DIMENSION, NON-LINEAR SPECTRA AND WIDTH

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Abstract

This talk presents a Morse-theoretic overview of some well known results and less known problems in spectral geometry and approximation theory.

§0 Motivation: Various Descriptions of the linear spectrum

The main object of the classical spectral theory is a *linear* operator Δ on a Hilbert space X. We assume Δ is a self-adjoint possibly unbounded (e.g., differential) operator and then consider the normalized *energy*

$$E(x) = \langle \Delta x, x\rangle / \langle x, x\rangle \ ,$$

which is defined for all non-zero x in the domain of Δ. Since the energy E is homogeneous,

$$E(ax) = E(x) \quad \text{for all} \quad a \in \mathbb{R}^{\times} \ ,$$

it defines a function on *the projective space* P consisting of the lines in the domain $X_{\Delta} \subset X$ of Δ,

$$P = P(X_{\Delta}) = X_{\Delta} \backslash \{0\} / \mathbb{R}^{\times} \ .$$

This function on P is also called the *energy* and denoted by $E : P \to \mathbb{R}$. Notice that since Δ is a *linear* operator the function E on P is *quadratic*, that is the ratio of two quadratic functions on the underlying linear space.

Now, the spectrum of Δ can be defined in terms of the energy E on P. To simplify the matter we assume below that Δ is a *positive* operator with *discrete* spectrum and then we have the following three ways to characterise the spectrum of Δ, that is the set of the eigenvalues $\lambda_0 \leq \lambda_1 \leq \ldots$ of Δ appearing with due multiplicities.

0.1 The Morse-theoretic description of the spectrum. Denote by $\Sigma = \Sigma(E) \subset P$ the *critical* set of E where the differential (or gradient) of E on P vanishes. A trivial (and well known) argument identifies Σ with the union of the 1-dimensional eigenspaces of Δ. In other words, if $x \in X$ is a non-zero vector from the line in X representing a point $p \in P$, then $p \in \Sigma$ if and only if $\Delta x = \lambda x$ for some real λ. Then

$$E(x) = \langle \Delta x, x \rangle / \langle x, x \rangle = \lambda$$

and so $E(p) = \lambda$ as well. It follows that the *spectrum of* Δ *equals the set of critical values of the energy* $E : P \to \mathbb{R}$. It is equally clear that the critical point of E corresponding to a *simple* eigenvalue λ_i is nondegenerate and has *Morse* index i. More generally, the multiplicity of λ_i equals $\dim \Sigma_i + 1$ for the component $\Sigma_i \subset \Sigma$ on which E equals λ_i, since Σ_i consists of the lines in the eigenspace $L_i \subset X$ associated to λ_i.

Notice that the definition of critical values of E is purely topological and applies to not necessarily quadratic functions on P. In fact, the set of critical values serves as a nice substitute for the spectrum for some non-quadratic energy functions (e.g., for the energy on the loop space in a compact *symmetric* space). But the essentially *local* nature of the critical values and non-stability of these under small perturbations (every point can be made critical by an arbitrary small C^0-perturbation of the energy function) forces us to look for another candidate for the non-linear spectrum.

0.2 Characterization of the spectrum by linear subspaces contained in the level sets $X_\lambda = \{x \in X \mid E(x) \leq \lambda\}$. Denote by $L_i \subset X$ the linear subspace spanned by the eigenvectors corresponding to the first $i+1$ eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_i$ of Δ and observe that

$$L_i \subset X_{\lambda_i} \ .$$

This signifies the inequality

$$\langle \Delta x, x \rangle \leq \lambda_i \langle x, x \rangle$$

for all $x \in L_i$, as $E(x) = \langle \Delta x, x \rangle / \langle x, x \rangle$.

The following *extremal property* of X_{λ_i} is more interesting. *If* $\lambda < \lambda_i$, *then* X_λ *contains no linear subspace of dimension* $i+1$. In fact, let $L \subset X$ be a linear subspace of dimension $i+1$. Then there is a non-zero vector $x \in L$ which is orthogonal to the subspace $L_{i-1} \subset X$. That is $\langle x, x_j \rangle = 0$ for the first i eigenvectors $x_0, \ldots, x_j, \ldots, x_{i-1}$ of Δ. It is trivial to prove that this x satisfies

$$\langle \Delta x, x \rangle \geq \lambda_i \langle x, x \rangle \ ,$$

which shows $L \not\subset X_\lambda$ for $\lambda < \lambda_i$.

Let us summarize this discussion in terms of the projective space $P = P(X_\Delta)$ and the energy of E on P.

The eigenvalue λ_i is the minimal number, such that the level

$$P_\lambda = E^{-1}[0,\lambda] = \{x \in P \mid E(x) \leq \lambda\} \subset P$$

contains a projective subspace of dimension i.

Remark. (a) The above characterization of λ_i is geometrical rather then topological as it makes use of the projective (linear) structure of P. On the other hand this projective definition of the spectrum obviously generalizes to *non-quadratic* energies E on P.

(b) An advantage of the projective definition of λ_i over the Morse-theoretic one (see 0.1) is the stability under small perturbations of the energy. Besides, the above existence proof of a "λ_i-hot" vector x in an arbitrary subspace $L \subset X$ of (asymptotically large) dimension $i+1$ gives a glimpse of general methods used for obtaining *lower bounds* for λ_i.

(c) An interesting generalization of the projective view on λ_i consists in replacing P by another geometrically signficant (homogeneous) space with a distinguished class of subspaces. The most obvious candidate for such a space is the Grassmann manifold $G = G_k(X)$ of the k-dimensional subspaces on X. Distinguished subspaces in G are Grassman manifolds $G_k(L) \subset G = G_k(X)$ for all linear subspaces $L \subset X$. (If $k = 1$, then $G = P$.) Now "the lower bound for λ_i" (see the above (b)) takes the following shape: any linear subspace $L \subset X$ contains "an interestingly hot" k-dimensional subspace $K \subset L$, where K becomes hotter and hotter as $\dim L \to \infty$ for $k = \dim K$ being kept fixed. (Compare Dvoretzky's theorem discussed in 1.2.)

0.3 Topological characterization of the eigenlevels $P_{\lambda_i} \subset P$. If we denote $\operatorname{pro} P_\lambda$ the maximal dimension of projective subspaces contained in P_λ, then we can say that the spectrum points λ_i are exactly those (see 0.2) where the function $\operatorname{pro} P_\lambda$ is *strictly* increasing in λ. In fact if λ_i is an eigenvalue of multiplicity m_i, then $\operatorname{pro} P_\lambda$ jumps up at λ_i by m_i.

Now we want to replace $\operatorname{pro} P_\lambda$ by a purely topological invariant of P_λ.

0.3.A Essential dimension. Consider a subset A in a topological space P and define the *essential dimension of A in P*,

$$\operatorname{ess} A = \operatorname{ess}_P A$$

as the smallest integer i, such that A is contractible in P onto an i-dimensional subset $A' \subset P$. This means there exists a continuous map (homotopy) $h : A : [0,1] \to P$, such that h on A at

$t = 0$ is the identity map,

$$h \mid A \times 0 : A \underset{\mathrm{Id}}{\subset} P$$

and such that

$$\dim h(A \times 1) \leq i \ ,$$

that is the image $h(A \times 1) \subset P$ admits arbitrarily fine coverings by open subsets where no $i+2$ among these subsets intersect.

0.3.B Basic example. If P is a projective space and A is a *projective* subspace, then

$$\operatorname{ess} A = \dim A \ . \tag{$*$}$$

Notice that the inequality $\operatorname{ess} A \leq \dim A$ is trivial while the opposite inequality $\operatorname{ess} A \geq \dim A$ amounts to the following (simple but not totally trivial).

0.3.C Topological fact. *The dimension of a projective subspace $A \subset P$ cannot be decreased by a homotopy of A in P.* (See 4.1 for the proof and further discussion.)

0.3.D. Now we return to our positive quadratic energy function E on P and observe that the level $P_\lambda = \{x \in P \mid E(x) \leq \lambda\}$ can be contracted in P onto the projective subspace corresponding to the linear span of the eigenvectors belonging to the eigenvalues $\lambda_i \leq \lambda$. (This is more or less obvious.) This property combined with 0.3.C and the discussion in 0.2 implies that

$$\operatorname{ess} P_\lambda = \operatorname{pro} P_\lambda$$

for all λ. Therefore the definition of λ_i for *quadratic functions* on P can be formulated purely topologically, *the eigenvalue λ_i is the minimal number λ, such that the level $P_\lambda \subset P$ has $\operatorname{ess} P_\lambda \geq i$, which means P_λ cannot be contracted onto an $(i-1)$-dimensional subset in P.*

0.3.D Remarks. (a) The notion of ess makes sense for subsets in an *arbitrary topological space Q* and therefore one can speak of the ess-spectrum of an energy E on Q.

(b) If an energy E on Q is amenable to Morse theory, then the number $M(\lambda)$ of *λ-cold eigenpoints* of E, that are critical points q of E where $E(q) \leq \lambda$, can be bounded from below in terms of the ess-spectrum by

$$M(\lambda) \geq N(\lambda) = \operatorname{ess} E^{-1}[0, \lambda] \ .$$

(See $[\mathrm{Gr}]_1$ for another estimate of this nature for spaces of closed curves in Riemannian manifolds.)

0.4 Definitions of "dim"-spectrum for any "dimension". Let "dim" be a monotone increasing function on subsets A of a given space P, that is

$$A_1 \subset A_2 \Longrightarrow \text{"dim"} A_1 \leq \text{"dim"} A_2 \; .$$

If such a "dimension" is originally defined only on a certain class of admissible subsets, we agree to extend "dim" to all subsets A in P by taking all admissible subsets $A' \subset A$ and by setting

$$\text{"dim"} A = \sup_{A'} \text{"dim"} A' \; .$$

For example, the ordinary dimension on linear (or projective) subspaces extends in this way to all subsets of a linear (projective) space.

Now, with a given "dim" we define the "dim"-*spectrum* $\{\lambda_i\}$ of an energy $E : P \to [0, \infty]$, *as follows,* λ_i *is the upper bound of those* $\lambda \in \mathbb{R}$*, for which the level* $E^{-1}[0, \lambda]$ *has "dim"* $< i$.

In "physical" terms, every $A \subset P$ with "dim"$A \geq i$ contains a λ-hot point ($a \in A$, where $E(a) \geq \lambda$) for every $\lambda \leq \lambda_i$ and λ_i is the maximal number with this property.

The spectrum $\{\lambda_i\}$ can be more conveniently defined via the *spectral function* which, roughly speaking, counts the number of eigenvalues (or rather, of energy levels) of E below λ for all $\lambda \geq 0$. More precisely, this number $N(\lambda)$ is defined by

$$N(\lambda) = \text{"dim"} E^{-1}[o, \lambda] \; .$$

0.4.A Remarks on the range of E. (a) We allow infinite values for the energy in order not to bother with the domain of definition of E (and Δ as in §0.1). Namely, if E is originally defined on a dense subset $P_0 \subset P$ we extend E to P by

$$E(p) = \limsup_{u \to p} E \mid U \cap P_0$$

over a fundamental system of neighbourhoods U of p.

(b) There is no reason to restrict oneself to $[0, \infty]$-valued energies. In fact, for an arbitrary map $E : P \to T$ one can define the spectral function on the subsets $S \subset T$ by

$$N(S) = \text{"dim"} E^{-1}(S) \; .$$

(According to the physical terminology such an E should be called *observable*. The standard example of this is the position $P \to \mathbb{R}^3$ of a particle in $\mathbb{R}^3$.)

Example. Let $\|x\|_1, \dots, \|x\|_m$ be norms on a linear space X. These naturally define a map E of the projective space $P = P(X)$ to the $(m-1)$-simplex $\Delta^m = \mathbb{R}_+^m / \mathbb{R}_+^\times$. A typical case of interest is $\|x\|_i = \|D_i x\|_{L_{p_i}}$ for some differential operators D_i on a function space X.

0.4.B Dimension-like properties of pro and ess. Let us axiomatize certain common features of the "dimensions" pro and ess by calling a function "dim" on subsets in a projective space P *dimension-like* if it has the following six properties.

(i) INTEGRALITY AND POSITIVITY. If $A \subset P$ is a non-empty subset, then "dim"A may assume values $0, 1, 2, \dots, \infty$. If A is empty then "dim"$= -\infty$.

(ii) MONOTONICITY. If $A \subset B$ then "dim A"$\leq$ "dim"B for all A and B in P.

(iii) PROJECTIVE INVARIANCE. If $f : P \to Q$ is a projective embedding between projective spaces, then

$$\text{“dim”} f(A) = \text{“dim”} A$$

for all $A \subset P$.

(iv) INTERSECTION PROPERTY. *If $P' \subset P$ is a projective subspace of codimension k, then*

$$\dim A \cap P' \geq \dim A - k$$

for all $A \subset P$.

(v) NORMALIZATION PROPERTY. If A is a *projective* subspace in P then "dim A" equals the ordinary dimension $\dim A$.

(vi) THE $*$-ADDITIVITY. Let $A_1 * A_2 \subset P$ denote the union of the projective lines meeting given subsets A_1 and A_2 in P. Then

$$\text{“dim”} A_1 * A_2 = \text{“dim”} A_1 + \text{“dim”} A_2 + 1 ,$$

provided A_1 and A_2 are *projectively disjoint*. This means the *projective spans* PA_1 and PA_2 do not intersect, where the *projective span* PA indicates the minimal projective subspace in P containing A. (Notice that this additivity implies the above normalization property, as $P^{m+n+1} = P^m * P^n$.)

Remark. It is obvious that pro satisfies (i)-(vi) and that ess satisfies (i) and (ii). The properties (iii)-(vi) for ess follow from

0.4.B$_1$ Subadditivity of ess. The following property makes the "dimension" ess especially useful,

$$\operatorname{ess} A \cup B \leq \operatorname{ess} A + \operatorname{ess} B + 1$$

for all subsets A and B in P. See 4.1 for the proof.

0.5 Codimension and width. Define the projective *codimension* $\operatorname{copro} A$ for A in P as the minimum of the codimensions of projective subspaces P' contained in P. Then define the *coprojective dimension* by

$$\operatorname{pro}^{\perp} A = \operatorname{copro} P \backslash A \ .$$

Observe that $\operatorname{pro}^{\perp}$ satisfies the "dimension" properties (i)-(vi) in 0.4.B. In fact (essentially because of (iv)) this $\operatorname{pro}^{\perp}$ is the *maximal* set function on P satisfying (i)-(vi). (Notice that pro is the minimal such function.)

0.5.A Definition of i-width. Let B be a subset in a Banach space X and define the *width function* of B on the dual space X' by

$$\operatorname{Wid}(B, y) = \sup_B y - \inf_B y$$

for all linear functions y on X. Then define *the i-width* of B by

$$\operatorname{Wid}_i B = (\lambda_i')^{-1} \ ,$$

where λ_i' is the i-th $\operatorname{pro}^{\perp}$-eigenvalue of the energy

$$E' = \| \quad \|' / \operatorname{Wid}(B, \quad) : P(X') \longrightarrow [0, \infty] \ .$$

For example,

$$\operatorname{Wid}_0 B = (\min E)^{-1} = \max \operatorname{Wid}(B, \quad) / \| \quad \|' = \operatorname{Diam} B \ .$$

In the special case, where B is a *centrally symmetric* subset in X our definition is equivalent to the usual one,

$\operatorname{Wid}_i B$ *equals the lower bound of those $\delta > 0$ for which there exists an i-dimensional linear subspace L in X whose $(\delta/2)$-neighbourhood contains B, that is*

$$\operatorname{dist}(b, L) \le \delta/2$$

for all $b \in B$.

