldots, x_{n-1}) \mid |x_k| \le \mathrm{Diam}_k V,\ k = 0,\ldots,n-1\} \subset \mathbb{R}^n.$$

One believes that every compact manifold V with (possibly empty) convex boundary and with $K(V) \ge 0$ roughly looks like V'. For example, the volume of V' seems a good approximation to Vol V and the spectrum of the Laplace operator on V' might approximate that on V. Namely, the corresponding numbers of eigenvalues $\le \lambda$ are conjectured to satisfy,

$$N'(C_n\lambda) \ge N(\lambda) \ge N'(C_n^{-1}\lambda).$$

A similar rough approximation is expected for small balls in manifolds with $K(V) \le 1$. Here the case $|K(V)| \le 1$ looks easy.

2.D. Manifolds with $S_k(V) \ge \alpha$ and $R_k(V) \ge \alpha$.

Write $S_k(V) \ge \alpha$ if the average of the sectional curvatures over the 2-planes in every tangent k-dimensional surface in $T(V)$ is $\ge \alpha$. Write $R_k(V) \ge \alpha$ if the sum of the first k eigenvalues of Ricci on $T_v(V)$ is $\ge \alpha$ for all $v \in V$. One does not know the geometric significanse of the inequalites $S_k > 0$ for $3 \le k \le n-1$ and $R_k > 0$ for $2 \le k \le n-1$, unless some additional conditions are imposed on V. What one wishes is an upper bound like $\mathrm{Diam}_\ell \le C/\sigma$ for $S_{\ell+2} \ge \sigma^2$. Here is a simple fact supporting this conjecture.

Let V *be a complete manifold with* Ricci ≥ 0 *and* $R_k \ge \sigma^2$ *for some fixed* $k \le n$. *Then* $\sup \mathrm{Vol}(V; \rho) \le C\rho^{k-1}/\sigma$ *provided* $|K(V)| \le \mathrm{const} < \infty$ *and* Inj Rad $V \ge \varepsilon > 0$.

This is shown by a limit argument as in 2.B.

Observe, that the inequality $R_k \ge \alpha$ defines a *convex* subset in the space of the curvature tensors on every space $T_v(V)$. This insures the stability of this inequality under certain (weak) limits of metrics.

3. VERY LARGE MANIFOLDS.

Define $\mathrm{Vol}_k(V)$ as the lower bound of those $s \geq 0$ for which there exists a simplicial map $f: V \longrightarrow P$ for some smooth triangulation of V and some $(n-k)$-dimensional polyhedron P, such that the k-dimensional volume of the pull-back $f^{-1}(p) \subset V$ is $\leq s$ for all $p \in P$. It is known that

$$(\mathrm{Vol}_k V)^{1/k} \geq C_n \text{ Fill Rad } V,$$

for all complete manifolds V (see $[G]_2$ P.134), but a similar inequality with C_k instead of C_n (here $n = \dim V$) is unknown.

Next, let

$$h_k(V;\rho) = \inf_{v \in V} \log \mathrm{Vol}_k B_v(\rho)$$

for the ball $B_v(\rho) \subset V$ and define the entropy $h_k(V)$ by

$$h_k(V) = \liminf_{\rho \to \infty} \rho^{-1} h_k(V,\rho).$$

The most interesting is the entropy of the universal coverings $\tilde{V}$ of compact manifolds V. Here one expect the ratios such as $h_k(\tilde{V})/(\mathrm{Vol}\ V)^{1/n}$ or as $h_k(V)/\mathrm{Diam}_\ell V$ to bound some topological invariants of V. It is known, for instance, that

$$(h_n(\tilde{V}))^n/\mathrm{Vol}\ V \geq C_n \|V\| \tag{7}$$

where $\|V\|$ denotes the <u>simplicial volume</u> of V, that is, roughly speaking, the minimal number of simplices needed to trianglate the fundamental classes of V (see $[G]_1$ P.245).

If $\tilde{V}$ is contractible, then one expects a similar bound for Pontryagin numbers and for the L_2-Betti numbers of V (see $[G]_1$ P.293 for related results).

A complementary problem is to bound h_k by some curvature condition on V. For example, does the inequality $S(V) \geq -\sigma^2$ implies $h_2(V) \leq C\sigma$? Here is a closely related.

3.A. Conjecture.

Every closed manifold V with $S(V) \geq -\sigma^2$ satisfies

$$||V|| \leq C_n \sigma^n \operatorname{Vol} V. \qquad (8)$$

Remarks.

(A) The inequality (8) for Ricci $V \geq -\sigma^2$ follows from (7), but the best constant C_n is unknown for $n \geq 3$.