0.5.B Coprojective dimension and width. Recall the *duality correspondence D* which maps subsets $Y \subset X'$ to those in X by

$$D(Y) = \bigcup_{y \in Y} D(y) \ ,$$

for

$$D(y) = \{x \in X \mid |y(x)| = \|y\|'\|x\|\} .$$

We use the same notation D for the associated correspondence on the projective spaces, $P(X') \rightsquigarrow P(X)$, and call a subset $Q \subset P(X)$ an *i-coplane* if it is the D-image of an i-codimensional projective subspace in $P(X')$. Then we define *copro*$^\perp A$ for all $A \subset P(X)$ as the maximal i such that the complement $P(X)\setminus A$ contains no i-coplaine. In other words $\text{copro}^\perp A \le i \iff A$ meets every i-coplane in $P(X)$. One easily sees with Bezout's theorem (compare §4) that

$$\text{ess}\, A \cap Q \ge \text{ess}\, A - i$$

for all $A \subset P(X)$ and all i-coplanes Q. In particular, if $\text{ess}\, A \ge i$, then A meets every i-coplane in $P(X)$, which is equivalent to

$$\text{copro}^\perp A \ge \text{ess}\, A$$

for all A. It follows that

$$\text{copro}^\perp A \ge \text{pro}\, A. \tag{$*$}$$

Notice that $(*)$ is a reformulation of the following

Tichomirov Ball Theorem. *Let $B^{i+1}(\varepsilon) \subset X$ be the ε-ball in some linear $(i+1)$-dimensional subspace of X and let L be a linear i-dimensional subspace in X. Then there exists a point $b \in B$, such that* $\text{dist}(b, L) = \varepsilon$.

In fact, the projectivization of the subset L^* of non-zero vectors $x \in X$ for which

$$\text{dist}(x, L) = \|x\| ,$$

is an i-coplane in $P(X)$, and every i-coplane comes from some L. Now, both $(*)$ and the ball theorem claim that L^* meets every $(i+1)$-dimensional linear subspace in X.

Coming back to the width of B, where B is the unit ball of some (semi)norm $\| \ \|_0$ on X, we see that

$$\text{Wid}_i\, B = 2(\lambda_i^\perp)^{-1}$$

for the copro$^\perp$-spectrum $\{\lambda_i^\perp\}$ of $E = \| \ \|/\| \ \|_0$ and the above discussion relates these $\lambda_i^\perp$ to the ess and pro$^\perp$-spectra by the inequalities

$$\lambda_i^\perp \le \lambda_i^{\text{ess}} \le \lambda_i^{\text{pro}} \tag{$**$}$$

Remark. The number $(\lambda_i^{\text{pro}})^{-1}$ is called in [I-T] *the Bernstein i-width* of the unit ball of $\| \ \|_0$ in $(X, \| \ \|)$.

0.6 Complementary dimensions and $\{\lambda_{ij}\}$. Let d be a "dimension" function on subsets $A \subset P$ and take $i = 0, 1, \ldots$. Represent A as the difference of subsets, $A = B \backslash C$, and let

$$d^i A = \sup_{B,C}(i - dC + 1)$$

over all B and C, where $dB = i$. If d is subadditive (as ess, see 0.4.B$_1$). That is, if

$$dB \leq dA + dC + 1 \ ,$$

then $d^i \leq d$ (and usually $d^i A = dA$, for $dA \leq i$) but in general d^i can be greater than d.

Next, for a given energy we define λ_{ij} for all $j \leq i$ as the $(i-j)$-th d^i-eigenvalue of E. In other words λ_{ij} is the upper bound of those λ for which every i-dimensional subset $B \subset P$ contains a λ-hot subset $C \subset B$ of dimension $\geq j$, where "λ-hot" signifies $E \mid B \geq \lambda$.

0.7 Generalized dimension. There are many interesting situations, where the ordinary (pro or ess) "dimension" of the levels of E is infinite, but there is some additional structure which allows a "renormalization". Here are two examples.

(a) Suppose E is a perturbation of E_0 for

$$E_0 = E_0(x) = \langle \Delta_0 x, x\rangle / \langle x, x\rangle \ ,$$

where Δ_0 is a selfadjoint operator with discrete spectrum which is ***not assumed*** positive anymore. If Δ_0 has infinitely many negative eigenvalues (e.g., Δ is the Dirac operator), then $\operatorname{pro} E^{-1}(-\infty, \lambda] = \infty$ for all λ. Yet one can define a *finite difference*

$$\operatorname{pro} E^{-1}(-\infty, \lambda] - \operatorname{pro} E^{-1}(-\infty, \lambda']$$

(representing the number of eigenvalues between λ and λ') as the index of an appropriate *Fredholm correspondence* between maximal linear subspaces in $E^{-1}(-\infty, \lambda)$ and $E^{-1}(-\infty, \lambda')$. This kind of situation arises, for example, in the *symplectic Morse theory*, where E is a perturbation of the action (see [Z], [Fl]) and also in the recent unpublished work by Floer on 3-dimensional gauge theory.

(b) VON-NEUMANN DIMENSION. This is defined, for example, on Γ-invariant linear subspaces of a Hilbert space X, where Γ is a given subgroup of unitary operators acting on X. The classical spectral theory does generalize to the Von-Neumann (algebras) framework but one does not know yet if there are suitable delinearization and de-Hilbertization of this theory.

§1 The spectrum of the ratio (L_p-norm)/(L_q-norm) and the concentration phenomenon for measurable functions

Consider the measure space (V, μ) and let

$$E = E_{p/q}(x) = \|x\|_p / \|x\|_q \ ,$$

where $\|x\|_p$ is ordinary L_p-norm on functions x on V,

$$\|x\|_p = \Big(\int_V |x|^p \Big)^{1/p} \ ,$$

and where $1 \le q < p \le \infty$. It is well known that every "sufficiently large" space L of functions on V contains a function x "concentrated near a single point" in V, where the concentration is measured by the energy $E(x)$. We shall prove in this section the simplest (and the oldest) result of this kind, and refer to [Pi] for deeper theorems.

We assume below that (V, μ) is a probability space, that is $\mu(V) = 1$. Then we define the *projective eigenvalue* $\lambda_i = \lambda_i(L_p/L_q)$ of $E = E_{p/q}$ as the minimal λ, such that $\operatorname{pro} P_\lambda \ge i$ (compare 0.3). Notice that here the inequality $\operatorname{pro} P_\lambda \ge i$ is equivalent to the following property: there exists on $(i+1)$-dimensional linear space L' of L_p-functions on V, such that $\|x\|_p \le \lambda \|x\|_q$ for all $x \in L$. Observe that $1 = \lambda_0 \le \lambda_1 \le \ldots \le \lambda_i \le \ldots$ and let $\lambda_\infty = \lim\limits_{i \to \infty} \lambda_i$.

1.1 Theorem. *The number $\lambda_\infty = \lambda_\infty(L_p/L_q)$ is bounded from below by*

$$\lambda_\infty \ge \gamma_\infty(p,q) = \pi^{\frac{1}{2q} - \frac{1}{2p}} \left(\Gamma\left(\frac{p+1}{2}\right)\right)^{\frac{1}{p}} \Big/ \left(\Gamma\left(\frac{q+1}{2}\right)\right)^{\frac{1}{q}} \ , \qquad (*)$$

for the Euler Γ-function. Furthermore, if the measure μ is continuous (i.e., without atoms) then also the opposite inequality holds true,

$$\lambda_\infty(L_p/L_q) \le \gamma_\infty(p,q) \ , \qquad (**)$$

and thus $\lambda_\infty = \gamma_\infty$.

Proof: For a finite dimensional linear space L of functions of V we consider its dual L' and interpret functions $\ell \in L$ on V as linear functions on L'.

For a measure ν on L' we denote by $I_p(\ell, \nu)$ the integral

$$I_p(\ell, \nu) = \int_{L'} |\ell|^p d\nu$$

for all $\ell \in L$ and write

$$E_{p/q}(\ell,\nu) = I_p^{\frac{1}{p}}(\ell,\nu)/I_q^{\frac{1}{q}}(\ell,\nu) \ .$$

for $\ell \in L \backslash \{0\}$.

Then we observe that (almost) every point $v \in V$ defines a linear function ℓ' on L that is $\ell'(\ell) = \ell(v)$ for all $\ell \in L$. This gives us a canonical map $V \to L'$, such that every function $\ell \in L$ on V "extends" to a linear function on L'. We denote by μ' the probability measure on L' which is the push-forward of μ under this map and observe that the L_p-norms in L are recaptured by μ. Namely

$$\int_V |\ell(v)|^p d\mu = I_p(\ell,\mu')$$

for all $\ell \in L$ and all p and accordingly

$$E_{p/q}(\ell) = E_{p/q}(\ell,\mu') \ .$$

If the measure μ on V is continuous, then obviously, for every i and every probability measure ν on $\mathbb{R}^{i+1}$ there exists an $(i+1)$-dimensional space L of functions on V such that the measure μ' on L' is linearly isomorphic to ν. That is μ' goes to ν by some linear isomorphism between L' and $\mathbb{R}^{i+1}$. In particular, such an L exists for the *normalized Gauss measure*

$$d\nu = dt_0 \dots dt_i \exp \sum_{j=0}^{i} t_j^2 \Big/ \pi^{\frac{i+1}{2}} \ .$$

A straight forward computation shows for this ν that

$$E_{p/q}(\ell,\nu) = \gamma_\infty(p,q)$$

for all $i = 0,1,\dots,\ 1 \le q < p \le \infty$, and all $\ell \in L \backslash \{0\}$. Since $E_{p/q}(\ell) = E_{p/q}(\ell,\nu)$ for $\nu = \mu'$, we obtain with the definition of λ_i the inequality

$$\lambda_i(L_p/L_q) \le \gamma_\infty(p,q) \quad \text{for} \quad 1 = 0,1,\dots,$$

which is equivalent to inequality $(**)$ of the theorem.

Now we turn to the proof of $(*)$ and start with the case where either p or q equals two and where we shall give a *sharp* bound for *each* λ_i. To do this we need the normalized measure ν_ρ on the sphere $S_\rho^i \subset \mathbb{R}^{i+1}$ of radius ρ. In other words ν_ρ is the probability measure on $\mathbb{R}^{i+1}$

which is invariant under the orthogonal group $O(i+1)$ and has support S^i_ρ. The $O(i+1)$-invariance of μ_ρ implies that $E_{p/q}(\ell,\nu_\rho)$ is constant in $\rho > 0$ and in ℓ for all non-zero linear functions ℓ on $\mathbb{R}^{i+1}$, which allows us to define

$$\gamma_i(p,q) = E_{p/q}(\ell,\nu_\rho) \ .$$

This agrees with our γ_∞ defined earlier as $\gamma_i(p,q) \to \gamma_\infty(;,q)$ for $i \to \infty$ by a straightforward computation.

Now, observe that the proof of $(**)$ also yields the following

1.1.A Trivial Proposition. *If the measure μ is continuous then*

$$\gamma_i(L_p/L_q) \le \gamma_i(p,q) \tag{++}$$

for all $i = 0,1,\ldots,$ and $1 \le q < p \le \infty$.

Notice that $(++)$ is stronger than $(**)$ as $\gamma_i < \gamma_\infty$ for $i < \infty$.

A more interesting fact is that $(++)$ is sharp if either p or q equals two.

1.1.B Theorem. *If p or q equals two then*

$$\lambda_i(L_p/L_q) \ge \gamma_i(p,q) \ , \tag{+}$$

for all $i = 0,1,\ldots$.

Proof: Let L be an arbitrary $(i+1)$-dimensional linear space of functions on V. To prove $(+)$ we must show that

$$E(L) \overset{\text{def}}{=} \sup_{\ell \in L \setminus \{0\}} E_{p/q}(\ell) \ge \gamma_i(p,q) \ .$$

First we recall (L',μ') and observe that

$$E(L) = \sup_{\ell \in L \setminus \{0\}} E_{p/q}(\ell,\mu') \ .$$

Then we invoke the group G of linear isometries of L with the L_2-norm (induced from $L_2(V,\nu) \supset L$) and consider the natural action of G on L' and on measures on L'. Notice that the dual L_2-norm on L' turns L' into a Euclidean space and G becomes the orthogonal group $O(i+1)$ acting on $\mathbb{R}^{i+1} = L'$ in the usual way. Then we average μ' over G and set

$$\overline{\mu}' = \int_G g\mu' dg$$

for the normalized Haar measure dg on G. Notice, that $\overline{\mu}'$ is a $O(i+1)$-invariant measure on $\mathbb{R}^{i+1} = L'$ and so the energy $E_{p/q}(\ell,\overline{\mu}')$ is independent on ℓ for all $\ell \in L\setminus\{0\}$.

1.1.B$_1$ Basic Lemma. *If p or q equals 2, then*

$$E(L) \geq E_{p/q}(\ell, \overline{\mu}') \tag{$*$}$$

for $\ell \in L \backslash \{0\}$.

Proof: Recall that $E_{p/q}$ is the ratio

$$E_{p/q}(\ell, \nu) = I_p^{1/p}(\ell, \nu) / I_q^{1/q}(\ell, \nu)$$

for

$$I_p(\ell, \nu) = \int_{L'} |\ell|^p d\nu \ .$$

To be specific let $p = 2$. Then the integral $I_p(\ell, \nu')$ is invariant under the action of G on μ' that is $I_p(\ell, g\mu')$ is constant in g as follows from the definition of G. Thus

$$E_{p/q}(\ell, g\mu') = C I_q^\alpha(\ell, g\mu')$$

for $\alpha = -\frac{1}{q}$ and some $C > 0$. This implies that

$$\sup_{g \in G} E_{p/q}(\ell, g\mu') \geq C \overline{I}_q^\alpha \ ,$$

where

$$\overline{I}_q = \int_G I_q(\ell, g\mu') dg \ .$$

Now, by the linearity of $I_q(\ell, \nu)$ in ν,

$$\overline{I}_q = I_q(\ell, \overline{\mu}')$$

and by the transitivity of G on the sphere $S^i \subset \mathbb{R}^{i+1} = L'$,

$$E(L) = \sup_{g \in G} E_{p/q}(\ell, g\mu') \ .$$

This all together yields $(*)$ for $p = 2$ and the same argument works for $q = 2$.

Now, the proof of $(+)$ follows from $(*)$ and the following simple lemma applied to the measure $\nu = \overline{\mu}'$,

1.1.B$_2$. *Let ν be a rotationally invariant (i.e., $O(i+1)$-invariant) probability measure in $\mathbb{R}^{i+1}$. Then*

$$E_{p/q}(\ell, \nu) \geq E_{p/q}(\ell, \nu_\rho) = \gamma_i(p, q)$$

for all non-zero linear functions ℓ on $I\!R^{i+1}$, all $\rho > 0$ and all $1 \le q < p \le \infty$.

Proof: We shall need the following trivial

1.1.B$'_2$ Calculus lemma. *Let $A_1(t) = a_1 t + b_1$ and $A_2(t) = a_2 t + b_2$ be linear functions in t whose derivatives A'_i are non-zero of same sign, that is $a_1 a_2 > 0$, and let $A_1(t_0)$ and $A_2(t_0)$ be positive at some point t_0. If $0 \le q < p < \infty$, then t_0 is not a local minimum point of the ratio $A_1^{\frac{1}{p}}/A_2^{\frac{1}{q}}$.*

We are going to apply this lemma to $E_{p/q} = I_p^{\frac{1}{p}}(\nu)/I_q^{\frac{1}{q}}(\nu)$ keeping in mind that I_p and I_q are linear in ν. We observe that every extremal point ν in the space of $O(i+1)$-invariant measures on $I\!R^{i+1}$ is a measure supported on a single sphere $S^i_\rho \subset I\!R^{i+1}$ for some $\rho > 0$, that is $\nu = \nu_\rho$. We also notice that the derivatives in ρ

$$I'_p(\nu_\rho) \quad \text{and} \quad I'_q(\nu_\rho)$$

are strictly positive. Now, 1.1.B$'_2$ shows that $E_{p/q}$ has no local minimum point apart from $\{\nu_\rho\}$ and so by an obvious compactness argument $E_{p/q}$ assumes the minimum exactly on the set $\{\nu_\rho\}_{\rho>0}$. Q.E.D.

1.1.B$_3$ Example. The best known and most useful case of Theorem 1.1.B is that where $p = \infty$ and $q = 2$. In this case $\gamma_i = \sqrt{i+1}$ and so 1.1.B amounts to the following property.

Let L be an $(i+1)$-dimensional linear space of functions on a probability space (V, μ). Then there exists a non-zero $\ell \in L$, such that

$$\sup_{v \in V} |\ell(v)| \ge \sqrt{i+1}\Big(\int_v |\ell(v)|^2 d\mu\Big)^{\frac{1}{2}} \tag{+}$$

Besides the case where $(V, \mu) = (I\!R^{i+1}, \mu_\rho)$ the equality is achieved for the finite measure space V consisting of $i+1$ equal atomes. This suggests that the averaging is not indispensible for the proof and the following (standard) argument gives a confirmation.

Let $\ell_0, \dots, \ell_i$ be an L_2-orthonormal basis in L. Then every L^2-unit vector $\ell \in L$ is a linear combination

$$\ell = \sum_{i=0}^{i} a_i \ell_i$$

for $\sum a_i^2 = 1$. Therefore the inequality $|\ell(v)| \le \lambda(v)$ for a given $v \in V$ and all unit vectors $\ell \in L$ is equivalent to the inequality

$$\sum_{i=1}^{i} \ell_i^2(v) \le \lambda^2(v) \ .$$

Hence,

$$\int_v \lambda^2(v) \geq \int_V \sum \ell_i^2(v) = i+1 \ ,$$

which implies the required inequality

$$\sup_{v\in V} |\lambda(v)| \geq \sqrt{i+1} \ .$$

1.1.B$_3$ The above (+) frequently applies to spaces of solutions x of an elliptic equation $\Delta x = 0$ (see [Ka], [Me], [G-M]). For example, if V is a Riemannian manifold of ***bounded (local) geometry*** and Δ on V is invariantly related to the geometry of V, then

$$\|x\|_\infty \leq \text{const}\, \|x\|_2 \ ,$$

where the constant depends only on the implied bound on the geometry. Then the above (+) applied to the normalized Riemannian volume of V, yields

$$\dim \operatorname{Ker}\Delta \leq \text{const}^{\,2} \operatorname{Vol} V \ .$$

If V is complete non-compact of *infinite* volume and L is an *infinite dimension* space of solutions x of $\Delta x = 0$, then one can sometimes make sense of the inequality $\dim L/\operatorname{Vol} V > 0$ and use (+) to prove the existence of a non-zero L_2 -solution x on V. (For example, see [Ka].)

1.1.C The proof of 1.1 for all p and q. The basic averaging argument (see 1.1.B$_1$) applies, in principle, to the linear isometry group of $(L, \| \ \|_p)$ for all p, but for $p \neq 2$ this group is usually two small to be useful. However, by Dvoretzky theorem (see 1.2), there exists a j-dimensional subspace $M \subset L$ whose L_p-norm is ε invariant under the L_2 isometry group $G = O(j)$ of $(M, \| \ \|_2)$,

$$(1-\varepsilon)\|x\|_p \leq \|gx\|_p < (1+\varepsilon)\|x\|_p$$

for all $x \in M$ and $g \in G$, where ε admits an universal bound in terms of $j = \dim M$ and $i = \dim L - 1$,

$$\varepsilon \leq \varepsilon_0(i,j) \ ,$$

such that for every fixed j,

$$\varepsilon_0(i,j) \longrightarrow 0 \quad \text{for} \quad i \to \infty \ .$$

Now the L_2-argument applies up to an ε-error to $(M, \| \ \|_p)$ and the error goes to zero for $i \to \infty$. Q.E.D.

1.1.D Remarks. (a) The above argument using Dvoretzky theorem also applies to the spectrum $\{\lambda_{ij}\}$ (see 0.6) and shows that for every fixed j

$$\lim_{i\to\infty} \lambda_{ij} \geq \gamma_\infty(p,q) \ .$$

In other words, every i-dimensional subspace $L \subset L_q(V,\mu)$ contains a j-dimensional subspace $M \subset L$, such that

$$E_{p/q} \mid M \geq (1-\varepsilon_{ij})\gamma_\infty(p,q)$$

where $\varepsilon_{ij} \to 0$ for $i \to \infty$.

(b) To prove 1.1 one actually needs only the *weak* Dvoretzky *theorem* (see 1.2.C) whose proof is obtained by an integration argument similar to that used in 1.1.B$_1$. (See §9.3 of [Gr] for yet another application of this argument.)

(c) Theorems 1.1 and (especially) 1.1.B look a century old but I made no effort to find early references. (The earliest frequently cited papers I know of are [Ru] and [Ste].) A very interesting use of 1.1.B$_3$ appears in [Ka] and the averaging argument of 1.1.B$_1$ can also be found in [G-M].

(d) If the measure space V in question is finite and consists of N atoms, then the i-th eigenvalue λ_i of L_p/L_q is related to the $(N-i)$-width of the unit ball $B_{p'} \subset L_{p'}$ with respect to the $L_{q'}$-norm by

$$(N-i)\text{-width}(B_{p'}, L_{q'}) = 2\lambda_i^{-1} \ ,$$

where p' and q' are determined by

$$\frac{1}{p'} + \frac{1}{p} = 1 \ , \qquad \frac{1}{q'} + \frac{1}{q} = 1 \ .$$

In the case where $N = \lambda i$ and the atoms of the underlying measure space V have unit mass the width, and hence λ_i, were estimated by Kaŝin (see [Pi]) as follows

$$\lambda_i \asymp \begin{cases} 1 & \text{for } p > 1 \geq 2 \\ i^{\frac{1}{q}-\frac{1}{2}} & \text{for } q < 2 < p \\ i^{\frac{1}{q}-\frac{1}{p}} & \text{for } q \leq p \leq 2 \end{cases}$$

where $a_i \asymp b_i$ signifies that a_i/b_i is pinched between two positive constants for $i \to \infty$. Similar (but more difficult) estimates for all N are due to Gluskin (see [Pi] and [Kaŝ]).

(e) **Question.** Let H be a homogeneous function in k variables of degree zero. Then for a given k-tuple $(p_1, \ldots, p_k)$ one defines the energy

$$E(x) = H(\|x\|_{p_1}, \ldots, \|x\|_{p_2}) \ ,$$

and asks what the spectrum of this E is. If $k = 2$, the question reduces to L_p/L_q. If $k = 3$, the simplest energy is $\|x\|_{p_1}\|x\|_{p_2}/\|x\|_{p_3}^2$.

In fact one is interested in the spectrum of the multi-parametric "energy"

$$x \longmapsto (\|x\|_{p_1}, \ldots, \|x\|_{p_k}) ,$$

as it is defined in 0.4.

1.2 Dvoretzky theorem. We state below for reader's convenience several versions of Dvoretsky thoerem and we refer to [Mi-Sh] for the proofs.

The classical version of the theorem claims that the ratio $E(x) = \|x\|'/\|x\|$ of two norms on a linear space L becomes "nearly constant" when restricted to an "appropriate" subspace $M \subset L$, provided $\dim L$ is sufficiently large. Here the non-constancy of E is measured by *the logarithmic oscillation*

$$\operatorname{los} E = \log(\sup E/\inf E)$$

and the precise statement is as follows.

1.2.A. *For every $j \leq i = \dim L$ there exists a linear subspace $M \subset L$ of dimension j, such that*

$$\operatorname{los} E \mid M \leq \varepsilon(i,j) , \tag{$*$}$$

where $\varepsilon(i,j)$ is a universal constant depending on i and j, such that for every fixed j, $\varepsilon(i,j) \to 0$ for $i \to \infty$.

1.2.B Remark. The most important special case of 1.2.A is where $L = \mathbb{R}^i$ and $\| \quad \|$ is the Euclidean norm on $\mathbb{R}^i$. In this case the theorem applied to $E = \| \quad \|'$ restricted to the unit sphere in $\mathbb{R}^i$. Notice that this special case (applied first to L and then to M) yields the general case.

1.2.C Weak Dvoretzky. In this version of the theorem the constant ε is allowed to depend on $C = \operatorname{los} E \mid L$. Namely, one assumes $\operatorname{los} E \mid L = C < \infty$ and only claims the existence of an $M \subset L$, such that

$$\operatorname{los} E \mid M \leq \varepsilon(i,j,C) ,$$

where $\varepsilon \to 0$ for $i \to \infty$ and j and C fixed. Here again the most important case is $(L, \| \quad \|) = \mathbb{R}^i$. This Euclidean Dvoretzky is equivalent (this is easy, see [Mi-Sh]) to the following subadditivity of the function $\operatorname{pro} X$ for $X \subset P$, which is, we recall, the maximal dimension of projective subspaces contained in X,

$$\operatorname{pro}(X \cup Y) \leq A\big(\operatorname{pro} X, \operatorname{pro}(Y + \varepsilon), \varepsilon^{-1}\big) ,$$

where X and Y are subsets in P, where $Y+\varepsilon\subset P$ denotes the ε-neighbourhood of Y with respect to the standard (Euclidean) metric in P, and where A is some function in three real variables. This is worth comparing with the subadditivity of the essential dimension,

$$\operatorname{ess}(X\cup Y)\leq \operatorname{ess}X+\operatorname{ess}Y+1\ ,$$

(see 0.4.B_1).

1.2.D Non-symmetric Dvoretzky. The Dvoretzky theorem remains true if we drop the symmetry requirement for the norms $\|\ \ \|$ and $\|\ \ \|'$. This is possible due to the following version of Bezout (Borsuk-Ulam) theorem (compare §4).

Let $E:\mathbb{R}^i\to\mathbb{R}$ be a continuous function and $x_1,\dots,x_k$ be some vectors in $\mathbb{R}^i$. If $k<i$, then there exist an orthogonal transformation g of $\mathbb{R}^i$, such that

$$Eg(x_\nu)=E\big(-g(x_\nu)\big)$$

for all $\nu=1,\dots,k$.

1.2.E Dualization. Dvoretzky theorem can be stated as the existence of an ε-round j-dimensional section of a convex subset K in $\mathbb{R}^i$. This yields, by duality, the existence of ε-round projections of K. Since projections commute with taking convex hulls one can drop the convexity assumption on K and arrive at the following proposition.

Let K be a compact subset in $\mathbb{R}^i$ which linearly spans $\mathbb{R}^i$. Then for every $j\leq i$ there exists a surjective linear map $A:\mathbb{R}^i\to\mathbb{R}^j$, such that K goes into the unit Euclidean ball in $\mathbb{R}^j$,

$$A(K)\subset B_1^j=\{x\in\mathbb{R}^j\mid \|x\|\leq 1\}\ ,$$

and $A(K)$ is ε-dense in B_1^j, where as earlier, for each j,

$$\varepsilon=\varepsilon(i,j)\longrightarrow 0\quad\text{as}\quad i\to\infty\ .$$

(Recall, that a subset of a metric space is called ε-dense in B if its ε-neighbourhood contains B.) *Moreover, one can find the above A of form λp, where $p:\mathbb{R}^i\to\mathbb{R}^j\subset\mathbb{R}^i$ is an orthogonal projection onto a subspace and λ is the multiplication by a scalar $\lambda>0$.*

1.2.F Projection of measures. With little extra effort the above discussion applies to projection of measure on $\mathbb{R}^i$ rather than of subsets. Namely, let μ_i be a probability measure on $\mathbb{R}^i$, for all $i=1,2,\dots$, such that the support of μ_i linearly spans $\mathbb{R}^i$.

Then for every $j=1,2,\dots$, there exists an orthogonally invariant measure $\overline{\mu}$ on $\mathbb{R}^j$ and a sequence of linear maps $A_i:\mathbb{R}^i\to\mathbb{R}^j$, such that the push-forward measures $A_*(\mu_i)$ on

$\mathbb{R}^j$ weakly converge to $\overline{\mu}$. Moreover one can choose $A_i = \lambda_i p_i$ as in 1.2.E. (This version of Dvoretzky theorem nicely fits the fixed-point philosophy of Fürstenberg, see [Gr-Mi].)

1.3 On the topological version of the $E_{p/q}$-spectra. If the measure space (V,μ) is infinite then the Ess-spectrum for $E_{p/q}$ collapses to the single point $\lambda_0 = 1$. This is immediate with the following.

Trivial observation. *Let $C_\varepsilon \subset P$ be the subset of (the projective classes of) functions x on V, such that $|x(v)| \leq 1$ for all $v \in V$ and*

$$\mu\{v \in V \mid |x(v)| = 1\} \geq 1 - \varepsilon \ .$$

Then $\operatorname{ess} C_\varepsilon = \infty$ *for all $\varepsilon > 0$.* (Notice that $\operatorname{pro} C_\varepsilon = 0$ for all $\varepsilon > 0$.)

Now let us compute the topological spectrum of $E_{p/q}$ on the finite measure space (V,μ) consisting of n equal atoms of mass $1/n$.

1.3.A. *The* ess-*spectrum of $E_{p/q}$ on V is*

$$\lambda_i = \left(\frac{n-i}{n}\right)^{\frac{1}{p}-\frac{1}{q}} \ . \tag{$*$}$$

Proof: A trivial computation shows that the critical points of the function E of index i are the baricenters of i-codimensional faces (which are $(n-i-1)$-dimensional simplices) of the L_1-sphere $\{\|x\|_1 = 1\} \subset L_1(V,\mu)$ and $E_{p/q}$ equals the above λ_i (given by $(*)$) at these baricenters. Hence $(*)$ follows by the Morse theory.

Remark. One can avoid using Morse theory by applying the following simple topological facts (A) and (B) to the unit L_1- and L_∞-balls

$$\left\{\|x\|_1 = \frac{1}{n}\sum_{i=1}^{n}|x_i| \leq 1\right\} \subset \mathbb{R}^n$$

and

$$\left\{\|x\|_\infty = \sup_i |x_i| \leq 1\right\} \subset \mathbb{R}^n \ .$$

(A) *Let $Q \subset P = P(\mathbb{R}^n)$ satisfy* $\operatorname{ess} Q \geq i$ *and let $B \subset \mathbb{R}^n$ be a convex centrally symmetric polyhedron with non-empty interior. Then the cone $\widetilde{Q} \subset \mathbb{R}^n$ over Q meets some $(n-i-1)$-dimensional face of B.*

Notice that a similar fact for $\operatorname{pro} Q \geq i$ holds true *without* assuming B is symmetric. In fact the meeting points of an $(i+1)$-plane $L \subset \mathbb{R}^n$ with the $(n-i-1)$-faces of B are exactly the extremal points of $B \cap L$.

(B) *Let $\overline{B}_i \subset P$ be the projection to P of the union of i-faces of B. Then*

$$\operatorname{ess} \overline{B}_i = i$$

Notice that $\operatorname{pro} \overline{B}^i = 0$ for the L_∞-ball B and $i \le n-2$ (this is the case for our ess-spectral discussion), which explains the sharp discrepency between the ess- and pro-spectra.

§2 Variation, oscillation and ess-spectra for spaces of continuous maps

The measure theoretic conentration phenomenon of the previous section has the following topological counterpart.

A "large" subspace in the space of continuous maps between topological spaces V and W must contain a "topologically complicated" map $x : V \to W$.

If $W = \mathbb{R}$ and V is connected, then the complexity of a function $x : V \to \mathbb{R}$ can be measured by the *variation* of X,

$$\operatorname{Var} x = \int_{\mathbb{R}} b_0\big(x^{-1}(t)\big)\, dt$$

where b_0 is the zero Betti number, that is the number of connected components of the pull-back $x^{-1}(t)$ for all $t \in \mathbb{R}$.

Notice that every map $x : V \to \mathbb{R}$ can be uniquely factorized as follows, $V \xrightarrow{\bar{x}} \overline{V} \xrightarrow{y} \mathbb{R}$, where $\overline{V}$ is a 1-dimensional space (graph) and $\bar{x}$ is a *connected* map of V onto $\overline{V}$, that is $\bar{x}^{-1}(\bar{v}) \subset V$ is connected for all $\bar{v} \in \overline{V}$. Then

$$\operatorname{Var} x = \operatorname{Var} y\ ,$$

where $\operatorname{Var} y$ may be thought of as the "total length" of $\overline{V}$ with the metric induced from $\mathbb{R}$. For example, if $V = [0,1]$, then $V = \overline{V}$, $x = y$ and

$$\operatorname{Var} x = \int_0^1 |x'(v)|\, dv\ .$$

Remarks. (a) The variation of $x : V \to \mathbb{R}$ is not an especially good measure of complexity as it is unstable under small perturbations of x. But one can stabilize $\operatorname{Var} x$ by introducing for every $0 < \varepsilon < 1$,

$$\operatorname{Var}_\varepsilon x = \inf_y \operatorname{Var}(x+y)$$

over all continuous functions $y : V \to \mathbb{R}$ satisfying

$$\|y\|_\infty \leq \varepsilon \|x\|_\infty$$

for the norm

$$\|x\|_\infty = \sup_{v \in V} |x(v)| \ .$$

(b) We are mainly concerned here with functions on $[0,1]$, but we have presented the definitions keeping an eye on possible generalizations. (Compare §2.1.3.B in $[\mathrm{Gr}]_3$).