(B) One can imagine a stronger version of (8), namely

$$||V|| \leq C_n \int_V |S_v^-(V)|^{n/2} dV \qquad (8')$$

where $S_v^- = \min(0, S_v)$. But this is unknown even with $K(V)$ in place of $S(V)$. In fact, the only known lower bound for the total curvature $\int_V |K|_v^{n/2} dv$ comes from characteristic numbers of V. One does not know, for example, if every hyperbolic 3-manifold admits a sequence of metrics such that $\int_V |S_v|^{3/2} dv \longrightarrow 0$, even if one insists on $K < 0$ for these metrics.

3.B. Specific Entropy $sh_k V$.

Let $sh_k(V; \rho)$ be the upper bound of the numbers $\ell \geqq 0$ with the following property. There exists a C^1-map $f: V \longrightarrow V$, such that $\operatorname{dist}(f, \operatorname{Id}) \leq \rho$ and every k-dimensional submanifold V' in V satisfies

$$\log \operatorname{Vol}_k V' - \log \operatorname{Vol}_k f(V') \geq \ell .$$

Then set

$$sh_k V = \liminf_{\rho \to \infty} \rho^{-1} sh_k(V; \rho).$$

Observe, for the universal covering $\tilde{V}$ of a compact manifold V, that $sh_k \tilde{V} = 0$ iff the fundamental group $\pi_1(V)$ is amenable and that $sh_2 \tilde{V} > 0$ iff $\pi_1(V)$ is hyperbolic (e.g. V admits a metric with $K < 0$). Furthermore, every symmetric space with $K \leq 0$ and rank $= 2$ has $h_k > 0$ and $sh_k > 0$ if and only if $k > 2$.

Conjecture. Let V be a complete geometrically contractible manifold with $S(V) \geq -\sigma^2$. Then

$$sh_2 V \leq C_n |\sigma|.$$

A related question is as follows. Let V be a compact manifold with $S(V) \geq \sigma^2$. Does there exist a (possibly singular) 2-dimensional surface (or a varifold) $V' \subset V$, such that Area $V' \leq C_n \sigma^{-2}$? In fact, one expects that

$$\mathrm{Vol}_2 V \leq C_n \sigma^{-2}.$$

4. NORMS ON THE COHOMOLOGY AND ON THE K-FUNCTOR.

The L_∞-norm on $H^*(V; \mathbb{R})$ is obtained by minimizing the L_∞-norm $= \sup_{v \in V} \|\omega\|_v$ of closed forms ω representing classes in H^* (see §7.4 in $[G]_2$ for details and references). Next, for an isomorphism class α of an orthogonal or unitary vector bundle $X \longrightarrow V$ we define $\|\alpha\|$ by minimizing the L_∞-norm of the curvature forms of (orthogonal or unitary) connections on X. An alternative "norm", called $\|\alpha\|^+$, is obtained by minimizing the Lipschitz constant of classifying maps of V into the pertinent Grassmann manifold G. Clearly

$$\|\alpha\| \leq C\|\alpha\|^+$$

for $C = C(n, \dim \alpha)$. Furthermore, if α is the class of a complex line bundle, then $\|\alpha\| = \|c_1(\alpha)\|$ for the first Chern class $c_1(\alpha)$. In fact, every closed 2-form ω on V in an integral cohomology class is the curvature form of some line bundle with curvature $= \omega$.

4.A. Theorem (see $[G_1L]$, $[G]_1$ P.294 and references therein).

Denote by $s = s(V)$ the minimal norm $\|\gamma\|$ for all orthogonal bundles with $w_2(\gamma) = w_2(V)$ for the second Stiefel Whitney class w_2. Then every unitary β satisfies

$$|\{\mathrm{ch}\, \beta \cdot \hat{A}(V)\}[V]| \leq C_n N(C'_n(s + \|\beta\|) - C'_n \sigma) \qquad (9)$$

where $\sigma = \inf S(V)$, where V is assumed compact and oriented, and where C_n, C'_n and C''_n are some universal positive constants. (Recall that $N(\lambda)$ denotes the number of eigenvalues $\leq \lambda$ of the Laplace operators on functions on V).

Corollaries.

(a) No metric g on V with $S(V, S) \geq \sigma > 0$ can be too large.

Proof. Take some β for which the left hand side of (9) does not vanish and observe that $s \longrightarrow 0$ and $\|\beta\| \longrightarrow 0$ as g is getting large. If n is odd, apply the above to $V \times S^1$ for a long circle S^1.