2.1. The number of oscillations of a function. For a function $x : V \to \mathbb{R}$ we write

$$\operatorname{Osc} x = \sup_{v \in V} x(v) - \inf_{v \in V} x(v)$$

and then for every positive $\gamma \leq 1$ define the number of γ-oscillations of x as follows. First we say that subsets V_1 nd V_2 and V are *x-independent* if there exists no *connected* subset $U \subset V$ on which x is constant and which meets both subsets V_1 and V_2. Then we define $\#_\gamma \operatorname{Osc} x$ as the maximal integer k for which there exists x-independent subsets $V_j \subset V$ for $j = 1, \ldots, k$, such that

$$\operatorname{Osc} x \mid V_j \geq \gamma \operatorname{Osc} x$$

for $j = 1, \ldots, k$. We abbreviate $\# \operatorname{Osc} = \#_1 \operatorname{Osc}$ and call this *the number of full oscillations* of x. If $V = [0,1]$ then $\# \operatorname{Osc} x$ equals the maximal number k such that $[0,1]$ can be partitioned into k subintervals with equal x-images.

Also notice that

$$\operatorname{Var} x \geq (\gamma \#_\gamma \operatorname{Osc} x) \operatorname{Osc} x$$

and that $\#_\gamma \operatorname{Osc} x$ enjoys an obvious kind of stability under perturbations of x.

2.2 Theorem. *Let Q be a subset in the projectivized space P of continuous functions on $[0,1]$. Then there exists a function $x \in Q$, such that*

$$\# \operatorname{osc} x \geq \operatorname{ess} Q \ .$$

Proof: Apply 4.3 A and A_1 to the space T of partitions of $[0,1]$ into $k+1 = \operatorname{ess} Q$ subintervals. This gives us a partition $[0,1] = \bigcup\limits_{i=0}^{k} I_i$, and an $x \in Q$, such that

$$\operatorname{osc} x \mid I_i = \operatorname{osc} x \ ,$$

for $i = 0, \ldots, k$. Q.E.D.

2.2.A Remarks and corollaries. (a) The above theorem applies, in particular to every $(k'+2)$-dimensional *linear* (sub)space L of functions on $[0,1]$ and claims the existence of a non-zero $x \in L$ having

$$\# \operatorname{osc} x \geq k+1 \ .$$

(b) One easily sees with 2.2 that the ess-spectrum (as well as the pro-spectrum) of the energy $E(x) = \operatorname{var} x / \|x\|_\infty$ is $\lambda_i = i$ for all $i = 0, 1, \ldots$.

(c) The divergence $\lambda_i \to \infty$ of the pro-spectrum can also be derived from the Dvoretzky theorem (see 1.2.E) as follows. Given an $(i+1)$-dimensional space L of functions on $[0,1]$, we have a continuous map of $[0,1]$ into the dual L', such that the functions from L appear as the restrictions of linear functions on L' to $[0,1]$ (see the proof of 1.1). As $i \to \infty$, we can find a k-dimensional subspace $L_0 \subset L$, such that $k \to \infty$ and the corresponding image of $[0,1]$ is ε-dense in the unit ball of L'_0 for some Euclidean metric in L'_0 where $\varepsilon \to 0$ for $i \to \infty$. Then obviously $E(x) \to \infty$ for all $x \in L_0$ and $i \to \infty$.

(d) Theorem 2.2 and its corollaries must be as old as the Bezout-Borsuk-Ulam theorem, but I have not checked the literature.

§3 Asymptotic additivity and homogeneity of Dirichlet energies

3.1 Examples of Dirichlet energies. The classical Dirichlet energy is defined on functions x on a bounded Euclidean domain V by

$$E(x) = \|dx\|_2 / \|x\|_2$$

where d denotes the differential of a function and where the L_2-norms of dx and x are taken with the ordinary Lebesgue measure in V. A more general class of integro-differential energies can be defined as follows. Let X and Y be smooth vector bundles over a manifold V and $D : x \mapsto y$ a linear (or non-linear) differential operator between the sections of X and Y. In order to define what we call the *L_pD/L_q-energy*

$$E(x) = \|Dx\|_p / \|x\|_q \ ,$$

we need the following additional structures (a) and (b).

(a) norms in the vector bundles X and Y. With these we have the point-wise norms $\|x(v)\|$ and $\|y(v)\|$ of sections of X and Y on V.

(b) A measure μ on V which is also denoted dv. With this we have the L_p-norm on sections of X and Y,

$$\|x\|_p = \left(\int_V \|x(v)\|^p dv \right)^{1/p} .$$

Notice that for the $L_\infty D/L_\infty$-energy one only needs the measure class of μ rather than the measure itself.

Remarks. (a) if the operator D has infinite dimensional kernel, then, in order to have an "interesting" spectrum, one should either restrict D to a subspace of sections where the kernel is finite dimensional or to pass to an appropriate quotient space. For example, if D is the exterior differential on *forms* (rather than on functions), then one should work modulo closed (sometimes exact) forms.

(b) It is sometimes interesting to use different measures in defining the norms of x and Dx. For example, one may bring into the picture some measure μ' concentrated on a "subvariety" $V' \subset V$ and then to look at $E = L_p D/L_q(\mu')$.

Let us look more closely at the case were $D = d$ is the exterior differential on functions x on V. Here $X = V \times \mathbb{R} \to V$ is the trivial bundle and $Y = T^*(V)$ is the cotangent bundle. We do not have to worry about a norm on X as we already have one, $\|x(v)\| = |x(v)|$, for the ordinary absolute value on $\mathbb{R}$. On the other hand there is no canonical norm on $T^*(V)$ and so we have to choose one. If V is connected, such a norm defines a metric on V by

$$\mathrm{dist}(v_1, v_2) = \sup_x |x(v_1) - x(v_2)|$$

over all C^1-functions x on V, such that

$$\|dx\|_\infty \underset{\text{def}}{=} \sup_{v \in V} \|dx(v)\| \leq 1$$

This distance (and sometimes the norm itself) is called a *Finsler* metric on V. A Finsler metric is called *Riemannian*, if the norm in each fiber $T_v'(V)$, $v \in V$ is Euclidean.

Remark. Usually one starts with a (dual) norm in the tangent bundle and define the distance as the length of the shortest path $p : [0,1] \to V$ between v_1 and v_2. Namely, the norm in $T(V)$ allows one to measure the tangent vectors $\frac{dp(t)}{dt} \in T_{p(t)}(V)$ and thus to define *the maximal stretch of p,*

$$\|Tp\| = \sup_{t \in [0,1]} \left\| \frac{dp(t)}{dt} \right\| .$$

Then one gets $\mathrm{dist}(v_1, v_2)$ as $\inf\limits_p \|Tp\|$ over all paths p with $p(0) = v_1$ and $p(1) = v_2$.

To conclude the definition of the $L_p d/L_q$-energy on a Finsler manifold V we need a measure on V. Usually one uses the Finsler norm on $T^*(V)$ to provide V with a measure as follows. One considers the *determinant bundle* $\Lambda(V)$ that is the top exterior power $\Lambda^n T^*(V)$ for $n = \dim V$. There are many (unfortunately too many) natural ways to define a norm on $\Lambda(V)$ starting from our norm on $T^*(V)$. Since $\Lambda(V)$ is one-dimensional, a norm on $\Lambda(V)$ is |section| of $\Lambda(V)$, that is a density on V which integrates to a measure on V.

3.1.A Dirichlet on metric spaces. For a function x on a metric space V we define the Lipschitz constant $\operatorname{Lip} x$ as the supremum of $|x(v_1) - x(v_2)|/\operatorname{dist}(v_1, v_2)$ over all pairs of distinct points v_1 and v_2 in V. Then for a point $v \in V$ we restrict x to the ε-balls $B_\varepsilon \subset V$ around v and set

$$|dx(v)| = |\operatorname{Lip}_v x| = \limsup_{\varepsilon \to 0} \operatorname{Lip} x \mid B_\varepsilon ,$$

and $\|dx\| = \sup_{v \in V} |dx(v)|$. Notice that $\|dx\| \le \operatorname{Lip} x$ and state the following

Trivial Lemma. *The following two conditions are equivalent*

(i) $\|dx\| = \operatorname{Lip} x$ *for all functions* x *on* V.

(ii) For every two points v_1 *and* v_2 *with some distance* d *in* V *and every* $\varepsilon > 0$ *there exists a (*ε*-middle) point* $v_\varepsilon \in V$, *such that* $\operatorname{dist}(v_i, v_\varepsilon) \le \varepsilon + \frac{1}{2} d$ *for* $i = 1, 2$.

Metrics satisfying (ii) are called *geodesic*. (They are also called inner metrics, length metrics and local metrics.) Observe that Finsler metrics are geodesic.

Now, with a measure on V we have the $L_p d/L_q$-energy

$$E(x) = \left(\int_V |dx|^p \right)^{1/p} \Big/ \left(\int_V |x|^q \right)^{1/q} .$$

If V is a Finsler space this agrees with the earlier definition. The same can be said for *Carnot spaces* defined below

3.1.B. Carnot spaces. Consider a first order differential operator D on functions x on V, where the range bundle Y is equipped with a norm. The issuing seminorm on C^1-functions,

$$x \longmapsto \|Dx\|_\infty$$

is called a *Carnot structure* on V, provided $D(\text{const}) = 0$, that is $D = h \circ d$ for some homomorphism $h : T^*(V) \to Y$. If h has a constant rank k, then the Carnot structure is uniquely determined by the image bundle of the adjoint homomorphism $h^* : Y^* \to T(V)$, called $\theta = \operatorname{Im} h^* \subset T(V)$, and a norm on θ.

Define Carnot's (semi)metric on V by

$$\operatorname{dist}(v_1, v_2) = \sup_x |x(v_1) - x(v_2)|$$

over all x with $\|Dx\|_\infty \leq 1$. One can equivalently define this "dist" with paths in V tangent to θ. Thus one sees, in particular, that dist is an honest metric, i.e., everywhere $< \infty$, if and only if every two points v_1 and v_2 in V can be joined by a path in V tangent to θ.

Remark. Carnot metrics are sometimes called *Carnot-Caratheodary* (see [G-L-P]) or *subelliptic* (see [St]). Here we reserve the word "*sub-Riemannian*" for the case where the above norm on θ is Euclidean.

3.1.C Alternative definitions of $\|dx\|_p$. Let us recall that the *coboundary* δx of a function x on V is the function on $V \times V$ defined by

$$\delta x(v_1, v_2) = x(v_1) - x(v_2) \ .$$

Next consider the following function K_ε on $V \times V$,

$$K_\varepsilon(v_1, v_2) = \begin{cases} 0 & \text{if } \operatorname{dist}(v_1, v_2) > \varepsilon \\ \varepsilon^{-1} & \text{if } \operatorname{dist}(v_1, v_2) \leq \varepsilon \end{cases}$$

and let $\delta_\varepsilon x$ be the product $K_\varepsilon \delta x$. In other words we restrict δx to the ε-neighbourhood of the diagonal in $V \times V$ and then divide it by ε. Notice that

$$\limsup_{\varepsilon \to 0} \|\delta_\varepsilon x\|_\infty = \|dx\|_\infty \ .$$

Denote by μ' the measure $\mu \times \mu$ on $V \times V$ and let μ'_ε denote the measure of the ε-neighbourhood of the diagonal, that is

$$\mu'_\varepsilon = \varepsilon \int\limits_{V \times V} K_\varepsilon d\mu' \ ,$$

and let

$$\|x\|'_p = \limsup_{\varepsilon \to \infty} \|\delta_\varepsilon x\|_p / \mu'_\varepsilon \ .$$

Notice that for sufficiently smooth Riemannian (and sub-Riemannian) spaces V, $\|x\|'_p = \mathrm{const}_{n,p} \|x\|_p$, where n is the dimension of V (which should be properly defined in the sub-Riemannian case). An advantage of $\|x\|'_p$ over $\|x\|_p$ for non-smooth spaces is clearly seen for $p = 2$ as the norm $\|x\|'_2$ is always Hilbertian and the deviation of $\|dx\|_2$ from being Hilbertian (as well as non-constancy of the norm ratios $\|dx\|_p / \|dx\|'_p$) measures non-smoothness of V. If V is a Finsler manifold (e.g., a domain in a finite dimensional Banach space) this measures how far V is from a Riemannian space. The picture is less clear for nowhere smoth (e.g., fractal) spaces V.

One can generalize the definition of K_ε by taking any function $e(t)$ and by letting

$$K_e = e\big(\operatorname{dist}(v_1, v_2)\big)$$

A classical choice of e is

$$e(t) = \exp -\varepsilon^{-1}t$$

which for $\varepsilon \to 0$ gives us (after a normalization) a regularized version of the above $\|dx\|'_p$.

Finally observe that the functions $K(v_1, v_2) = e\big(\operatorname{dist}(v_1, v_2)\big)$ define integral operators on V,

$$x \longmapsto K * x = \int_V K(v_1, v_2)x(v_1)dv_1 \ .$$

Spectra of such operators are similar to those of the energies $\|dx\|'_p / \|x\|_p$.

Example. Let $x \mapsto K^0_\varepsilon * x$ be the averaging of x over the ε-balls $B(v, \varepsilon)$ in V, that is

$$K^0_\varepsilon(v_1, v_2) = \begin{cases} 0 & \text{for } \operatorname{dist}(v_1, v_2) \geq \varepsilon \\ \big[\mu B(v_2, \varepsilon)\big]^{-1} & \text{for } \operatorname{dist}(v_1, v_2) < \varepsilon \end{cases}$$

(Notice that this K^0_ε is not of the form $e(\text{dist})$, unless the measure $\mu\big(B(v, \varepsilon)\big)$ is constant in v). If V is "sufficiently smooth" then the operator

$$A_\varepsilon = \varepsilon^{-2}(Id - K^0_\varepsilon)$$

converges for $\varepsilon \to 0$ to the Laplace operator $\Delta = d^*d$ on V. In particular, the eigenvalues of the operator $|A^*_\varepsilon A_\varepsilon|^{1/4}$ converge to those of the energy $\|dx\|_2 / \|x\|_2$. This suggests the definition of the norms $\|\Delta x\|_p = \limsup\limits_{\varepsilon \to \infty} \|A_\varepsilon x\|_p$ for an arbitrary metric space V. Probably, the existence of sufficiently many x with $\|\Delta x\|_p \leq \infty$ implies certain smoothness of X. Otherwise one may try norms associated to more regular operators, for example $\varepsilon^{-\rho}(Id - K^0_\varepsilon)$ for $\rho < 2$.

3.1.D. The above relation between metrics in V and norms on function spaces is of quite general nature. Namely, every seminorm on the space X of (say, continuous) functions x on V defines a norm in the dual X'. As V is canonically mapped into X' by Dirac's $v \mapsto \delta_v$, we get an induced (Caratheadory) metric on V. More generally, if X is the space of sections of a k-dimensional vector bundle over V, then V is naturally mapped into the Grassmanian of the k-planes in X', which again induces a (Bergman) metric in V from a seminorm in X.

The major problem of the geometric spectral theory is to relate the properties of such metrics on V with the spectra of the (ratios between) norms in question.

Remark. The above metric on V may degenerate. For example if we use the norm $\|L_p d\|$ on an n-dimensional manifold V, then the resulting metric on V is degenerate for $n \geq p$. In such a case it is useful to consider the following distance between subsets (rather than points) in V,

$$\operatorname{dist}(V_1, V_2) = \sup_x \|x\|^{-1} ,$$

where x sums over all functions which are equal zero on V_1 and one on V_2. Notice that dist^{-1} is called the *capacity* (associated to the norm $\| \quad \|$) and it has been extensively studied for the above norm $\|L_p d\|$ (see [M-H]).

3.2 Dirichlet energy under cutting and pasting. Start with the simplest case where V is the disjoint union of V_1 and V_2 and canonically decompose each function x on V into the sum $x_1 + x_2$ where $x_1 \mid V_2 = 0$ and $x_2 \mid V_1 = 0$. One trivially has

3.2.A Lemma. *If $p \leq q$ then the energy $E(x) = \|Dx\|_p / \|x\|_q$ satisfies*

$$E(x) \geq \min\big(E(x_1), E(x_2)\big) . \tag{$*$}$$

On the contrary, if $p \geq q$, then

$$E(x) \leq \max\big(E(x_1), E(x_2)\big) . \tag{$**$}$$

In particular, if $p = q$ and say $E(x_1) \leq E(x_2)$, then

$$E(x_1) \leq E(x) \leq E(x_2) .$$

This implies the following sub-additivity of the number $N(\lambda)$ of the eigenvalues $\leq \lambda$, that is

$$N(\lambda) = \text{“dim”} E^{-1}(-\infty, \lambda] + 1$$

for a given “dim” (see 0.4).

3.2.A$_1$. *If $p \geq q$ then $N(\lambda) \geq N_1(\lambda) + N_2(\lambda)$ where $N_i(\lambda) = N(\lambda, E \mid V_i)$ for $i = 1, 2$.*

Proof: The inequality $(**)$ shows that the $*$-product (see 0.4) of $E_1^{-1}(-\infty, \lambda) * E_2^{-1}(-\infty, \lambda)$ is contained in $E^{-1}(-\infty, \lambda)$ for all λ (here $E_i = E_i \mid V_i$), and 3.2.A$_1$ follows.

3.2.A$_2$. Suppose our “dim” is sub-additive,

$$\text{“dim”} A \cup B \leq \text{“dim”} A + \text{“dim”} B + 1 .$$

Then for $p \le q$,

$$N(\lambda) \le N_1(\lambda) + N_2(\lambda) \ ,$$

as

$$E^{-1}[0,\lambda] \subset E_1^{-1}[0,\lambda] \cup E_2^{-1}[0,\lambda] \ .$$

Remind, that ess and $\text{pro}^{\perp}$ are subadditive which implies the above inequality for the respective $N(\lambda)$.

3.2.A$_3$ Remark. If "dim" is not subadditive one can bound the $\{\lambda_{ij}\}$-spectrum (see 0.6) rather than $\{\lambda_i\}$ as follows. Let $M(\lambda, N)$ be the maximal number, such that every N-"dimensional" subset A in P (where the energy E lives) satisfies

$$\text{"dim"}\big(A \cap E^{-1}[\lambda,\infty)\big) \ge M \ .$$

(Notice that this M can be obviously expressed in terms of λ_{ij}.) Then for $p \le q$ one trivially has

$$M(\lambda, N) \ge M_2\big(\lambda, M_1(\lambda, N)\big)$$

for all N and λ, where M_1 and M_2 refer to $E \mid V_1$ and $E \mid V_2$ correspondingly

Let us summarize the previous discussion for $p = q$ and "dim"$=$ ess.

3.2.B Additivity of the spectrum for the energy $E(x) = \|Dx\|_p/\|x\|_p$. *If* V *is the disjoint union of* V_1 *and* V_2 *then the number*

$$N(\lambda) = \operatorname{ess} E^{-1}(-\infty, \lambda] + 1$$

is the sum of those for V_1 *and* V_2,

$$N(\lambda) = N_1(\lambda) + N_2(\lambda) \ .$$

Remarks. (a) *According to our notation this includes the case* $E(x) = \|dx\|_p/\|x\|_p$ *on an arbitrary metric space* V.

(b) The above additivity property trivially generalizes to the case where the measure μ underlying $E(x)$ is decomposed into a sum of measures, $\mu = \mu_1 + \mu_2$, such that the supports of μ_1 and μ_2 are disjoint.

3.2.C Monotonicity of $E(x)$. Let $f : V' \to V$ be a locally homeomorphic map. Then vector bundles on V induce those on V' and a given operator D on V lifts to D' on V'. Now, if our measure μ on V is the push-forward of some μ' on V', then the pull-back map $x \mapsto x' = f^*(x)$ preserves $E = L_p D/L_q$ for $E'(x') = E(x)$, and this remains valid for $E = L_p d/L_q$ on metric spaces.

3.2.C$_1$ Corollary. *Let $\{V_j\}$, $j = 1, \dots, k$ be an open cover of V and functions $p_j : V_j \to \mathbb{R}_+$ form a partition of unity. Then the counting function $N(\lambda)$ for $E = L_p D/L_q$ on (V, μ) is bounded by the functions $N_j(\lambda)$ on $(V_j, p_j\mu)$,*

$$N(\lambda) \leq \sum_{j=1}^{k} N_j(\lambda) \ ,$$

provided $p \leq q$ and "dim" is subadditive (compare 3.1.A$_2$).

3.2.D Energy and $N(\lambda)$ on V/V_0. Denote by $P_0 \subset P$ the space of functions (or sections) vanishing on a given subset $V_0 \subset V$. An important example is where $V_0 = \infty$ and then P_0 by definition of this ∞ consists of functions with compact supports. The energy E restricted to P_0 is also called E on V/V_0 and the corresponding counting function is denoted $N(\lambda, V/V_0)$ or just $N^0(\lambda)$. If V_0 is not specified then $N^0(\lambda)$ refers to $N(\lambda, V/\infty)$.

It is obvious that

$$N^0(\lambda) \leq N(\lambda)$$

and that

$$N(\lambda, U/\infty) \leq N(\lambda, V/\infty)$$

for all open subsets $U \subset V$. It follows (see $(*)$ in 3.2.A) that for $p \geq q$

$$N^0(\lambda) \geq \sum_{j=1}^{k} N_j^0(\lambda)$$

where $N_j^0 = N^0(V_j)$ for disjoint open subsets $V_1, \dots, V_j, \dots, V_k$ in V.

3.2.E A bound on the counting function $N(\lambda)$ on V by those on V/V_0 and V_0. Let $V_\varepsilon \subset V$ denote the ε-neighbourhood of V_0,

$$V_\varepsilon = \left\{ v \in V \mid \operatorname{dist}(v, V_0) \leq \varepsilon \right\} ,$$

and $\|x\|_q^\varepsilon$ denote the L_2-norm of the restriction $x \mid V_\varepsilon$. Let $E_\varepsilon(x) = \|Dx\|_p / \|x\|_q^\varepsilon$ and denote by $N_\varepsilon(\lambda)$ the corresponding counting function. Notice that $E_\varepsilon(x) \geq E_\varepsilon(x \mid V_\varepsilon)$ and $N_\varepsilon(\lambda) \leq N(\lambda, V_\varepsilon)$.

Next we recall $N^0(\lambda) = N(\lambda, V/V_0)$ and we assume that $D = d$ and $p = q$. Thus the functions $N(\lambda)$, $N^0(\lambda)$ and $N_\varepsilon(\lambda)$ count the energy levels for $L_p d/L_p$.

3.2.E$_1$ Lemma. *If the implied dimension is subadditive then*

$$N(\lambda) \le N^0(\lambda') + N_\varepsilon(\lambda'')$$

for

$$\lambda = \lambda'\lambda''/(\lambda'' + \lambda' + \varepsilon^{-1})$$

and for all positive λ', λ'' *and* ε.

Proof: Let $a_\varepsilon(v) = \varepsilon^{-1}\,\mathrm{dist}(v, V_0)$ for $v \in V_\varepsilon$ and $a_\varepsilon(v) = 1$ outside V_ε. Then

$$\|d(a_\varepsilon x)\|_p \le \|Dx\|_p + \varepsilon^{-1}\|x\|_p^\varepsilon \ .$$

Now the inequalities

$$\|d(a_\varepsilon x)\|_p \ge \lambda'\|a_\varepsilon x\|_p \ ,$$

$$\|dx\|_p \ge \lambda''\|x\|_p^\varepsilon$$

and

$$\|a_\varepsilon x\|_p + \|x\|_p^\varepsilon \ge \|x\|_p$$

imply

$$\|dx\|_p \ge \lambda\|x\|_p$$

for $\lambda = \lambda'\lambda''/(\lambda'' + \lambda' + \varepsilon^{-1})$ and the proof follows.

3.2.F Asymptotic additivity of the function $N(\lambda)$**.** Call a subset $V_0 \subset V$ *thin* if for every $C \ge 0$ there exist $\varepsilon > 0$ and $\lambda_0 \ge 0$, such that N_ε defined in 3.1.E satisfies for all $\lambda \ge \lambda_0$,

$$CN_\varepsilon(C\lambda) \le N(\lambda) \ .$$

We call $N(\lambda)$ asymptotically equivalent to $M(\lambda)$ and write

$$N(\lambda) \sim M(\lambda)$$

if

$$N(C\lambda) \ge M(\lambda) \ge N(C^{-1}\lambda)$$

for every $C > 1$ and all sufficiently large (depending on C) λ.

3.2.F Weyl additivity theorem. *Let the metric space V be decomposed into the union of closed subsets $V = V_1 \cup V_2$, where the intersection $V_0 = V_1 \cap V_2$ is thin. Then the implied counting function $N(\lambda)$ for the energy $E = L_p d/L_p$ and "dim" = ess satisfies*

$$N(\lambda) \sim N_1(\lambda) + N_2(\lambda)$$

where N_i for $i = 1, 2$ are the corresponding functions for V_1 and V_2.

Proof: This follows from 3.2.E$_1$ and 3.2.B.

Remark. Instead of using the specific cut-off function $a_\varepsilon = \varepsilon^{-1}$ dist, one could postulate the existence of such a function with an appropriate notion of *capacity* of V_0 (compare 3.1.D). Thus one would obtain a more general (and more conceptual) version of the additivity theorem.

3.3 The function $N(\lambda)$ and the covering numbers. For a metric space V we consider the numbers $\mathrm{COV}(\varepsilon)$, which is the minimal number of ε-balls needed to cover V, and the number $\mathrm{IN}(\varepsilon)$, which is the maximal number of disjoint ε-balls in V. Notice that

$$\mathrm{COV}(\varepsilon) \geq \mathrm{IN}(\varepsilon) \geq \mathrm{COV}(2\varepsilon)$$

for all $\varepsilon \geq 0$.

Also notice that these numbers asymptotically for $\varepsilon \to 0$ are additive as $N(\lambda)$ and in some cases $N(\lambda)$ can be roughly estimated in terms of $\mathrm{COV}(\lambda^{-1})$. First we give such estimates in the easiest case $E = L_\infty d/L_\infty$.

3.3.A Observation. *The function $N(\lambda)$ = "dim"$E^{-1}(-\infty, \lambda]+1$ for $E(x) = \|dx\|_\infty/\|x\|_\infty$ on a geodesic (see 3.1) metric space V satisfies for all $\lambda > 0$,*

$$\mathrm{IN}(2\lambda^{-1}) \leq N(\lambda) \leq \mathrm{COV}(\lambda^{-1}) \ .$$

Proof: Given disjoint ε-balls $B_1, \ldots, B_N$ in V we consider the linear space L of functions generated by the constants and the functions $\mathrm{dist}(v, V \backslash B_i)$, $i = 1, \ldots, N$. Then the (obvious) inequality

$$2\|x\|_{L_\infty} \geq \varepsilon \|dx\|_\infty$$

for all $x \in L$ yields the lower bound on $N(\lambda)$.

To get the upper bound we observe that every "N-dimensional" subspace in the projective space P of functions on V contains (see 0.4) a function x vanishing on a given subset $S \subset V$ consisting of N-points. Since V is geodesic, such an x is bounded by

$$\|x\|_\infty \leq \|dx\|_\infty \sup_{v \in V} \mathrm{dist}(v, S) \ ,$$

which trivially yields the desired upper bound on $N(\lambda)$.

3.3.B The μ-regularity constant and an upper bound on $N(\lambda)$ for $E = L_p d/L_p$. Denote by $\delta = \delta(V, \mu)$ the minimal number such that every two concentric balls on V of radii R and $2R$ satisfy

$$\mu(B(2R)) \leq 2^\delta \mu(B(R))$$

for the given measure μ on V.

Example. If $V = \mathbb{R}^n$ then $\delta = n$. Moreover, if V is a complete Riemannian manifold with non-negative Ricci curvature then also $\delta = \dim V$.

3.3 B$_1$ Observation. *The function $N(\lambda)$ for $E = L_p d/L_p$ satisfies*

$$N(\lambda) \geq \mathrm{IN}(C\lambda^{-1})$$

for

$$C = 2^{2+\delta/p} .$$

Proof: Consider the linear space L of functions on V generated by constants and the functions $\operatorname{dist}(v, V\backslash B_i)$ for disjoint ε-balls B_i in V. Every $x \in L$ obviously satisfies

$$\|dx\|_p \leq C\varepsilon^{-1}\|x\|_p$$

which immediately yields what we want.

3.3.C Local and global lower bounds on the spectrum. Let V be μ-partitioned into closed subsets V_j, $j = 1, \ldots, k$, that is $V = \bigcup_j V_j$ and doubly covered points in V have measure zero. *If "dim" is subadditive and $p = q$, then, as we know,*

$$N(\lambda, V) \leq \sum_j N(\lambda, V_j) . \tag{$*$}$$

In particular, if

$$\lambda = \min_j \lambda_1(V_j) \tag{$*$}$$

then $N(\lambda) \leq k + 1$. More generally, if

$$\lambda = \min_j \lambda_{i_j}(V_j) ,$$

then

$$N(\lambda) \leq \sum_j i_j + 1 . \tag{$**$}$$

Remark. The presence of constant functions makes $\lambda_0 = 0$ which forces us to use $\lambda_{i_j}(V_j)$ for $i_j \geq 1$. On the other hand the number $\lambda_1(V_j)$ for "nice" small subsets V_j is expected to be $\sim (\operatorname{diam} V_j)^{-1}$. For example, smooth domains in $\mathbb{R}^n$, and more generally, compact Riemannian manifolds do admit arbitrarily fine "nice" partitions. Unfortunately, the construction of "nice" partitions may be quite difficult (if at all possible) for general spaces X. (A trivial obstruction to the "niceness" is disconnectedness. In fact, a set with $m+1$ connected components have $\lambda_0 = \lambda_1 = \ldots = \lambda_m = 0$.) To alleviate this problem we introduce the following.

3.3.C$_1$ Mollified spectrum. Take a neighbourhood $U \subset V$ of a subset $V_0 \subset V$ and let $\widetilde{x}$ denote extensions to $U \supset V_0$ of functions x on V_0. Then we define

$$\|\widetilde{dx}\|_p = \inf_{\widetilde{x}} \|d\widetilde{x}\|_p$$

and study the corresponding $\widetilde{E}(x) = \|\widetilde{dx}\|_p / \|x\|_q$ and $\widetilde{N}(\lambda)$ for functions x on V_0.

Remark. This $\widetilde{E}$ is a special case of an energy E where one uses two different measures for the definition of $\|dx\|_p$ and $\|x\|_q$. The properties of such energies are quite similar to those where there is only one measure. In fact one can often reduce two measures to one by modifying the operator D in question.

Now, consider a covering $V = \bigcup_j V_j$ and let $U_j \supset V_j$ be neighbourhoods such that the multiplicity of the covering of V by U_j is at most m. Then the function $N(\lambda, V)$ for $E = L_p d / L_p$ and "dim"$=$ ess satisfies

$$N(m^{-\frac{1}{p}}\lambda) \leq k+1$$

where

$$\lambda = \min_j \widetilde{\lambda}_1(V_j) \tag{$\tilde{*}$}$$

for the mollified $\widetilde{\lambda}_1$ of V_j in U_j. This is proven the same way as above $(*)$ and $(**)$ also generalizes to

$$N(m^{-\frac{1}{p}}\lambda) \leq \sum_j i_j + 1 \tag{$\widetilde{**}$}$$

for $\lambda = \min_j \widetilde{\lambda}_{i_j}(V_j \subset U_j)$.

3.3.C$_2$ Corollary. *Let the μ-constant $\delta(V) < \infty$ (see 3.3.B) and let for every ε-ball $B(\varepsilon)$ in V the mollified eigenvalue $\widetilde{\lambda}_1(B(\varepsilon) \subset B(\rho\varepsilon))$, for the concentric $\rho\varepsilon$-ball satisfies $\widetilde{\lambda}_1 \geq \tau\varepsilon^{-1}$ for given constants $\rho \geq 1$ and $\tau > 0$ and for all $\varepsilon > 0$. Then*

$$N(\lambda) \leq a\,\mathrm{COV}(\lambda^{-1})$$

for some constant $\underline{a} = a(\delta, \rho, \tau) > 0$.

Proof: The inequality $\delta < \infty$ gives us a control over multiplicities of coverings of V by $\rho\varepsilon$-balls, where V is already covered by the concentric ε-balls.

Besides, δ controls the growth of $\mathrm{COV}(\varepsilon)$ which is sufficient for our purpose. We leave the (trivial) details to the reader.