(b) Let (V, g) be a closed oriented manifold, such that, for a fixed metric g_0 on V, one has $g \wedge g \geq g_0 \wedge g_0$, that is the identity map $(V, g) \longrightarrow (V, g_0)$ decrease areas of the surfaces in V. Then the Laplace operator on (V, g) satisfies for all $\lambda \geq 0$

$$N^{2/n}(\lambda) \geq C_n\lambda + C'_n\sigma - C'', \qquad (9')$$

where $\sigma = \inf S(V, g)$ and where the constant C'' depends on (V, g_0). Furthermore, if V is spin, then

$$N^{2/n}(\lambda) \geq C_n\lambda + C'_n\sigma - C'''_n\rho^{-2},$$

where $\rho = \text{HS Rad}_2(V, g)$.

Proof. Apply (9) with appropriate β and γ.

Remarks.

(1) The inequalities (9') and (9") can be applied to the universal covering of V where the dimension $N(\lambda)$ is understood in the sense of Von Neumann algebras.

(2) The best constants C'' in (9') seems an interesting invariant of (V, g_0).

The norm of an appropriate β (as well as of $s(V)$) can be often made arbitrary small by passing to the universal covering $\tilde{V}$ of V where some version of (9) still holds true (see [GL]). This is so, for instance, if $\tilde{V}$ is a hyperspherical manifold with $w_2(\tilde{V}) = 0$. In this case (9) implies $\inf S(V) \leq 0$ for every metric on V. Furthermore, the norm $\|\beta\|^+$ also becomes arbitrary small in the hyperspherical case. Thus, by combining [GL]-twisting with [VW]-untwisting (see [VW]), one gets the following result.

4.B.

Let the universal covering $\tilde{V}$ of a compact manifold V be spin and hyperspherical. Then the spectrum of the Dirac operator on $\tilde{V}$ contains zero.

Remark. A similar argument applies to the Laplace operator on forms on $\tilde{V}$. However, the Laplace on functions on $\tilde{V}$ contains zero in the spectrum iff $sh_n \tilde{V} = 0$.

Question. Let V be a "large" manifold, e.g. V is contractible and covers a compact manifold V'. Does the spectra of Dirac and Laplace (on forms!) contain zero ? This is likely if $\pi_1(V')$ satisfies the strong Novikov conjecture.

4.C. Symplectic Forms.

Let ω be a symplectic (i.e. closed and nonsingular) 2-form on a closed manifold V. Write $g \geq \omega$ if the L_∞-norm of ω with respect to (the metric) g is ≤ 1 and set

$$\|\omega\|_S = \sup_{g \geq \omega} \sigma_g$$

for $\sigma_g = \inf S(V, g)$. If V is spin and if some real multiple of ω represents an integral class in $H^*(V; \mathbb{R})$ then (9) implies $\|\omega\|_S < \infty$. Furthermore all metrics $g \geq \omega$ on V satisfy

$$N^{2/n}(\lambda) \geq C_n \lambda + C'_n \sigma_y - C'' \qquad (10)$$

for some (interesting ?) constant $C'' = C''(V, \omega)$ (compare (9')).

Question. Are the spin and the integrality conditions essential ?

How can one evaluate $\|\omega\|_S$ for known examples of symplectic manifolds ?

Observe the following useful property of the L_∞-norm on the image $I^* = f^*(H^*(K; \mathbb{R})) \subset H^*(V; \mathbb{R})$ for an arbitrary continuous map $f: V \longrightarrow K$ where $K = K(\Gamma/1)$ for a residually finite group Γ.

4.C'.

For every $\alpha \in I^*$ and every $\varepsilon > 0$, there exists a finite covering $\tilde{V} \longrightarrow V$ and some integral classes $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_p$ in $H^*(\tilde{V}; \mathbb{Z}) \subset H^*(V; \mathbb{R})$ such that $\|\alpha_i\| \leq \varepsilon$ for $i = 1, \ldots, p$ and the pull-back $\tilde{\alpha} \subset H^*(\tilde{V}; \mathbb{R})$ of α is representible by some real combination of $\tilde{\alpha}_i$.

4.C". Corollary.

If a closed even dimensional spin manifold V possesses a 2-dimensional class $\alpha \in I^*$, such that $\alpha^{n/2} \neq 0$ (for $n = \dim V$), then V admits no metric with $S > 0$, provided the implied group Γ is residually finite.

Proof. Apply (9) to some line bundles $\tilde{\beta}_i$ on $\tilde{V}$ with $c_1(\tilde{\beta}_i) = \tilde{\alpha}_i$.

Probably, one can drop the residual finiteness condition by elaborating on non-compact thechniques in [GL]. It also would be interesting to eliminate spin by Schoen-Yau minimal manifolds techniques (see [S] and references therein).

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