3.3.C_3 Remarks. (a) If V satisfies the assumptions of 3.3.C_2, then 3.3.B_1 also applies, which shows that $N(\lambda)$ has the same order of magnitude for $\lambda \to \infty$ as $\mathrm{COV}(\lambda^{-1})$. In particular, a subset V_0 is thin (see 3.2.F) if and only if its covering number satisfies

$$\mathrm{COV}(\varepsilon, V_0)/\,\mathrm{COV}(\varepsilon, V) \to 0 \quad \text{for} \quad \varepsilon \to 0\,.$$

Another consequence of the above discussion is the existence of constants $d = d(V) \geq 0$ and $b_i = b_i(V) > 0$ for $i = 1, 2$, such that

$$b_1 \lambda^d \leq N(\lambda) \leq b_2 \lambda^d\,.$$

We shall see later on that for $\lambda \to \infty$ one can take $b_1 \to b_2$, provided the space V is "infinitesimally renormalizable" (see 3.4).

(b) The conclusion of 3.3.C_2 remains valid if the bound $\widetilde{\lambda}_1 \geq \tau\varepsilon^{-1}$ is replaced by $\widetilde{\lambda}_j \geq \tau\varepsilon^{-1}$ for a fixed $j \geq 1$ and if one uses $a = a(\delta, \rho, \tau, j)$.

(c) Lower bounds on λ_1 often come under the name of *Poincaré-Sobolev inequalities*. By Cheeger's theorem, the first eigenvalue of $E(x) = \|dx\|_2/\|x\|_2$ on a Riemannian manifold can be bounded from below by the *isoperimetric* constant (see below) and Cheeger's argument (based on the coarea formula) can be generalized to non-Riemannian geodesic spaces. Let us indicate several examples where $\lambda_1 \geq \mathrm{const}\,\mathrm{Diam}\,V$.

(c_1) V is the interval with the standard metric and measure. The lower bounds on all λ_i are immediate here.

(c_2) V is the Euclidean ball or cube. Then the inequality $\lambda_1 \geq \mathrm{const}_n\,\mathrm{Diam}$ follows from the following multiplicativity of λ_1

$$\lambda_1(V_1 \times V_2) \geq \mathrm{const}\min\big(\lambda_1(V_1), \lambda_1(V_2)\big)\,.$$

In fact λ_1 of certain "fibered spaces" V can be bounded from below by those of the base and the fibers. We shall show this in another paper where we shall generalize Kato's inequality to non-linear spectra.

(c_3) Recall that a (geodesic) *segment* $[v_1, v_2] \subset V$ for v_1 and v_2 in V is the image of an isometric map $[0, d] \to V$ for $d = \mathrm{dist}(v_1, v_2)$ which sends $-1 \to v_1$ and $1 \to v_2$. A subset $V_0 \subset V$ is called a *d-cone* from $v_0 \in V$ if it is a union of segments of length d issuing from v_0. If V_0 is a cone, one naturally defines *αd-cones* $\alpha V_0 \subset V$ for $\alpha \in [0, 1]$.

For a μ-measureable cone V_0, we consider the function $\mu(\alpha) = \mu(\alpha V_0)$ which is monotone in α and so almost everywhere differentiable. Divide the measure of the complement $V_0 \backslash \alpha V_0$ by the derivative of $\mu(\alpha)$ and let

$$b(V_0) = \sup_{\alpha \geq \frac{1}{2}} \mu(V_0 \backslash \alpha V_0)/\mu'(\alpha) .$$

Then take the supremum over all d-cones V_0 in V,

$$b_d = b_d(V) = \sup_{V_0} b(V_0) .$$

It is shown in $[\mathrm{Gr}]_4$, for Riemannian manifolds V, that $\widetilde{\lambda}_1$ of $B(\varepsilon) \subset B(10\varepsilon)$ can be bounded from below by $\lambda_1 \geq C\varepsilon$ where $C > 0$ depends only on $\sup\limits_{d \leq 20\varepsilon} b_d$. In fact the argument in $[\mathrm{Gr}]_4$ extends to all metric spaces and (as we shall prove elsewhere) yields the following more general (and especially useful for Carnot spaces) lower bound on $\widetilde{\lambda}_1$.

(c_4) Instead of joining points by segments we join them by *random paths*. Namely, to each pair of points $(v_1, v_2) \in V \times V$ we assign a probability measure $\widetilde{\mu}_{v_1,v_2}$ in the space of continuous maps $[0,1] \to V$ joining v_1 and v_2. By integrating this measure over $V \times V$ we get a measure on the space of maps $[0,1] \to V$, called $\widetilde{\mu}$. Similarly, for each $v_0 \in V$ we have the integrated measure $\widetilde{\mu}_{v_0}$ in the space P_{v_0} of paths issuing from v_0.

Next consider a "hypersurface" in V that is a subset H whose ε-neighbourhoods H_ε satisfy

$$A(H) \underset{\text{def}}{=} \limsup_{\varepsilon \to 0} \varepsilon^{-1} \mu(H_\varepsilon) < \infty$$

and denote by $P_{v_0}(H) \subset P_{v_0}$ the subset of path $p : [0,1] \to V$, such that $p(t) \in H$ for some $t \geq \frac{1}{2}$. Define

$$\widetilde{b} = \sup_{H, v_0} \widetilde{\mu}(P_{v_0}(H))/A(H) .$$

Notice that this $\widetilde{b}$ (as well as b_d of the previous section) is an essentially local invariant in the space of paths.

It is nearly obvious (compare $[\mathrm{Gr}]_4$) that the inequality $\widetilde{b} = \widetilde{b}(\mu) < \infty$ for some $\widetilde{\mu} = \mu_{v_1,v_2}$ gives us the following.

Isoperimetric inequality. *Let V be a compact metric space and let V_1 and V_2 be compact subsets in V separated by a hypersurface H in V (i.e., V_1 and V_2 lie in different components of $V \backslash H$). Then* $\min\big(\mu(V_1), \mu(V_2)\big) \leq 4\widetilde{b}A(H)$.

By Cheege's theorem this suffices to bound λ_1 (and $\widetilde{\lambda}_1$) from below.

Notice that the "geodesic cone" set-up (see (C_3)) corresponds to the Dirac δ-mass supported on a geodesic segment between v_1 and v_2 (at least for those v_1 and v_2 where such a segment is unique).

3.3.C_4 Reduction of the (isoperimetric) Sobolev inequality to Poincaré inequality. Let every ball $B \subset V$ satisfy the following two conditions

(1) POINCARÉ PROPERTY. Every hypersurface H in B dividing B into two pieces of equal measure satisfies

$$A(H) \geq C\left[\mu(B)\right]^{\alpha}$$

for some constant $C > 0$ and $0 < \alpha < 1$.

(2) UNIFORM COMPACTNESS. There are at most k points in B whose mutual distances are all $\geq$ radius of B.

If V is a geodesic space, then the boundary of every subset W with $\mu(W) \leq \frac{1}{2}\mu(V)$ satisfies

$$A(\partial W) \geq K^{-1}C\big(\mu(W)\big)^{\alpha}. \qquad (*)$$

Proof: To simplify the matter, assume that $\mu\big(W \cap B_v(r)\big)$ is continuous in the radius r of the ball around each point $w \in W$. Then there exists a ball of maximal radius say B_1, such that $\mu(B_1 \cap W) = \frac{1}{2}\mu(W)$. Then we take the second such largest ball B_2 with center outside B_1, then B_3 with center outside the union $B_1 \cup B_2$ and so on. Thus we obtain balls $B_1, \ldots, B_i, \ldots$ covering W. If some of these balls intersect at $w \in W$, then their centers, say $v_1, \ldots, v_\ell$, satisfy for all $1 \leq i < j \leq \ell$

$$\operatorname{dist}(v_i, v_j) \geq \max\big(\operatorname{dist}(v_i, w), \operatorname{dist}(v_j, w)\big)\ .$$

Since V is a *geodesic* space, there exist points $v_i' \in B_i$, such that

$$\operatorname{dist}(v_i', w) = \delta = \min_{1 \leq i \leq \ell} \operatorname{dist}(v_i, w)$$

and

$$\operatorname{dist}(v_i', v_i) = \operatorname{dist}(v_i, w) - \delta\ .$$

Clearly,

$$\operatorname{dist}(v_i', v_j') \geq \delta$$

and so $\ell \leq k$. (This argument reproduces the standard proof of Besicoviĉ covering lemma.)

Now, we apply (1) to $H_i = B_i \cap \partial W$ and obtain

$$A(H_i) \geq C\mu(B_i \cap W)$$

and then $(*)$ by adding these inequalities over all $i = 1, 2, \ldots$.

Application to λ_1. By Mazia-Cheeger inequality our $(*)$ implies

$$\|x\|_{L_q} \leq \text{const}\,\|dx\|_{L_1}$$

for const $=$ const$(k^{-1}C, \alpha)$, for $q = \alpha^{-1}$ and all functions x on V whose both levels V_+ where $x \geq 0$ and V_- have measures $\geq \frac{1}{2}\mu(V)$. It follows, that the first eigenvalue λ_1 of $E = L_1 d/L_q$ for $q = \alpha^{-1}$ is $\geq \text{const}^{-1} > 0$. (Notice that the inquality (1) we started with expresses a kind of lower bound on the first eigenvalue of $L_1 d/L_1$ on the ball B.)

3.3.D Spectra of disjoint unions $V = \bigcup\limits_k V_k$ for $p \neq q$. As we have seen earlier, the spectral function $N(\lambda) =$ "dim"$E^{-1}[0,\lambda]$ of V is the sum of the corresponding functions $N_k(\lambda)$ of V_k, provided "dim" is subadditive (e.g., "dim" $=$ ess) and $E = L_p d/L_q$ for $p = q$. If $p \neq q$, then the determination of best bounds on $N(\lambda)$ in terms of $N_k(\lambda)$ is a non-trivial problem which is closely related to the spectrum of L_p/L_q (compare §1.). To see this relation we consider several examples, where we assume for simplicity's sake that all pieces V_k, $k = 1, \ldots, \ell$, have the same measure $\mu(V_k) = \ell^{-1}$.

3.3.D$_1$. *Let $N_k(\alpha) \geq 1$ for some $\alpha > 0$ and all $k = 1, \ldots, \ell$ and let $N'(\lambda)$ be the spectral function for the energy $E'(y) = \|y\|_{L_p}/\|y\|_{L_q}$ on the measure space consisting of ℓ atoms of mass ℓ^{-1}. Then*

$$N(\lambda) \geq N'(\beta\lambda)\ ,$$

for $\beta = \alpha^{-1}\ell^{\frac{1}{p}-\frac{1}{q}}$.

Proof: Take functions x_k on V_k for $k = 1, \ldots, \ell$, such that $E(x_k) \leq \alpha$, and observe that the restriction of E to the span of these x_k is bounded by $\beta E'$.

3.3.D$_2$. Let us apply the above to the spectrum of $L_p d/L_q$ on a metric space V, which satisfied the following strong regularity assumption. Every two (not necessarily) concentric balls B_1 and B_2 in V of radii R and $2R$ satisfy

$$C^{-1} \leq \mu(B_1)/\mu(B_2) \leq C$$

for all $R > 0$ and a fixed $C = C(V) > 0$. We recall the maximal number $\mathrm{IN}(\varepsilon)$ of disjoint ε-balls in V and look at linear combinations of standard functions supported in such balls. Then *for the ess-spectral function $N^{\mathrm{ess}}(\lambda)$ we obtain with the following lower bound*

$$N^{\mathrm{ess}}(\lambda) \geq b\,\mathrm{IN}(\lambda^{-1})$$

for some constant $b > 0$ depending only on C.

Remark. If one wants to estimate the *pro-spectrum* of $L_p d/L_q$ one should invoke estimates by Kaŝin and Gluskin of the pro-spectrum of L_p/L_q (see [Pi]).

3.3.D$_3$. Suppose that the (mollified if necessary) spectral function of every ε-ball satisfies for given p and q,

$$N^{\mathrm{ess}}(\lambda_0, B_\varepsilon) \le \mathrm{const}\, \varepsilon^{-1} \big(\mu(B_\varepsilon)\big)^{\frac{1}{p}-\frac{1}{q}} ,$$

for some fixed $\lambda_0 > 0$ and all $\varepsilon > 0$. *Then for $p \ge q$ the function $N^{\mathrm{ess}}(\lambda)$ of V is bounded by*

$$N^{\mathrm{ess}}(\lambda) \le c\,\mathrm{IN}(\lambda^{-1}) ,$$

by the earlier additivity argument. Thus

$$N^{\mathrm{ess}}(\lambda) \asymp \mathrm{IN}(\lambda^{-1}) .$$

To grasp the meaning of this asymptotic relation, let ε_i be the maximal number for which there are i disjoint ε_i-balls $B_i, B_2, \dots, B_i$ in V and let x_i denote the distance function to the complement of these balls,

$$x_i(v) = \mathrm{dist}\Big(v, V \setminus \bigcup_{j=1}^{i} B_j\Big) .$$

Then the above discussion amounts to saying that x_i approximately equals the i-th "eigenfunction" of the energy $E(x) = \|dx\|_{L_p}/\|x\|_{L_q}$, that is

$$\lambda_i^{\mathrm{ess}} \asymp E(x_i) .$$

3.3.E Pro-spectra for $p > q$. Let us show that pro-spectrum in most cases grows faster than the ess-spectrum for $p > q$. Namely $\lambda_i^{\mathrm{pro}}/\lambda_i^{\mathrm{ess}} \to \infty$ for $i \to \infty$.

Start with the simplest case, where $p = \infty$ and $q = 2$. Assume that V can be covered by i balls of radius $\varepsilon = \varepsilon_i$ and show that

$$\lambda_{2i}^{\mathrm{pro}} \ge \sqrt{i}\,\varepsilon_i$$

provided $\mu(V) = 1$. In fact, let L be a $2i$-dimensional linear space of functions on V and $L' \subset L$ an i-dimensional subspace of the functions vanishing at the centers of the covering balls. Then every $x \in L'$ has $\|x\|_{L_\infty} \le \varepsilon^{-1}\|dx\|_{L_\infty}$ and our claim follows from 1.1.B.

This argument applies to all $q < \infty$ and yields the relation $\lambda_i^{\mathrm{pro}}/\lambda_i^{\mathrm{ess}} \to \infty$ under the regularity assumption on V.

3.3.E$_1$. In order to make the above argument work for $p < \infty$ we must first project our L to some finite dimensional L_p-space, and then apply the results of Kašin and Gluskin cited earlier. Such a projection is customarily constructed either with *spline approximations* (discretization) or with *smoothing operators*. Recall that the set S of functions on V is called an (ε, d)-*spline* if the restriction of S on each ε-ball in V is at most d-dimensional. In what follows we shall only use very primitive piece-wise constant splines which correspond to the smoothing with the kernel K_ε in 3.1.C. (A discussion on deep smoothing of Nash can be found in [Gr]$_3$.)

Let us assume every ε-ball $B_\varepsilon \subset V$ satisfies the following:

Mollified Poincaré L_p-lemma. *If a function x on B_ε has $\int\limits_{B_\varepsilon} x dv = 0$, then the L_p-norm B_ε of x on the concentric ball B_δ is bounded by the L_p-norm of dx on B_ε as follows*

$$\|x \mid B_\delta\|_{L_p} \leq C\varepsilon^{-1}\|dx\|_{L_p} ,$$

for a fixed $C > 0$ and all δ satisfying

$$\delta \leq C^{-1}\varepsilon.$$

Let us also assume V is regular as earlier and prove the following:

Theorem. *If $q < p$ then*

$$\lambda_i^{\text{pro}} \geq \text{const}\, i^\theta \lambda_i^{\text{ess}}$$

for some positive const and θ, and all $i = 1, 2, \ldots$.

Proof: Let L be a $2i$-dimensional linear space of functions on V. Take the minimal $\varepsilon = \varepsilon_i$, such that some δ-balls for $\delta \leq C^{-1}\varepsilon$, say $B_1(\delta), \ldots, B_i(\delta)$ cover V. Notice that we may assume the covering by the concentric ε-balls has bounded (independent of i) multiplicity. Denote by $L' \subset L$ the i-dimensional subspace defined by the equations

$$\int\limits_{B_j(\varepsilon)} x\, dv = 0 , \qquad j = 1, \ldots, i ,$$

and let

$$\mu_i' = \sup_{x \in L'} \|dx\|_p / \|x\|_p .$$

Notice that

$$\mu_o' \geq \text{const}\, \varepsilon_i^{-1} ,$$

by the earlier discussion.

Now we take

$$\varepsilon' = \varepsilon'_i = (C'\mu'_i)^{-1}$$

for large (but independent of i) constant, and consider a covering of V by i' balls of radius $\delta' = C^{-1}\varepsilon^1$. We may assume (slightly changing the covering if necessary), that there exists a partition of V into i' subsets V_j of equal mass $= \mu(V)/i'$, such that each subset is contained in a δ-ball of the covering.

Let $x \mapsto \overline{x}$ be the linear operator, which averages x over each V_j, $j = 1, \ldots, i'$. Namely $\overline{x}$ is constant and equal $\int_{V_j} x/\mu(V_j)$ on every V_j. Now we see that

$$\lambda_{2i}^{\text{pro}} \geq \lambda'_i \mu'_i \ ,$$

where λ'_i is the i-the eigenvalue of $E' = L_p/L_q$ on the i'-dimensional space, and the theorem easily follows from the known bound on λ'_i (see [Kaŝ] and [Pi]).

3.4 Selfsimilarity and asymptotics $N(\lambda) \sim \text{const}\,\lambda^d$. This signifies the existence of the limit,

$$\text{const} = \lim_{\lambda\to\infty} N(\lambda)/\lambda^d \ ,$$

and one is most happy when $0 < \text{const} < \infty$. Notice that the relation $N(\lambda) \sim \text{const}\,\lambda^d$ is equivalent to the *asymptotic homogeneity* of $N(\lambda)$, that is

$$N(a\lambda) \sim a^d N(\lambda)$$

for every fixed $a > 0$ and $\lambda \to \infty$. We shall see below that in certain cases this asymptotics follows from (infinitesimal) homogeneity of the energy.

3.4.A Example. Let V_ε denote the ε-cube $[0,\varepsilon]^n$ and

$$aV_\varepsilon = V_{a\varepsilon} \quad \text{for} \quad a > 0 \ .$$

We also denote by $a : V_\varepsilon \to aV_\varepsilon$ the obvious (scaling) map which transforms functions x on V_ε to those on aV_ε. Namely $x(v) \mapsto x(a^{-1}v)$, that is $x \mapsto x \circ a^{-1}$. It is obvious that the energy $E(x) = \|dx\|_p/\|x\|_p$ is homogeneous

$$E(x \circ a^{-1}) = a^{-1}E(x) \ .$$

Next we observe that for every $k = 1, 2, \ldots$, the cube V_ε can be partitioned into k^n cubes $k^{-1}V_\varepsilon$. Then the asymptotic additivity of $N(\lambda)$ (see 3.1.F$_1$) implies for "dim" = ess that $N(k\lambda) \sim k^n N(\lambda)$ for all *integers* $k > 0$.

3.4.B Asymptotic homogeneity of $N^0(\lambda)$ for domain $V \subset \mathbb{R}^n$. Recall that $N^0(\lambda, V) = N(\lambda, V/\infty)$ refers to E on functions with compact supports in V, where V is an open subset in $\mathbb{R}^n$. We denote by $aV \subset \mathbb{R}^n$ the homothety (scaling) of V by $a \in \mathbb{R}$ and write

$$\sum_{i=1}^{k} a_i V \prec W ,$$

if there exist vectors $b_i \in \mathbb{R}^n$, such that the translates $a_i V_i + b_i \subset \mathbb{R}^n$ do not intersect and are all contained in W. Now the homogeneity of $E(x)$ together with the obvious superadditivity of $N^\circ(\lambda)$ imply the following property of $N^0(\lambda) =$ "dim"$E^{-1}(-\infty, \lambda)$ for $E = L_p d/L_q$, and $p \geq q$, and for all "dim" satisfying (i)-(vi) in 0.4.

(*) *The relation*

$$\sum_{i=1}^{k} a_k V \prec W$$

implies the inequality

$$\sum_{i=1}^{k} N^0(a_i \lambda, V) \leq N^0(\lambda, W) ,$$

for all open subsets V and W in $\mathbb{R}^n$ and all strings of real numbers a_i.

Now we recall the following

Trivial Lemma. *Let V be a bounded open supset in $\mathbb{R}^n$ and $N(\lambda)$ a positive function in $\lambda \in (0, \infty)$, such that*

$$\sum_{i=1}^{k} N(a_j \lambda) \leq N(a_0 \lambda)$$

for all strings of real numbers a_i satisfying

$$\sum_{i=1}^{k} a_i V \prec a_0 V .$$

Then

$$\limsup_{\lambda \to \infty} \lambda^{-n} N(\lambda) = \liminf_{\lambda \to \infty} \lambda^{-n} N(\lambda) ,$$

that is

$$N(\lambda) \sim C \lambda^n$$

for some $C \in [0, \infty]$, provided the boundary $\partial V \subset \mathbb{R}^n$ has measure zero.

3.4.B$_1$. On Positivity and finiteness of constant C. The above discussion shows that the spectral function $N^0(\lambda) = N^0(\lambda, V/\infty)$ for $E = L_p d/L_q$ and $p \geq q$ satisfies Weil's relation $N^o(\lambda) \sim C\lambda^n$, where $C < \infty$ for all "dim" and $p \geq q$ by Poincaré's Lemma. It is obvious that $C > 0$ for $p = q$ and all "dim". Furthermore, if "dim" $=$ ess, then $C > 0$ for all $p \geq q$, as it follows from 3.3. On the other hand if $p > q$ and "dim" $=$ pro then $C = 0$. In fact

$$N^0(\lambda) \asymp \lambda^{n-\theta}$$

for some $\theta > 0$ which can be explicitly determined by the standard approximation techniques, (see [Kaŝ] and [Pi]). Probably, the $\sim$asymptotics also follows by those techniques.

3.4.B$_2$. Determination of $C_0 = C/\operatorname{Vol} V$. It is clear from the previous discussion that $C = C_0 \operatorname{Vol} V$ where $C_0 = C_0(n,p,q)$ a universal constant. If $p = q = 2$ one known this C_0 from the spectrum of the Laplace operator $\Delta = d^*d$, but apart from this case the exact determination of C_0 (or of the asymptotics for $n \to \infty$) seems to run into the same problem as for the covering constant of $\mathbb{R}^n$ by equal balls.

3.4.C. Asymptotics $N(\lambda) \sim C\lambda^n$ for Riemannian manifolds V. Small balls in V are almost isometric to those in $\mathbb{R}^n$ for $n = \dim V$. It follows that

$$N(\lambda) \sim C_0(\operatorname{Vol} V)\lambda^n$$

for the above C_0 and under the same conditions as p and q as for domains in $\mathbb{R}^n$. Notice, that for $p = q = 2$ one obtains much sharper asymptotics using heat and (or) wave equations. One might try to extend the heat equation method to other p and q by using some functional integral of $\exp -tE(x)$.

3.4.D Homogeneous Lie groups. Let V be a Lie group with a left invariant geodesic metric, such that for every $a > 0$, V admits an a-selfsimilarity, that is a map $a : V \to V$, such that

$$\operatorname{dist}(av_1, av_2) = a \operatorname{dist}(v_1, v_2)$$

for all v_1 and v_2 in V. It is well known that such a V is a nilpotent Lie group of Hausdorff dimension $d \geq n = \dim_{\mathrm{top}} V$, where $d = n$ iff $V = \mathbb{R}^n$. The argument of 3.4.B immediately yields Weyl's relation

$$N(\lambda) \sim C\lambda^d$$

for $p \geq q$. Furthermore, one knows (see [F-S], [Pa], [Var]) that this C behaves as that in 3.4.B$_1$.

3.4.E Smooth metric spaces. For metric spaces V_1 and V_2 one defines the Hausdorff distance, called $|V_1 - V_2|_H$, by the condition:

$|V_1 - V_2|_H \leq \varepsilon \iff$ their exists a metric on the disjoint union $V_1 \cup V_2$, which extend those on V_1 and on V_2, and such that the ordinary Hausdorff distance between the subsets V_1 and

V_2 in $V_1 \cup V_2$ is $\leq \varepsilon$. A more invariant but somewhat less convenient definition consists of mapping the Cartesian power V^N into $\mathbb{R}^M$ for $M = N(N-1)/2$ by $\{v_i\} \mapsto \operatorname{dist}(v_i, v_j)$ and then by measuring the Hausdorf distances of the images in $\mathbb{R}^M$ of V_1 and V_2 for all N.

If V_1 and V_2 carry some measures, we can incorporate these into the definition of the Hausdorff distance by either looking at the pushforward of the measures to $\mathbb{R}^M$, or with the following additional requirement on the metric in $V_1 \cup V_2$:

Every ε-ball B in $V_1 \cup V_2$ has $\mu_1(B) - \mu_2(B) \leq \varepsilon$, where μ_1 and μ_2 are the measures on V_1 and on V_2 respectively.

Now, for every metric space $V = (V, \operatorname{dist})$ we write

$$aV = (V, a \operatorname{dist}) ,$$

for all $a > 0$, and we call V (uniformly) C^1-*smooth*, if every two balls $B_{\varepsilon_1}(v_1)$ and $B_{\varepsilon_2}(v_2)$ in V satisfy

$$|\varepsilon_1^{-1} B_{\varepsilon_1}(v_1) - \varepsilon_2^{-1} B_{\varepsilon_2}(v_2)|_H \leq \delta \tag{$*$}$$

where δ depends only on $\operatorname{dist}(v_1, v_2)$ and $\delta \to 0$ for $\operatorname{dist}(v_1, v_2) \to 0$.

It is easy to show that every smooth geodesic space admits a *tangent cone* $T_v(V)$ at all $v \in V$, that is a homogeneous Lie group as in 3.4.D, such that $\varepsilon^{-1} B_\varepsilon(v)$ Hausdorff converges to the unit ball $B_1 \subset T_v(V)$,

$$|B_1 - \varepsilon^{-1} B_\varepsilon(v)|_H \longrightarrow 0 \quad \text{for} \quad \varepsilon \to 0 .$$

Next we say that V is μ-smooth for a given measure μ on V if $(+)$ incorporates the measure, where the ball $\varepsilon^{-1} B_2$ is given the measure of total mass one obtained by the normalization of $\mu \mid B_\varepsilon$. In this case $\varepsilon B_\varepsilon(v)$ converges to $B_1 \subset T_v$ together with μ and one can see that the spectrum D of $L_p d / L_q$ is semicontinuous that is $N(\lambda, T_v) \leq \liminf_{\varepsilon \to 0} N\big(\lambda, \varepsilon^{-1} B_\varepsilon(v)\big)$. Furthermore, if the (mollified) first eigenvalue of each ball B_ε in V is bounded from below by $\operatorname{const} \varepsilon^{-1} \mu(B)^{\frac{1}{p} - \frac{1}{q}}$, then the spectrum in continuous. It easily follows (under the same conditions as in 3.4.B) that

$$N(\lambda, V) \sim C \lambda^d$$

where d is the Hausdorff dimension is constant in v) and

$$C = \int_V C_0\big(T_v(V)\big) dv .$$

Remarks. (a) The asymptotics $N(\lambda) \sim C\lambda^d$ remains valid under milder (non-uniform) smoothness condition, where the tangent cone may not exist on some "thin" subset of V. In fact one can even replace the Hausdorff distance by another one which is concerned with the measure-images of V^N in $\mathbb{R}^M$ rather than the set-images. It would be interesting to find meaningful examples to justify such generalizations.

(b) The previous discussion has the following discrete counterpart, where D is a *difference operator* on a discrete set V. For example, we may consider the coboundary operator on 0-cochains on the set V of vertices of some graph. Then we consider an exhaustion of V by finite subsets V_i and study the asymptotics of the spectrum of $D \mid V_i$ for $i \to \infty$. The standard example is that of $V = \mathbb{Z}^n \subset \mathbb{R}^n$ where V_i is a ball of radius i around the origin. The smoothness of V must be now expressed in terms of the *tangent cone at infinity*, (which for metric spaces V refers to the Hausdorff limit of $(\operatorname{diam} V_i)^{-1} V_i$ for $i \to \infty$) and the spectral asymptotics are closely related to the thermodynamics limit in statistical mechanics. The existence of such limits in $\mathbb{R}^n$ is easy by the non-Abelian nilpotent case is non-trivial (see $[\mathrm{Pa}]_2$).

3.4.F Remarks on the case $p < q$. If $\dim V = 1$, then the energy $E = L_p d / L_q$ satisfies

$$N(\lambda) \asymp \lambda$$

for *all* p and q as it follows from 2.2. In general, if for example V is a domain in $\mathbb{R}^n$, one asks what happens for p and q in the range of the Sobolev embedding theorem, that is for

$$s = 1 - \frac{n}{p} + \frac{n}{q} \geq 0 \, .$$

Notice that the energy $E(x) = \|dx\|_p / \|x\|_q$ is *scale homogeneous* of degree s,

$$E(x \circ a) = a^s E(x) \ ,$$

and so the spectrum of E accumulates at zero for $s < 0$. On the other hand, by the embedding theorem the spectrum is discrete for $s > 0$ but the asymptotics (say for "dim" = ess) seems to be unknown for $p < q$. The most interesting case is that of $p = 1$ and $q = n/n - 1$ where $s = 0$ and the (non-compact) embedding theorem is still valid. This theorem bounds λ_1 away from zero (for all "dim") but I do not know if the spectrum is discrete (i.e., $\lambda_i \to \infty$), say for "dim" = ess.

3.4.G The asymptotics $N^0(\lambda) \sim C\lambda^{rn}$ for operators D of order r. Let D be a differential operator of *pure order* r on $\mathbb{R}^n$ with *constant coefficients*. In other words D is invariant under translations and

$$D(x \circ a) = a^r D(x) \ .$$

Then the previous argument implies that the corresponding $N^\circ(\lambda)$ for $E = L_p D / L_q$ and $p \geq q$ is asymptotic to $C\lambda^{\frac{n}{r}}$ where $0 \leq C \leq \infty$. If $D = \partial^r$, where $\partial^r x$ denotes the string of the partial derivatives of x of order r (e.g., $\partial^1 = d$) then $C < \infty$ by Poincaré's lemma.

The inequality $C < \infty$ remains true for all elliptic operators D and $1 < q \leq p < \infty$ as

$$\|\partial^r x\|_p \leq \text{const}\, \|Dx\|_p$$

for functions x with compact support in $\mathbb{R}^n$. In fact this is even true for pseudo-differential operators of order r which may be any *real* number, e.g., for $(\sqrt{\Delta})^r$ where Δ is the Laplace operator. On the other hand if one wishes to keep $p = \infty$, one should require that D has finite dimensional kernel on every open subset in $\mathbb{R}^n$ which is much stronger than ellipticity. Properties of such D are identical in most respects to those of ∂^r. (If $r = 1$ then $\partial^1 = d$ essentially is the only example, but for $r \geq 2$ there are plenty of such D. For example

$$D : x \longmapsto \left(\frac{\partial^2 x}{\partial u_1^2}, \frac{\partial^2 x}{\partial u_2^2}, \ldots, \frac{\partial^2 x}{\partial u_n^2} \right) \ .$$

3.4.G$_1$. The above discussion extends to homogeneous (nilpotent) Lie groups in place of $\mathbb{R}^n$. Here we look at left invariant operators of order r such that

$$D(x \circ a) = a^r D(x) \ .$$

Then the corresponding energy $E = L_p D / L_q$ is a-homogeneous of degree $s = r - \frac{d}{p} + \frac{d}{q}$, where d is the Hausdorff dimension of some (and hence any) left invariant and a-homogeneous geodesic metric on our group. Such homogeneity insures, as earlier, the asymptotics $N(\lambda) \sim \text{const}\, \lambda^{d/r}$. What is less trivial is the bound const $< \infty$ and, more generally, the discreteness of the spectrum for $s > 0$. For this we need some (hypo)-ellipticity of D. Probably, if D everywhere (formally as well as locally) has finite dimensional kernel, then the above spectrum is discrete. In fact this finiteness condition makes any mentioning of the group structure unnecessary but nilpotent groups enter through the back door anyway.

3.4.G$_2$. Another generalization consists of allowing *polylinear* operators on $\mathbb{R}^n$ of *pure degree* r, which means $D(x \circ a) = a^r D(x)$. Instead of the finite kernel condition, one should now postulate the discreteness of the spectrum of $L_\infty D / L_\infty$ (on all domains in $\mathbb{R}^n$). More interesting examples are provided by (elliptic) Monge-Ampere operators and the Yang-Mills operator.

3.4.G$_2$. Let us indicate some (very) non-elliptic operators D on $\mathbb{R}^n$ with (spectrally) interesting energy $E = L_pD/L_q$. First, let

$$Dx = \frac{\partial^n x}{\partial u_1, \dots, \partial u_n} ,$$

and restrict E to functions with compact support in a bounded domain in $\mathbb{R}^n$. This is especially attractive for $p = 1$ and $q = \infty$ where the problem is non-trivial even for $D = \partial^r$.

Another example is $D = \frac{\partial^3}{\partial u_1^3} + \alpha \frac{\partial^3}{\partial u_2^3}$ on $\mathbb{R}^2$ for some real α with the periodic boundary conditions. (Which means passing to the torus $\mathbb{R}^2/\mathbb{Z}^2$). Here the spectrum of E is intimately related to arithmetic properties of α. For example the discreteness of the spectrum for $\alpha = 2$ and $E = L_2D/L_2$ is a non-trivial theorem of Thue.

§4 Bezout intersection theory in P, Δ and $P \times \Delta$

4.1 Cohomological definition of ess. Recall that the $\mathbb{Z}_2$-cohomology of the projective space P^k is multiplicatively generated by a single 1-dimensional element, say α, such that $\alpha^i \neq 0$ for $i \leq k$ (and, of course $\alpha^i = 0$ for $i > k = \dim P^k$). With this one sees that $\operatorname{ess} P^k = k$, since the cohomology is homotopy invariant. In fact, one knows (and the proof is very easy) that for all locally closed (i.e., open $\cap$ closed) subsets $Q \subset P^k$, the "dimension" $\operatorname{ess} Q$ equals the greatest i, such that the class α^i does not vanish on Q. Now the subadditivity of ess follows from the fact that cohomology classes multiply like functions. Namely, *if α vanishes an A and β on B then the cup-product $\alpha \vee \beta$ vanishes on $A \cup B$*, where A and B are locally closed subsets in a topological space and α, and β are some cohomology classes of this space.

Here is another immediate corollary of the cohomological definition of "dimension" ess.

4.1.A. *Let S be a connected topological space with a continuous involution called $s \longleftrightarrow -s$, and let f be a symmetric continuous map of S into the sphere S^k, where symmetric means $f(-s) = -f(s)$. If the $\mathbb{Z}_2$-cohomology of S vanish in the dimensions $1, 2, \dots, i-1$, then the image in $P^k = S^k/\mathbb{Z}_2$ of the induced map $\overline{f} = f/\mathbb{Z}_2$ satisfies*

$$\operatorname{ess} \overline{f}(S/\mathbb{Z}_2) \geq i \ .$$

Remark. The vanishing assumption is satisfied for example, for the sphere S^j for $j \geq i$, since every $(i-1)$-dimensional subset is contactible in S^j for $j \geq i$. In particular, the existence of a symmetric map $f : S^j \to S^k$ implies that $j \leq k$. This fact is often called the *Borsuk-Ulam theorem.*

4.2 Bezout theorem. The Poincaré-Lefschetz duality between the cup-product and intersection shows that

$$\operatorname{coess} A \cap B \leq \operatorname{coess} A + \operatorname{coess} B\ , \tag{$*$}$$

where the coessence of a subset in P^k is $k - \text{ess}$, and where A and B are locally closed subsets in P^k.

Here is another form of Bezout theorem. Let $f : S^i \to S^k$ be a symmetric continuous map and $\overline{f} : P^i \to P^k$ be the induced map. Then

$$\operatorname{coess} \overline{f}^{-1}(A) \geq \operatorname{coess} A \tag{$**$}$$

for all $A \subset P^k$.

Example. Let $\varphi : S^i \to \mathbb{R}^j$ be a continuous map and $A \subset P^i$ consists of the pairs $(s, -s)$ such that $\varphi(s) = \varphi(-s)$. Then

$$\operatorname{coess} A \geq i - j\ . \tag{$***$}$$

In fact, let $\overline{f} : P^i \to P^{i+j-1}$ be defined by

$$\overline{f} : (s_0, \ldots, s_i) \longmapsto \big(s_0, \ldots, s_i, \varphi_1(s) - \varphi_1(-s), \ldots, \varphi_j(s) - \varphi_j(-s)\big)$$

where $s_0, \ldots, s_i$ are the coordinates of a (one out of two) point in S^i over $\overline{s} \in P^i$ and where $\varphi_1, \ldots, \varphi_j$ are the components of φ. Then A equals the pullback of the obvious j-codimensional subspace in P^{i+j+1} and Bezout theorem applies because $\overline{f}$ is covered by some f.

Remarks. (a) If $i \geq j$, then $(***)$ says that A is non-empty. This is another formulation of the Borsuk-Ulam theorem.

(b) Let us *define* coess$'$ of a subset A in a (possibly infinite dimensional) projective space P as the minimal i such that there exists a continuous map of P into another projective space, say $\overline{f} : P \to P'$, such that $\overline{f}$ can be covered by a symmetric map of the (spherical) double-coverings of P and P' and such that A contains the pull-back of a projective subspace in P' of codimension i. This coess$'$ satisfies $(*)$ and $(**)$ (almost) by definition. Moreover, by the Poincaré Lefschetz duality

$$\operatorname{coess}' \geq \operatorname{coess} = \dim P - \text{ess}$$

if P is finite dimensional.

Example. (a) Every i-coplane (see 0.5.B) obviously has coess$' = i$. Hence, it meets every i-plane by the above discussion.

(b) Let P be the projective space of continuous functions on V and $U \subset V$ be a measurable subset. Denote by $P_u \subset P$ the subset of functions x *equidividing* U, that is the subsets $x^{-1}(-\infty, 0] \cap U$ and $x^{-1}[0, \infty) \cap U$ have measure at least $\frac{1}{2}\mu(U)$. (If the zero level $x^{-1}(0) \subset V$ of X has measure zero, then these equal $\frac{1}{2}\mu(U)$.) Obviously coess$'$ $P_u \leq 1$.

4.2.A Corollary (Borsuk-Ulam again). *Let a subset $A \subset P$ have* ess $A \geq i$ *(e.g., A is projective of dimension i) and $U_1, \ldots, U_i$ are subsets in V. Then there exists a function $x \in A$ equidividing all i subsets.*

4.2.B An archetypical spectral application. Let V be a compact n-dimensional Riemannian manifold and $E(x)$ denotes the $(n-1)$-dimensional volume *of the zero set* $x^{-1}(0) \subset V$. Then *the spectrum* $\{\lambda_i\}$ *of this* E *satisfies*

$$\lambda_i \asymp i^{\frac{1}{n}} .$$

Proof: To bound λ_i from below partition V into i subsets U_i which are roughly isometric to the Euclidean ball of radius $\varepsilon = i^{-\frac{1}{n}}$. Then the above equidividing function x satisfies according to the (isoperimetric) Poincaré lemma,

$$E(x) \geq \text{const}\, i\varepsilon^{n-1} = \text{const}\, i^{\frac{1}{n}} .$$

Next, for the upper bound, first let V be a domain in $\mathbb{R}^n$. Then the space P_d of polynomials of degree $\leq d$ has $i = \dim P_d \asymp d^n$. Since the zero set Σ of a polynomial of degree $\leq d$ meets every line at no more than d points,

$$\operatorname{Vol}_{n-1}(\Sigma \cap V) \leq d(\operatorname{Diam} V)^n ,$$

which provides the required upper bound on λ_i for $V \subset \mathbb{R}^n$. In the general case, one may apply a similar argument to an algebraic realization (due to J. Nash) of V in some Euclidean space $\mathbb{R}^N$.

Question. Let P be the space of maps $x : V \to \mathbb{R}^m$ for some $m < n$ and

$$E(x) = \operatorname{Vol}_{n-m} E^{-1}(0) .$$

Then the above polynomial example shows that

$$\lambda_i \leq \text{const}\, i^{\frac{m}{n}} .$$

But I do not even know how to prove that $\lambda_i \underset{i \to \infty}{\longrightarrow} \infty$ for $m \geq 2$.

4.3 $\mathbb{Z}_2$-simplices. . Consider a topological space S and a continuous map π of S into a (finite or infinite dimensional) simplex. A *k-face*, say $S_k \subset S$ by definition is the pull-back of a k-face Δ_k in Δ and the boundary ∂S_k is the pull-back of the boundary of Δ_k. Recall that the $\mathbb{Z}_2$-cohomology of the pair $(\Delta_k, \partial\Delta_k)$ equals $\mathbb{Z}_2$ in dimension k and say that S is a *$\mathbb{Z}_2$-simplex* if the generator of this cohomology group, say $h(\Delta_k)$, goes by π^* to a *non-zero* element in $H^k(S_k, \partial S_k; \mathbb{Z}_2)$, say to $h(S_k)$, for all finite dimensional faces S_k of S.

Examples. (a) If S contains a subset S', such that $\pi : S' \to \Delta$ is a *homeomorphism*, then S is a $\mathbb{Z}_2$-simplex.

(b) Let $\pi : \Delta \to \Delta$, where π sends every face of Δ *into* itself. Such a map is homotopic to the identity (by an obvious linear homotopy) and so this is a $\mathbb{Z}^2$-simplex. hence, *the map π necessarily is surjective.*

In fact, one has the following obvious (modulo elementary homology theory):

4.3.A Proposition. *Let $\pi' : S \to \Delta$ be a continuous map sending each face $S_k = \pi^{-1}(\Delta_k)$ of S to Δ_k. Then π' is onto.*

4.3.B Basic example. Let S be the space of sequences $s = s_0, s_1, \ldots, s_k, \ldots$ of non-negative L_q-functions on V, such that the sum

$$\sigma = \sum_k \int s_k(v)dv$$

satisfies

$$0 < \sigma < \infty ,$$

and define $\pi : S \to \Delta$ by

$$\pi : s \longmapsto \left(\int s_0/\sigma, \int s_1/\sigma, \ldots, \int s_k/\sigma \ldots\right) .$$

If the implied measure μ on V is continuous, as we shall always assume below, then this is a $\mathbb{Z}_2$-simplex. Important subsimplices in S are:

(a) $S_\chi \subset S$, where every s_k equals 0 or 1, i.e., s_k is the characteristic function of the set where $s_k = 1$.

(b) $S_{co} \subset S_\chi$, where the implied subsets *cover* V.

(c) $S_{pa} \subset S_{co}$, where the subsets *partition* V, that is $\sum\limits_k s_k = 1$.

Denote by $S(k)$ the set of sequences with $s_j = 0$ for $j > k$ and look at the induced $\mathbb{Z}_2$-simplex structure over $\Delta^k \subset \Delta$.

4.3.B$_1$ Proposition. *Let $T \subset S(k)$ be $\mathbb{Z}_2$-subsimplex (over Δ^k) and x a bounded function on V. Then there exists $s = (s_0, \dots, s_k) \in T$, such that*

$$\int_V s_0 x = \int_V s_1 x = \dots = \int_V s_k x \,.$$

Proof: We may assume T is compact which provides a constant C, such that

$$\sigma_i = C + \int_V s_i x > 0$$

for all $i = 0, \dots, k$ and $s \in T$. Then 4.2.B applies to the map $T \to \Delta^k$ defined by $s \to (\sigma_0/\sigma, \dots, \sigma_k/\sigma)$, for $\sigma = \sum_{i=0}^{k} \sigma_i$.

4.4 $\mathbb{Z}_2$-simplices in $P \times \Delta$. Let $\overline{f} : P^k \times \Delta \to P$ be a continuous map which admits a lift to a continuous map $S^k \times \Delta \to S$, where S^k and S are the spheres double covering P^k and P respectively. Then by the elementary homology theory the pull-back $T = \overline{f}^{-1}(P') \subset P^k \times \Delta$ of every *k-coessential* subset $P' \subset P$ (i.e., $\operatorname{coess} P' \leq k$) is a $\mathbb{Z}_2$-simplex for the projection $T \to \Delta$. In fact the same conclusion remains valid for every k-essential subset $Q \subset P^\infty$ instead of P^k. This leads to the following unification of 4.2.A and 4.3.B$_1$.

4.4.A. *Let $T \subset S(k)$ be a $\mathbb{Z}_2$-simplex (over Δ^k) in the space of sequences of subsets $V_0, \dots, V_k$ in V and let Q be a $(k+1)$-essential (i.e., $\operatorname{ess} Q \geq k+1$) set of continuous functions on V. Then there exist a function $x \in Q$ and a sequence $(U_0, \dots, U_k) \in T$, such that*

(1) the zero level $x^{-1}(0) \subset V$ equidivides all $U_0, \dots, U_k$ (in the sense of 4.2.A).

(2) For a given $p < \infty$

$$\int_{U_0} |x|^p = \int_{U_1} |x|^p = \dots = \int_{U_k} |x|^p \,.$$

4.4.A$_1$ Remarks. (a) one can replace (1) by the following

$$(1') \qquad \sup_{v \in U_j} x(v) = - \inf_{v \in U_j} x(v) \quad \text{for} \quad j = 0, \dots, k \,.$$

In fact one can require any "equidivision property" in-so-far as the "division" is continuous in $x \in Q$.

(b) Suppose each open subset $U_j(t)$ is continuous in $t \in T$ for the Hausdorff metric is the space of subsets and assume that

$$\mu\big(U_j(t)\big) = 0 \Longleftrightarrow \operatorname{Diam} U_j(t) = 0$$

for all $t \in T$ and $j = 0, \dots, k$. Then the condition (2) can be replaced by

$$\sup_{v \in U_0} x(v) = \sup_{v \in U_1} x(v) = \dots = \sup_{v \in U_k} x(v) \ .$$

In fact, one can use here any notion of size of x on U_j, satisfying an obvious continuity condition.

4.4.B Spectra for Δ-dim. We have seen in 2.2 and 3.4.F how the above proposition is used to bound from below the spectrum of $L_1 d/L_\infty$ on the unit interval. To obtain a similar bound for a domain $V \subset \mathbb{R}^n$ (say for $L_1 d/L_q$ and $q = \frac{n}{n-1}$ or for $L_1 \partial^n / L_\infty$) one needs a $\mathbb{Z}_2$-simplex of partitions into "sufficiently round" subsets. One can also us coverings rather than partitions if one controls the multiplicity.

To be able to use our spectral language we say that a set Q of coverings of V by $k+1$ subsets $V_0, \dots, V_k$ has Δ-dim $Q \geq k$ if it contains (and hence is) a $\mathbb{Z}_2$-simplex over Δ^k. Now for every energy $E = E(s)$ we can define the Δ-dim-spectrum of E. Here are some interesting energies.

(a) $$E^\circ(s) = \sup_{0 \leq j \leq k} (\operatorname{Diam} V_j)/(\operatorname{Vol} V_j)^{1/n}$$

(b) $$E^\lambda(s) = \sup_{0 \leq j \leq k} \lambda_1(V_j)^{-1} \ ,$$

where λ_1 is the first eigenvalue of a pertinent energy on V_j, say of $L_q \partial^r / L_\infty$ on V_j. (One can generalize this by using any λ_i for $i \geq 1$.)

(c) $E^\mu(s) =$ the measure theoretic multiplicity of the covering, that is the L^∞-norm of the sum of the characteristic functions of U_j.

Questions. What are the spectra of $E^\circ + E^\mu$ and of $E^\lambda + E^\mu$? What are the spectra of E° and E^λ on the space of partitions (i.e., for $E^\mu = 1$)?

Example. For every compact smooth domain $V \subset \mathbb{R}^2$ one can easily construct a k-simplex of *partitions* for all $k = 1, 2, \dots$ such that $E^\circ(s) \asymp k$. It is unlikely that one can make $E^\circ \asymp 1$, (i.e, bounded) but something like $E^\circ(s) \asymp k^{1/2}$ might be possible.

Remark. The energy $E^\lambda + E^\mu$ (or E^λ on partitions) is designed to bound from below the spectrum of a pertinent energy on functions x on V but it is unclear how sharp such a bound might be. In other words we want to know how close E^λ is to the dual of E from where λ_1 (or λ_i) comes.

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